A CONTRIBUTION TO THE THEORY OF SELF-RENEWING AGGREGATES, WITH SPECIAL REFERENCE TO INDUSTRIAL REPLACEMENT

By Alfred J. Lotka

1. **Introduction.** The analysis of problems of industrial replacement forms part of the more general analysis of problems presented by "self-renewing aggregates."[1] While the subject could, therefore, be treated in general and consequently rather abstract terms, for the purpose of exposition it will be advantageous to relate the discussion to concrete applications. These, in the past, have been mainly of two kinds, namely, first, applications to population analysis with related problems in genetics on the one hand and actuarial problems on the other; and second, applications to industrial replacement. As the fundamental setting of the two types of problems is very similar, leading in each case to certain integral equations, it will be advantageous to consider together both problems, or both phases of the general problem. This will incidentally give us an opportunity to observe the analogy, but also certain points of difference, between the two aspects of the problem.

Historically, the investigation of an actuarial problem came first. L. Herbelot[2] (1909) examined the number of annual accessions required to maintain a body of N policyholders constant, as members drop out by death. He assumes an initial body of N "charter" members at time $t = 0$, all of the same age, which for simplicity may be called age zero, since this merely amounts to fixing an arbitrary origin of the age scale. He further assumes the same uniform age at entry for each "new" member.

Then, if $p(t)$ is the probability at the age of entry of surviving t years, the survivors of charter members at time t will number $Np(t)$; and if $f(\tau)$ is the rate per head at which members drop out by death at time τ, being then immediately replaced by a new member of the fixed age of entry, then the survivors at time t of "new" members will evidently be given by

$$N \int_0^t f(\tau)p(t - \tau)\, d\tau$$

[1] I use here an English equivalent, as nearly as possible, to the German phrase "sich erneuernde Gesamtheiten," used by Swiss actuaries.

[2] Herbelot's original paper is disfigured with a number of misprints. It is essentially reproduced, with the errors corrected, in a paper by R. Risser (1912). The same treatment of the problem is also given by Zwinggi (1931) and by Schulthess (1935), (1937).

Hence, the condition for a constant membership N is

$$Np(t) + N \int_0^t f(\tau)p(t - \tau)\, d\tau = N \tag{1}$$

or

$$p(t) + \int_0^t f(\tau)p(t - \tau)\, d\tau = 1 \tag{2}$$

Differentiating with regard to t, and remembering that $p(0) = 1$, we have

$$p'(t) + \int_0^t f(\tau)p'(t - \tau)\, d\tau + f(t) = 0 \tag{3}$$

Equation (3) may be written

$$f(t) = -p'(t) - \int_0^t f(\tau)p'(t - \tau)\, d\tau \tag{4}$$

or, putting $(t - \tau) = a$

$$f(t) = -p'(t) - \int_0^t f(t - a)p'(a)\, da. \tag{5}$$

For the solution of the integral equation thus obtained Herbelot uses the method of successive differentiations,[3] duly pointing out its limitations, and applying it to several specific expressions for the survival function $p(a)$.

There is nothing in Herbelot's treatment to limit its application to living organisms. It is directly applicable to the problem of industrial replacement of an equipment comprising N original units installed at time $t = 0$, and maintained constant by the replacement of disused units with new.

Next in chronological order, of publications dealing with the type of problem with which we are here concerned, is a paper by Sharpe and Lotka (1911), who use Hertz's form of solution for the integral equation involved.[4] To this I wish

[3] This method is also followed in dealing with the problem of renewal by Risser (1912), (1920); Zwinggi (1931); Schulthess (1935), (1937); Preinreich (1938). All these authors applied their reflections to arbitrarily assumed frequency distributions for the renewal function, of simple analytical form. For example, among the more recent applications is one by Schulthess, who uses the function $p(t) = \left(1 - \frac{t}{\omega}\right)^m$; and quite recently, Preinreich has suggested the use of a Type I Pearson frequency curve on the basis of Kurtz's observational data. It is to be noted, however, that when it comes to actual application, Preinreich does not use an ordinary Pearson Type I curve nor actual observational data of any kind, but very conveniently simplifies the Pearson formula by giving integral values, namely 1 and 2, to the exponents, thereby reducing to triviality the task of applying the method of differentiation. None of these authors makes any attempt to deal with actual numerical observations which, in practice, fall far wide of any of the simple analytical formulae employed by them.

[4] P. Hertz, *Mathematische Annalen*, 1908, vol. 65, pp. 84 to 86.

to refer in some detail, adding to the original exposition in the light of later developments. The treatment of the subject proceeds here along somewhat broader lines, but, with obvious changes in the meaning of the symbols, and with certain modifications and limitations which are themselves of interest, the development is immediately applicable to economic systems composed of units having a characteristic "mortality" in use.

A population of living organisms, unlike industrial equipment, has practically no beginning. We know its existence only as a continuing process. Accordingly the equation for its development is most naturally framed without *explicit* reference to any "charter members."

The basis of the analysis is as follows:

In a population growing solely by excess of births over deaths (i.e. in the absence of immigration and emigration), the annual female births $B(t)$ at time t are the daughters of mothers a years old, born at time $(t - a)$ when the annual female births were $B(t - a)$. If fertility and mortality are constant and such that a fraction $p(a)$ of all births survive to age a, and are then reproducing at an average rate $m(a)$ daughters per head per annum, then, evidently,[5]

$$B(t) = \int_0^\infty B(t - a)p(a)m(a)\,da \tag{6}$$

$$= \int_0^\infty B(t - a)\varphi(a)\,da. \tag{7}$$

This is the fundamental equation in its original form, and, as noted above, it does not explicitly refer to any initial state, though, as will be seen presently, in order to make the problem determinate, data regarding the system at some particular period must be given. For the present we note that (7) can be written

$$B(t) = \int_t^\infty B(t - a)\varphi(a)\,da + \int_0^t B(t - a)\varphi(a)\,da \tag{8}$$

$$B(t) = B_1(t) + \int_0^t B(t - a)\varphi(a)\,da. \tag{9}$$

It is to be noted that the right hand member of (8), splits the total births $B(t)$ into two sections, those in which $(t - a) < 0$, that is, births of daughters whose mothers were born *before* $t = 0$; and those for which $(t - a) > 0$, that is births of daughters whose mothers were both *after* $t = 0$. The former section is denoted by $B_1(t)$ in (9). The function $B_1(t)$ thus defined will be found, in the

[5] Here and elsewhere in these developments the limits of the integral have, for simplicity, been written 0 and ∞. This ensures the inclusion of all nonvanishing terms in the integrand; the inclusion of terms for which either $\varphi(a)$ or $B(t - a)$ vanishes does not, of course, affect the value of the integral. If $\varphi(a)$ is represented between the limits α, ω of the reproductive period by some analytical expression, such as a Pearson frequency function, it is, of course, understood that outside the range α, ω we must put $\varphi(a) = 0$.

further development, to play a significant rôle. Here it will suffice to point out that it vanishes for all values of t greater than ω, the upper limit of the reproductive period, because $\varphi(a)$ vanishes for these values of a.

2. **Special case.** A case of special interest is that in which $B_1(t)$ represents the births of daughters whose mothers were all born in an interval of time $t = -dt$ to $t = 0$. In that case the first integral in (8) reduces to a single term, so that

$$B(t) = B(0)\varphi(t)\,dt + \int_0^t B(t-a)\varphi(a)\,da \tag{10}$$

or, putting

$$B(0)\,dt = N_0 \tag{11}$$

$$B(t) = N_0\varphi(t) + \int_0^t B(t-a)\varphi(a)\,da. \tag{12}$$

This last equation holds also if a finite number of births take place (or are regarded as taking place) at a point of time $t = 0$.

Equations (10) and (12) are of interest as basic for the examination of the progeny of an infinitesimal population element,[6] that is, of a "zero" generation, born at time zero. In that case $B_1(t)$ is the annual rate of births in the "first" generation, and is simply proportional to $\varphi(t)$, i.e.

$$B_1(t) = N_0\varphi(t) \tag{13}$$

For the sake of greater generality the development has so far been given in terms of the phenomenon of replacement (reproduction) as it presents itself in a population of living organisms. But it should be noted here that, with appropriate changes in the meaning of $\varphi(a)$, equation (12) is directly applicable to the problem of industrial renewal in an installation originally installed at some point of time and maintained at a constant level by the replacement of each unit by a new one, the moment it is disused. In that case the "rate per head of reproduction" $m(a)$ at age a is evidently the same thing as the "death rate per head" at age a, namely

$$\mu(a) = -\frac{dp(a)}{p(a)\,da} = -\frac{p'(a)}{p(a)} \tag{14}$$

so that

$$\varphi(a) = p(a)\mu(a) \tag{15}$$

becomes

$$\varphi(a) = -p'(a). \tag{16}$$

[6] A. J. Lotka, (1928), (1929).

Reverting now to the fundamental equation in its first form (6), a trial substitution

$$B(t) = Qe^{rt} \tag{17}$$

is found to satisfy this equation, provided that r is a root of the characteristic equation

$$\int_0^\infty e^{-ra}\varphi(a)\,da = 1 \tag{18}$$

We may speak of (17) as a particular solution of (6) or (7). It is easily seen that the sum of such particular solutions is also a solution, i.e.

$$B(t) = Q_1e^{r_1t} + Q_2e^{r_2t} + \cdots \tag{19}$$

where r_1, r_2 etc., are roots of the characteristic equation (18).[7]

For real values of r the function

$$F(r) = \int_0^\infty e^{-ra}\varphi(a)\,da \tag{20}$$

decreases monotonically as r increases, since, from its nature, $\varphi(a) > 0$ for all values of a. Hence (18) can have only one real root r_1, and we shall have

$$r_1 \gtreqless 0 \quad \text{according as} \quad \int_0^\infty \varphi(a)\,da \gtreqless 1. \tag{21}$$

If $u + iv$ is a complex root of (18) then

$$1 = \int_0^\infty e^{-ua}\cos va\,\varphi(a)\,da \tag{22}$$

$$0 = \int_0^\infty e^{-ua}\sin va\,\varphi(a)\,da \tag{23}$$

and it is evident from (22) that $u < r_1$, since $\cos(va) \leqq 1$ for all values of a. The real part of any complex root of (18) is, therefore, algebraically less than the real root r_1.

This reasoning[8] is evidently quite independent of the particular form of $\varphi(a)$, and is thus equally true, whether $\varphi(a)$ be given in purely empirical form (defined by a table of values), or as a standard form of frequency curve, such as for example a Pearson curve of suitable type.

The roots of (18) can be determined directly, though rather laboriously, from

[7] For a discussion of the convergence of the series (19) see G. Herglotz, *Mathem. Annalen*, 1908, vol. 65, pp. 87 et seq.

[8] Adapted from P. Hertz, *Math. Annalen*, 1908, vol. 65, pp. 1–86; G. Herglotz, ibid. pp. 87–106. The Hertz solution is also applied to a similar problem by J. B. S. Haldane, *Proc. Cambridge Phil. Soc.*, 1926, vol. 23, p. 607. A particularly detailed development is given by H. T. J. Norton, *Proc. London Math. Soc.*, 1926, vol. 28, p. 21.

equations (22) and (23); or, they can be brought into relation with the Thiele semivariants μ of the function $\varphi(a)$ defined by

$$F(r) = \int_0^\infty e^{-ra}\varphi(a)\,da = m_0 e^{-\mu_1 r + \frac{1}{2!}\mu_2 r^2 - \cdots} \tag{24}$$

where m_n is the nth moment of $\varphi(a)$ and the seminvariants μ can be computed from the moments by the algorithm

$$\begin{cases} m_1 = \mu_1 m_0 \\ m_2 = \mu_1 m_1 + \mu_2 m_0 \\ m_3 = \mu_1 m_2 + 2\mu_2 m_1 + \mu_3 m_0 \\ m_4 = \mu_1 m_3 + 3\mu_2 m_2 + 3\mu_3 m_1 + \mu_4 m_0 \end{cases} \tag{25}$$

etc.

In terms of these seminvariants the characteristic equation (18) becomes

$$\mu_1 r - \mu_2 \frac{r^2}{2!} + \cdots - \log_e m_0 = \log_e 1 = 2\pi n i \tag{26}$$

where n takes on all positive and negative integral values. Separating the real and imaginary parts in (26), and retaining seminvariants up to the fourth,

$$\psi(u, v) = \frac{\mu_4}{4!}(u^4 - 6u^2v^2 + v^4) - \frac{\mu_3}{3!}u(u^2 - 3v^2) + \frac{\mu_2}{2!}(u^2 - v^2) - \mu_1 u + \log_e m_0 = 0 \tag{27}$$

$$\chi(u, v) = \frac{\mu_4}{3!}uv(u^2 - v^2) + \frac{\mu_3}{3!}v(v^2 - 3u^2) + \mu_2 uv - \mu_1 v = 2\pi n. \tag{28}$$

If $\varphi(a)$ does not differ too widely from the normal (Gaussian) distribution, so that seminvariants of higher than second order can be neglected for roots in the neighborhood of $u = 0$, $v = 0$, we shall have, approximately[9]

$$\frac{\mu_2}{2!}(u^2 - v^2) - \mu_1 u + \log_e m_0 = 0 \tag{29}$$

$$\left(u - \frac{\mu_1}{\mu_2}\right)v = \frac{2\pi n}{\mu_2} \tag{30}$$

[9] The relations which follow hold *exactly* if $\varphi(a)$ is actually a normal curve. It should be noted, however, that this can not be strictly the case, since the infinite tail of the curve on the negative side would imply replacement or reproduction antedating the original installation or zero generation. Nevertheless, a normal frequency curve will be admissible if the part of the curve extending into the negative age field is negligible. For a concrete example (electric light bulbs) see E. J. Gumbel, "Die Verteilung der Gestorbenen um das Normalalter," *Aktuarske Vedy* (Praze), 1933, p. 90.

or, putting

$$\left(u - \frac{\mu_1}{\mu_2}\right) = U \tag{31}$$

we have

$$U^2 - v^2 = \left(\frac{\mu_1}{\mu_2}\right)^2 - \frac{2 \log_e m_0}{\mu_2} \tag{32}$$

$$Uv = \frac{2\pi n}{\mu_2}. \tag{33}$$

It is thus seen that in these circumstances the roots u, v correspond to the points of intersection of the hyperbola (32) centered at $u = \frac{\mu_1}{\mu_2}$, $v = 0$, with a family of hyperbolas (33) concentric with (32), but with their axes at 45° to those of (32).

The intersections of the hyperbolas (33) with the axis of v are given by putting $u = 0$ in (30), namely

$$v = \frac{2\pi n}{\mu_1} \tag{30a}$$

This also gives, approximately, the frequency of the oscillatory components for which u is sufficiently small. In particular, for the first component, we have, in that case

$$v = \frac{2\pi}{\mu_1} \tag{30b}$$

so that its wave length is (approximately) μ_1, the mean of the $\varphi(a)$ curve.

These facts are illustrated in Fig. 1, drawn to scale according to the vital statistics of the United States, 1920, for which the requisite computations were available from prior publications (Lotka, (1928), (1929)). The diagram is drawn in full, showing four intersections of each hyperbola of the family (33). Actually values of v occur in pairs, corresponding to conjugate roots $u \pm iv$. The intersections in the two upper quadrants must be disregarded, as they do not correspond to roots of (18).

To simplify notation let us write (32), (33) in the form

$$U^2 - v^2 = K \tag{32a}$$

$$Uv = C. \tag{33a}$$

Solving for U^2, v^2 we find

$$U^2 = \tfrac{1}{2}\{K \pm \sqrt{K^2 + 4C^2}\} \tag{34}$$

$$v^2 = \tfrac{1}{2}\{-K \pm \sqrt{K^2 + 4C^2}\} \tag{35}$$

from which, incidentally, it is seen that

$$U^2 + v^2 = \sqrt{K^2 + 4C^2} \tag{36}$$

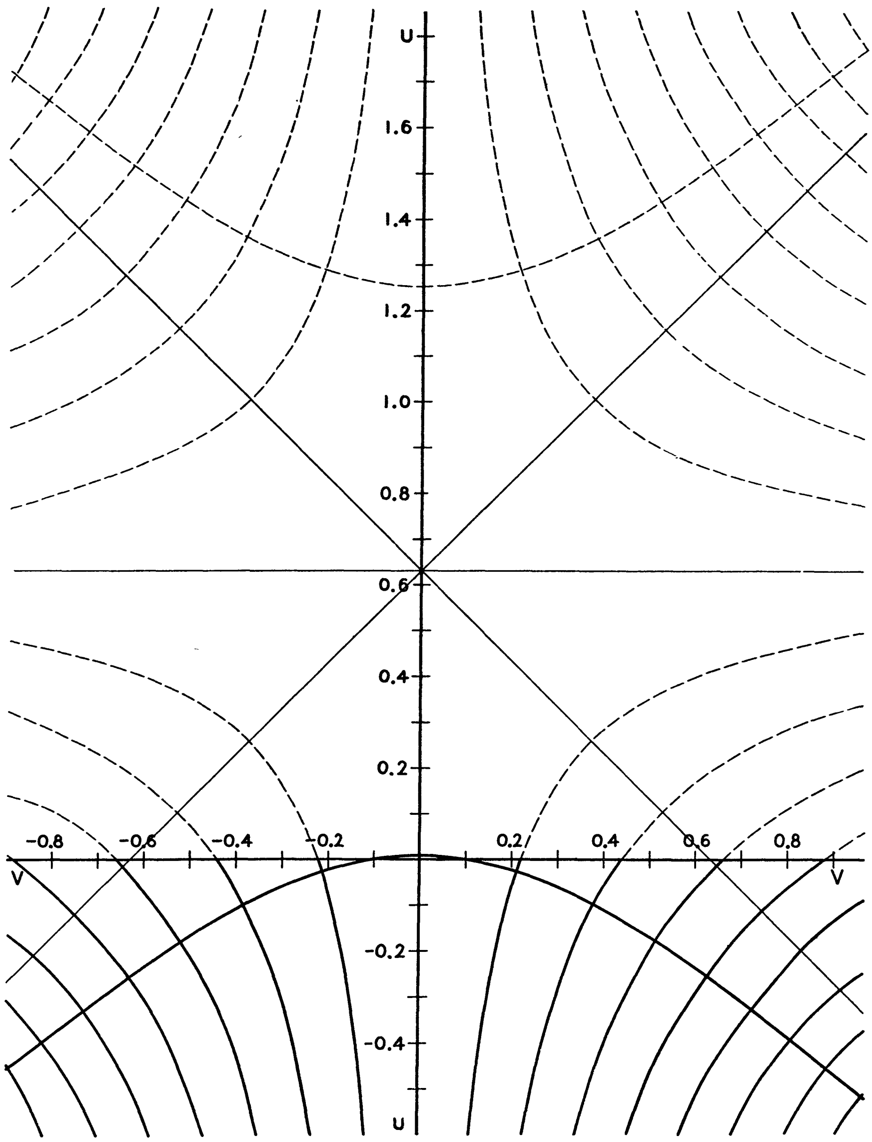

FIG. 1. ROOTS OF FUNDAMENTAL EQUATION (18) AS INTERSECTIONS OF CURVE (32) WITH FAMILY OF CURVES (33)

and hence, that the intersections of the hyperbola (32) with (33) lie on circles of radius

$$R = \sqrt[4]{K^2 + 4C^2}. \tag{37}$$

When the third and fourth moments (and therefore third and fourth seminvariants) are taken into account[10] the hyperbolas become distorted into new curves, though the general topographic features of the diagram tend to be preserved. In particular, the property of orthogonality of intersection of the curves (32) with (33) is preserved, in accordance with a well-known property of conjugate functions.[11] This is shown in the left hand panel of Fig. 2, drawn for the same data as Fig. 1, but including not only the hyperbolic curves, but also the corresponding modified curves obtained by retaining the third and fourth seminvariants in the computation.[12] Only the quadrant relevant to the location of the roots is shown.

3. **The coefficients Q in the solution** (19). These are determined by initial conditions, being, in fact related to the function $B_1(t)$. As their determination in the original paper by Hertz and Herglotz is rather complicated, the following relatively simple method, resembling that by which the constants in a Fourier series are determined, is of interest:

Multiplying equation (9) by $e^{-r_i t}$, where r_i is a root of (18), transposing terms, and integrating between the limits 0 and ω, where ω is the highest age for which $\varphi(a)$ has a value other than zero, we have

$$\int_0^\omega e^{-r_i t} B_1(t)\, dt = \int_0^\omega e^{-r_i t}\left\{B(t) - \int_0^t B(t-a)\varphi(a)\, da\right\} dt. \tag{38}$$

Introducing the solution (19) in the right hand member of (38), we obtain

$$\int^\omega e^{-r_i t} B_1(t)\, dt = \sum Q_j \int_0^\omega e^{-r_i t}\left\{e^{r_j t} - \int_0^t e^{r_j(t-a)}\varphi(a)\, da\right\} dt \tag{39}$$

$$= \sum P_{ij} \qquad (j = 1, 2, 3, \cdots). \tag{40}$$

Consider now a particular term P_{ij} in the sum $\sum$. Multiplying out the exponentials we obtain

$$P_{ij} = Q_j \int_0^\omega e^{-(r_i - r_j)t}\left\{1 - \int_0^t e^{-r_j a}\varphi(a)\, da\right\} dt \tag{41}$$

which, in view of the characteristic equation (18) reduces to

$$P_{ij} = Q_j \int_0^\omega e^{-(r_i - r_j)t} \int_t^\omega e^{-r_j a}\varphi(a)\, da\, dt \tag{42}$$

$$= Q_j \int_0^\omega e^{-r_j a}\varphi(a) \int_0^a e^{-(r_i - r_j)t}\, dt\, da. \tag{43}$$

Hence, if $i \neq j$

$$P_{ij} = \frac{Q_j}{r_j - r_i} \int_0^\omega e^{-r_j a}\varphi(a)\{e^{-(r_i - r_j)a} - 1\}\, da \tag{44}$$

[10] Which is as far as curve fitting by Pearson's method goes.

[11] See, for example, W. E. Byerly, *Integral Calculus*, 1888, p. 289.

[12] For a given value of u equation (27) is a biquadratic in v, and equation (28) is a cubic in v lacking the second degree term. The computation of the curves is in consequence relatively simple.

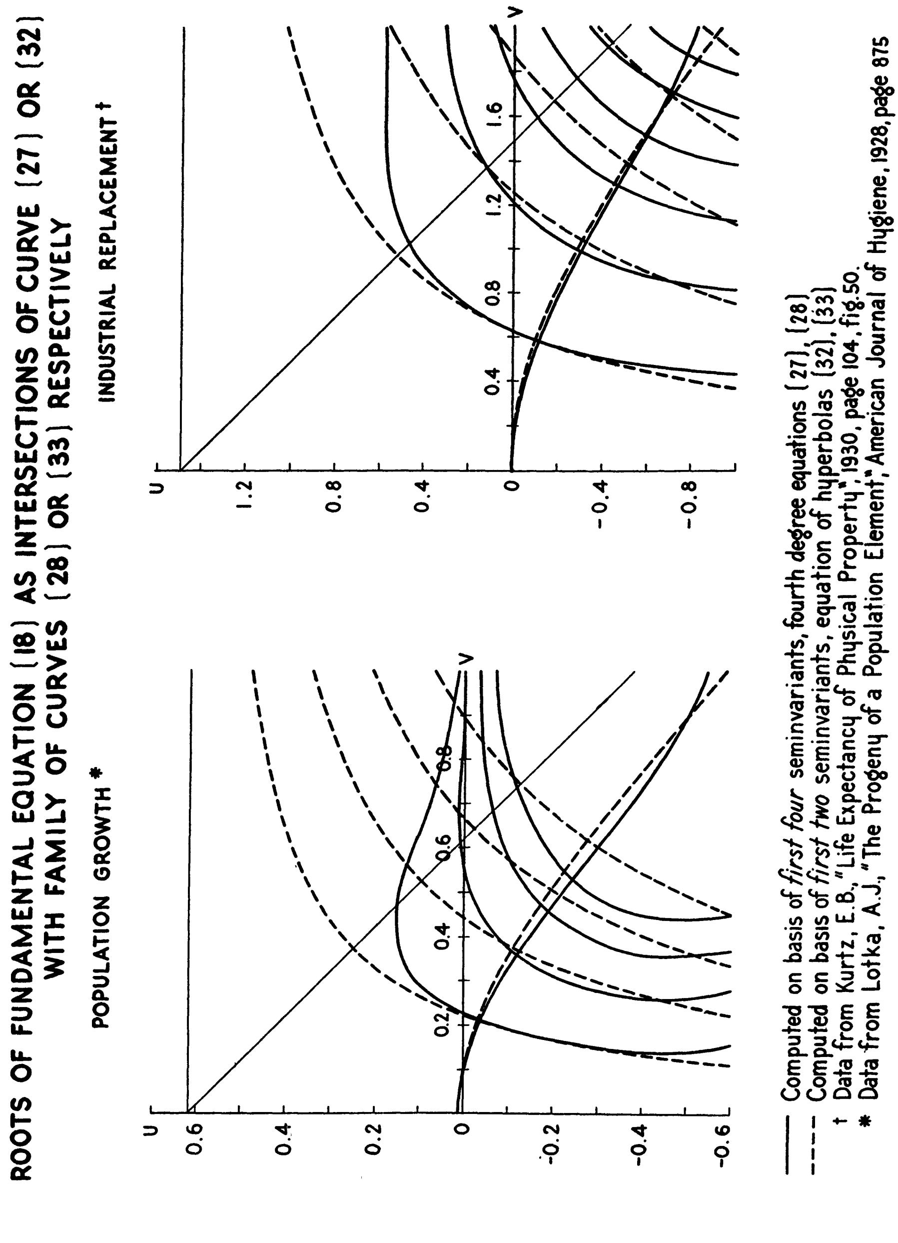
ROOTS OF FUNDAMENTAL EQUATION (18) AS INTERSECTIONS OF CURVE (27) OR (32)
WITH FAMILY OF CURVES (28) OR (33) RESPECTIVELY
POPULATION GROWTH *
INDUSTRIAL REPLACEMENT †
U
V
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0.2
0.4
0.6
0.8
1.2
0.8
0.4
0
-0.4
-0.8
0.4
0.8
1.2
1.6
—— Computed on basis of *first four* seminvariants, fourth degree equations (27), (28)
- - - Computed on basis of *first two* seminvariants, equation of hyperbolas (32), (33)
† Data from Kurtz, E.B., "Life Expectancy of Physical Property", 1930, page 104, fig. 50.
* Data from Lotka, A.J., "The Progeny of a Population Element", American Journal of Hygiene, 1928, page 875

Fig. 2

$$= \frac{Q_j}{r_j - r_i}\left\{\int_0^\omega e^{-r_i a}\varphi(a)\,da - \int_0^\omega e^{-r_j a}\varphi(a)\,da\right\} \tag{45}$$

$$= 0 \tag{46}$$

since r_i and r_j are both roots of (18). But if $i = j$, then (44) is of the indeterminate form 0/0 and we must refer back to equation (43), from which, with $i = j$, we obtain, instead of (44) a different expression, namely

$$P_{ii} = Q_i \int_0^\omega e^{-r_i a}\varphi(a) \int_0^a dt\,da \tag{47}$$

$$= Q_i \int_0^\omega a e^{-r_i a}\varphi(a)\,da \tag{48}$$

so that the only term in the sum $\sum$ in equation (40) that does not vanish is the term P_{ii} and finally

$$Q_i = \frac{P_{ii}}{\displaystyle\int_0^\omega a e^{-r_i a}\varphi(a)\,da} \tag{49}$$

$$= \frac{\displaystyle\int_0^\omega e^{-r_i t} B_1(t)\,dt}{\displaystyle\int_0^\omega a e^{-r_i a}\varphi(a)\,da} \tag{50}$$

$$= \frac{\displaystyle\int_0^\omega e^{-r_i t}\left\{B(t) - \int_0^t B(t-a)\varphi(a)\,da\right\}dt}{\displaystyle\int_0^\omega a e^{-r_i a}\varphi(a)\,da} \tag{51}$$

or, finally, in view of (20)

$$Q_i = \frac{\displaystyle\int_0^\omega e^{-r_i t}\left\{B(t) - \int_0^t B(t-a)\varphi(a)\,da\right\}dt}{-\{F'(r)\}_{r=r_i}}. \tag{52}$$

The coefficients Q are thus fully determined by (50) or its equivalents (51) or (52), when initial conditions are given, that is, when the function $B_1(t)$ is given for $0 < t < \omega$ or, what amounts to the same thing, when $B(t)$ is known for this range of values of t. For complex roots the denominator in (52) becomes,[13] in view of (27), (28)

$$-\frac{dF(r)}{dr} = -\left\{\frac{\partial\psi}{\partial u} + i\frac{\partial\chi}{\partial u}\right\} = G - iH \tag{53}$$

[13] Since r_i is a root of $F(r) = 1$, we have

$$\left[\frac{dF(r)}{dr}\right]_{r=r_i} = \left[\frac{dF(r)}{F(r)\,dr}\right]_{r=r_i} = \left[\frac{d\log_e F(r)}{dr}\right]_{r=r_i}$$

where G and H can be expressed in terms of the seminvariants by partial differentiation of (27), (28) with regard to u, namely

$$G = \mu_1 - \mu_2 u + \frac{\mu_3}{2!}(u^2 - v^2) - \frac{\mu_4}{3!}(u^3 - 3uv^2) + \cdots \tag{54}$$

$$H = \mu_2 v - \mu_3 uv + \frac{\mu_4}{3!}(3u^2 v - v^3) - \cdots. \tag{55}$$

In the special case that the "zero generation" is composed of N_0 individuals (or "units") all born (or "entering") at time zero, the coefficients Q are correspondingly simplified in form. For the term in the real root r we have

$$Q = \frac{N_0}{-F'(r)}. \tag{56}$$

Conjugate complex root terms unite in pairs,[14] giving

$$Q'e^{(u+iv)t} + Q''e^{(u-iv)t} = \frac{2N_0 e^{ut}}{G^2 + H^2}\{G \cos vt - H \sin vt\}. \tag{57}$$

Unless $\varphi(a)$ is a normal distribution, the computation of the roots, u, v, and the coefficients G, H, in terms of seminvariants becomes impracticable for higher order roots, which then have to be computed directly and laborously from equations (22), (23). In practice components of very high order will hardly be needed, nor will their use be warranted, since the high order seminvariants, which are then involved, are not usually known with sufficient accuracy. An exception occurs when the $\varphi(a)$ curve is essentially of the nature of a composite curve. This is what actually happened in the case of the curve of reproduction for a human population. For details on this point the reader must be referred to my paper "The Progeny of a Population Element".

4. **Alternative Representation of the Function $B(t)$.** By the application of the Hertz-Herglotz solution of the integral equation (6), the evolution of a population or aggregate is represented as the resultant of a series of damped oscillations.

Additional insight into the nature of the renewal process is gained by viewing the total renewals as composed of contributions from successive "generations".[15]

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[14] For details see A. J. Lotka, *The Progeny of a Population Element*, p. 892.

[15] In the case of a population the term "generation" calls for no explanation: mother, daughter and granddaughter, for example, represent three generations; in the case of industrial replacement, the term is to be understood in this sense, that the original installation constitutes the original or zero generation, the units introduced to replace disused units of the zero generation constitute the "first" generation, renewal of these the second, and so on.

This explanation may seem unnecessary. However, from some correspondence received by the writer it seems that perhaps some readers have confused the generations thus defined with successive "cycles" of duration equal to the extreme "length of life" of the units. With such "cycles" we are not here concerned.

This leads to an alternative representation, in which the evolution of the aggregate appears as the sum of a series of frequency curves, each corresponding to the contribution of one generation to the total births or replacements at time t.[16]

In order to realize this second representation we note, first of all, equation (7) applies not only to the total births at time t, but, with slight modification, also to the births in any particular generation. Here it will be convenient to consider the special case of a zero generation of N_0 individuals (or units) all born (or installed) at time $t = 0$.

The births (or renewals) in the "first" generation, that is offspring of the zero generation, or renewals of disused units of the zero generation, will be distributed in time according to the equation

$$B_1(t) = N_0\varphi(t). \tag{58}$$

For the second generation, or renewals of disused units of the first generation, we shall have

$$B_2(t) = \int_0^t B_1(t-a)\varphi(a)\,da \tag{59}$$

[16] This alternative approach of the problem bears some superficial resemblance to a method followed by R. Frisch in his article "Sammenhengen mellem primaerinvesteringen og reinvestering" (*Statsekonomisk Tidskrift*, 1927, p. 117). Frisch also follows up the distribution in time of first, second, and higher order replacements, and gives diagrams bearing a superficial resemblance to Fig. 4 in the present text. But Frisch's development has otherwise little in common with that here presented. He deals with equipment composed of various units, with expectation of life varying discontinuously or continuously from one unit to another, but fixed at a single value for a given unit. To use one of his own examples, it is as if a wooden hammer with a life of one year were always replaced by another wooden hammer, also with a life of one year, and so on: while a steel hammer, with a life of three years, were always replaced by another steel hammer, also with a life of exactly three years. The analogous case in population analysis would be presented by a population in which length of life were strictly hereditary, so that a man dying at age 50 would have a son, grandson, etc., each dying at age 50. In the field of industrial replacement and in population analysis alike this is a highly unrealistic supposition.

Needless to say, with these basic assumptions, Frisch's resulting equations differ fundamentally from those here given, and the distribution curves for successive orders of replacements, as shown in Frisch's Fig. 3 do *not* have the property that the j-th seminvariant of the k-th order replacement curve is k times that of the j-th seminvariant of the first order curve, except for $j=1$. The fact is that Frisch's curves in his Fig. 3 are all similar, except for a constant factor applied to the vertical scale and its reciprocal applied to the horizontal scale. In this case all the corresponding seminvariants, except the first, are evidently unchanged in passing from one curve to the next. Frisch, as a matter of fact, does not introduce seminvariants into his discussion at all. The Hertz solution he could not possibly introduce, since his fundamental equations are not of a form appropriate for the use of the Hertz solution.

The later sections of Frisch's paper deal with somewhat more complicated cases, but they all involve the assumption of "strict heredity," that is, the assumption that a unit with length of life v is replaced by another having exactly the same length of life v. At any rate, that is the understanding I have formed of the Danish text, studied with the assistance of a native of Scandinavia. All the formulae in the text bear out this understanding.

and, generally, for the $(j + 1)$th generation[17]

$$B_{j+1}(t) = \int_0^t B_j(t - a)\varphi(a)\, da. \tag{60}$$

Now, by a well-known property[18] of the Thiele seminvariants, it follows from (58), (59), (60), that the seminvariants of the distribution-in-time of the births (or replacements) in the jth generation are simply the j-tuple of the corresponding seminvariants of the first generation, that is, of $\varphi(t)$.

Furthermore, it is easily shown that as j, the order of generation, increases, the distribution of renewals approaches[19] the normal (Gaussian) frequency distribution.

By virtue of these properties the distribution curves for successive generations are easily constructed.[20]

The sum total of the contributions of successive generations should, of course, agree with the expression for the total annual births $B(t)$ at time t given by the fundamental equation (9). In point of fact, by summing the left and the right hand members of equations (58), (59), and (60) for all generations up to the highest, say the n-th, "reproducing" at time t, we find

$$B(t) = \sum_{j=1}^{j=n+1} B_j(t) = B_1(t) + \int_0^t \sum_{j=1}^{j=n} B_j(t - a)\varphi(a)\, da. \tag{61}$$

Since the n-th is the highest generation contributing,[21] the value of the integral in (61) is not changed by writing n instead of $n + 1$ as the upper limit of the summation sign on the right. But then (61) becomes simply

$$B(t) = B_1(t) + \int_0^t B(t - a)\varphi(a)\, da$$

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[17] The births in the j-th generation extend at most from $t = j\alpha$ to $t = j\omega$, but it is not necessary to take this into account in writing the limits of the integrals in (60) and corresponding equations, because the inclusion or exclusion of vanishing terms in the integrand does not affect the value of the integral. Similar remarks apply to the effect of the limited range of $\varphi(a)$. See also footnote 5.

[18] For details, see A. J. Lotka, "The Progeny of a Population Element," *American Journal of Hygiene*, 1928, vol. 8, p. 875; also "The Spread of Generations" *Human Biology*, 1929, vol. 1, p. 305.

[19] In practice quite rapidly, even if $\varphi(a)$ is far from normal.

[20] For the case in which $\varphi(a)$ is a Pearson Type I curve, details of the process are given in my paper "Industrial Replacement," *Skandinavisk Aktuarietidskrift*, 1933, p. 51. I may here remark that such a Pearson Type I curve for the distribution in the first generation does not strictly give again a Pearson Type I curve in the second generation, because the moments beyond the 4th are neglected in fitting such a curve. But it must be remembered that the same neglect is practiced in the original fit of the data, so that the fit in the second generation will in general be as adequate as that in the first, provided, of course, that proper attention is paid to Pearson's criteria.

[21] The special case that the limiting n so defined is ∞ would require special discussion, which, however, presents no great difficulty. As this case is of little if any practical importance, this discussion is here omitted.

that is, summation of the contributions of individual generations to the total annual births, leads us back to the fundamental equation (9), which confirms the correctness of our analysis.

TABLE I

Age Schedule of Survivorship and of Replacements[22] in First Generation

Age Interval	Survivors from Original Installation to Beginning of Specified Age Interval	Replacements Within Specified Age Interval
0–1	100,000	—
1–2	100,000	—
2–3	100,000	300
3–4	99,700	900
4–5	98,800	1,800
5–6	97,000	3,000
6–7	94,000	5,700
7–8	88,300	10,300
8–9	78,000	14,100
9–10	63,900	13,900
10–11	50,000	13,800
11–12	36,200	13,200
12–13	23,000	10,400
13–14	12,600	6,300
14–15	6,300	3,700
15–16	2,600	2,200
16–17	400	400
17–18	—	—

5. **Application to Kurtz's data.** An extensive collection of numerical data (mortality curves) on renewal of industrial equipment has been published by E. B. Kurtz (1930), (1931). By way of example the analysis developed above has been applied to the data "Group III," as fitted by him with a Pearson Type I curve, namely[23]

$$B_1(t) = 14{,}950\left(1 + \frac{t - 10}{12.67}\right)^{9.16}\left(1 - \frac{t - 10}{10.43}\right)^{7.54}. \tag{62}$$

[22] Data from E. B. Kurtz, *Life Expectancy of Physical Property*, 1930, Table 22, Cols. 5 and 6, p. 86, and p. 104, Fig. 50.

[23] The numerical values of the constants in the formula as here given differ slightly from those given by Kurtz, perhaps owing to the retention by him of higher decimals in his computations. There is also an inconsistency between Kurtz's use of 10 for the mean in his formula, whereas on his drawing the mean is placed at 100.

The aperiodic component is the number of units originally installed (arbitrarily assumed as 100,000) divided by the mean of the frequency curve (equation 62). Following Kurtz, this has also been arbitrarily made equal to 10, which simply implies a particular choice of time unit. The fundamental data and characteristics are set forth in Tables I and II. The first six oscillatory components, were computed retaining moments and seminvariants up to μ_4, with the results shown in Table III and in Figs. 2 (right hand panel), 3 and 4.

TABLE II

Moments and Seminvariants of Curve of Replacements in First Generation[24]

j	Moments[25] m_j	Seminvariants μ_j
0	100,000	
1	0	10 [26]
2	671,924	6.7192
3	130,070	−1.3007
4	12,323,200	−12.1228

TABLE III

Constants of the Series Solution (19) of Integral Equation (7) for First Six Oscillatory Components Computed from First Four Moments and Seminvariants of an Industrial Replacement Curve[27]

Order of Component n	u	v	G	H	$\frac{G}{G^2 + H^2}$	$\frac{H}{G^2 + H^2}$
0	0	0	10.0000	0	.10000	0
1	−.11009	.57767	11.1688	4.1458	.07869	.02921
2	−.30144	.98920	14.3353	7.6696	.05423	.02902
3	−.46500	1.28383	18.4982	10.4425	.04100	.02314
4	−.59500	1.51475	23.1094	12.7773	.03314	.01832
5	−.69800	1.70500	29.2088	14.8877	.02718	.01385
6	−.78000	1.86117	32.5165	16.7797	.02429	.01253

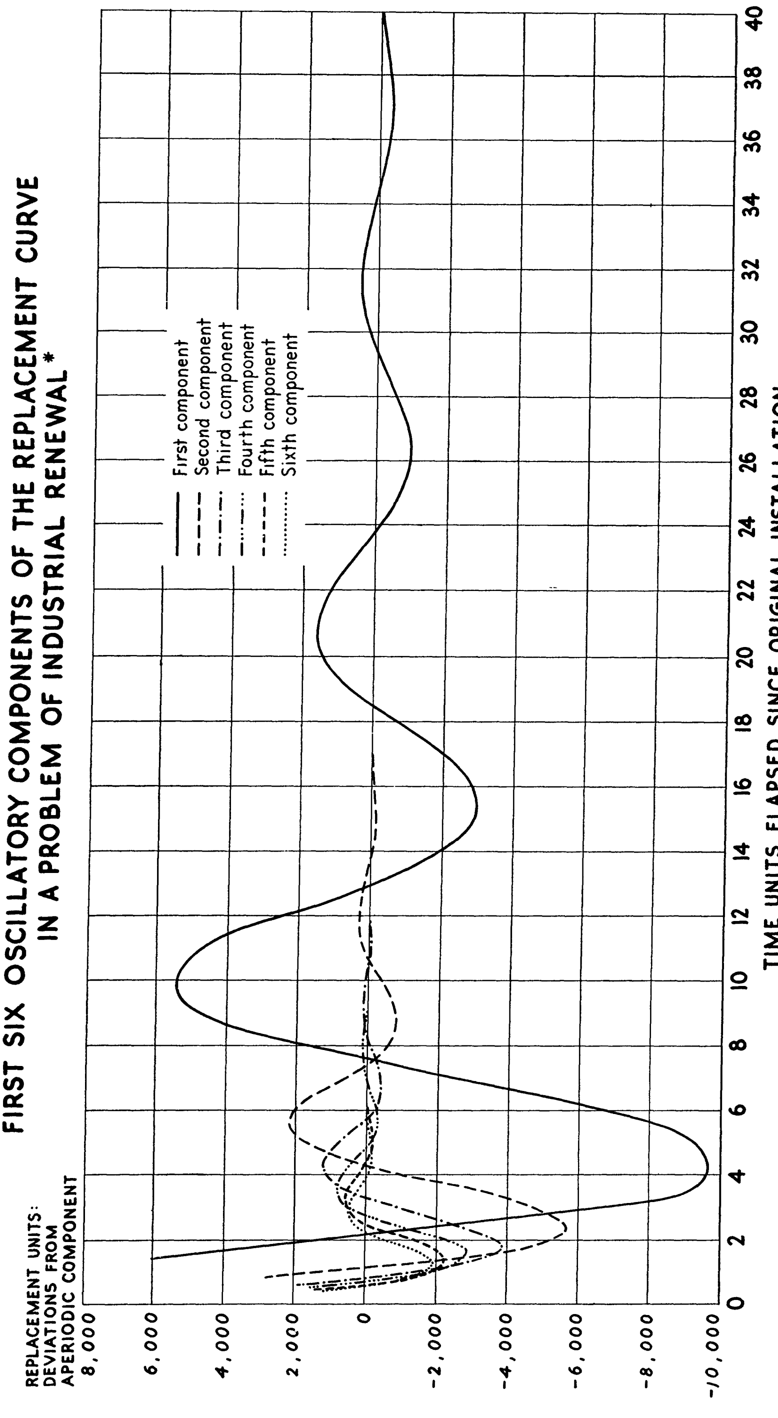
In particular, Fig. 4 shows the curve obtained by the summation of the first six oscillatory components superposed over the aperiodic (constant) component. It also shows the distribution curves of the first five generations within the range of the time scale on the diagram. Summation of these reproduces,

[24] Data from E. B. Kurtz, *Life Expectancy of Physical Property*, 1930, Table 22, p. 86, and Fig. 50, p. 104.

[25] Moments taken about age 10.

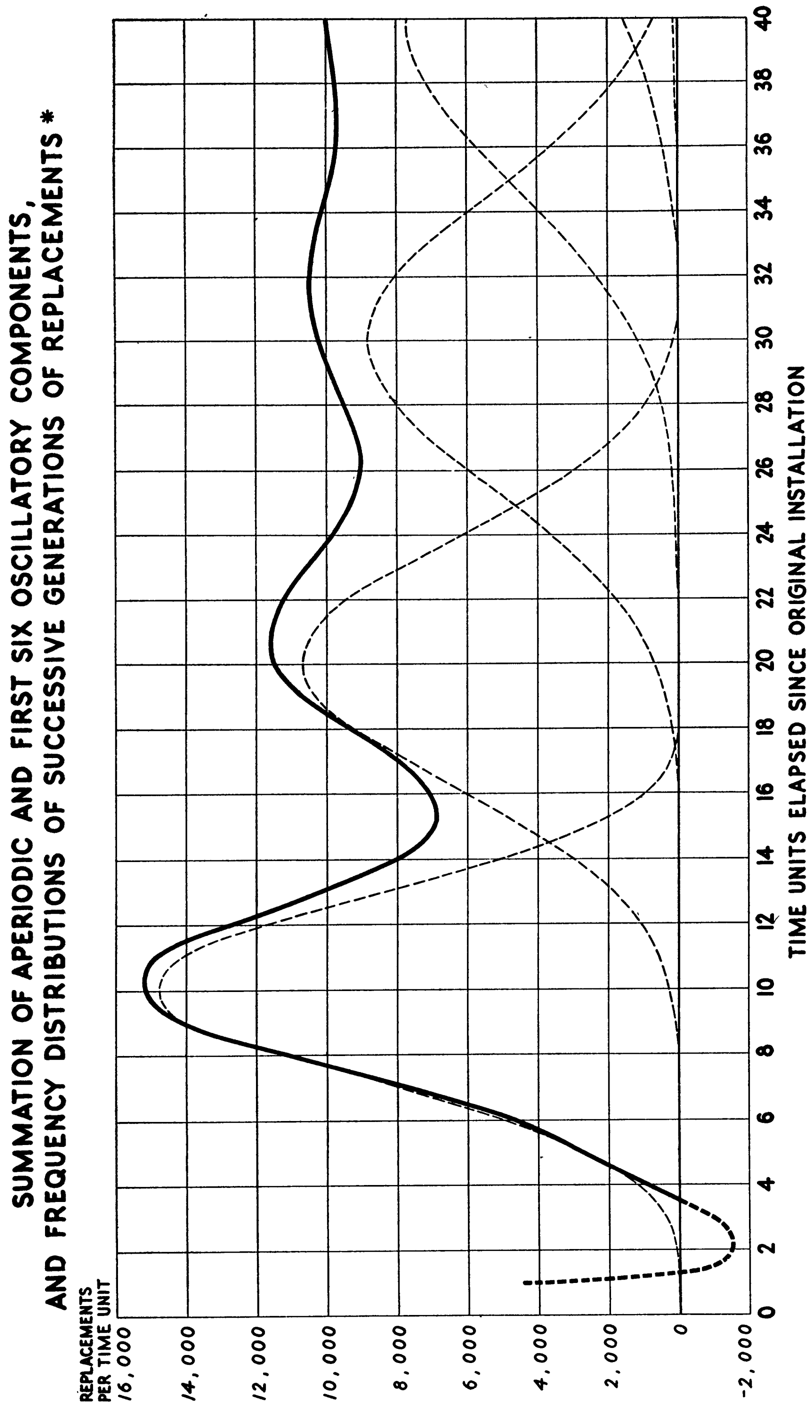
[26] This value of μ_1 is taken with reference to the origin.

[27] Data from E. B. Kurtz, *Life Expectancy of Physical Property*, 1930, p. 104, fig. 50.



* Graph of solution (19) of equation (12): First 6 oscillatory components
Data from Kurtz, E.B., "Life Expectancy of Physical Property", 1930, page 104, fig. 50.

FIG. 3



* Graph of Solution (19) of equation (12): summation of first six oscillatory components
Data from Kurtz, E. B., "Life Expectancy of Physical Property," 1930, page 104, fig. 50.

FIG. 4

within the errors of drawing, the resultant curve of the oscillatory solution, except for the very early stages of the process, where the oscillatory solution is of no practical interest, because the first generation alone dominates the whole process, and this is given by the observational data direct or after fitting with the curve such as (62).

It remains to consider briefly the relative advantages of the method of solution by differentiation, as originally applied by Herbelot, Risser, and others, on the one hand, and the use of the Hertz-Herglotz expansion, as introduced for the treatment of this type of problem by Sharpe and Lotka.

One obvious advantage of the method of differentiation *when it is applicable*, is that the result is obtained in the form of a closed, finite expression *for each cycle*.

Against this is to be reckoned, first, that the range of application of the method is severely limited. Preinreich in a recent issue of *Econometrica* (1938) uses for an illustration of the method a Pearson Type I curve, but in the very special and trivial form that the exponents are integers, namely 1 and 2. In practice the exponents will always be fractional, and then successive differentiations do not terminate as obligingly as in Preinreich's case. As already noted, Preinreich, though citing Kurtz's observational data on industrial replacement, discreetly abstains from using these for his numerical example.

Secondly, the disadvantage of a solution in form of an infinite series is more apparent than real. In practice the first few terms of the series obtained by the Hertz-Herglotz method will usually give an adequate representation of the facts, except for a short period immediately following the first installation. It is true that here this method, unless carried to high order components, may give an imperfect representation of industrial replacements, and may, in fact, give impossible negative values in this region, as in the example exhibited in Fig. 4. But this is practically unimportant, because in practice there will actually be few, if any, such very early replacements in an installation of finite dimensions. In fact, second and higher order replacements immediately after first installation are obviously out of the question in practice. For example, it may well happen once in a while that a telegraph pole is demolished on the very first day of service by collision with a truck. It is even imaginable that its replacement, put up the same day, might again be immediately demolished. But even in a country-wide installation one would hardly expect a third, fourth or fifth replacement to be required on the day of installation. In other words, that part of the replacement curve which relates to the very early period after first installation, is composed practically of first replacements only.

So for example in the diagram, Fig. 4, the curve of total replacements, up to about $t = 8$, is simply the curve of first replacements, which is given directly by the data of the problem. Within the range of errors of drawing the influence of higher components are quite unobservable in this region.

The case is even more favorable in the application of the method to the problem of population growth, for here there is actually no reproduction whatever until age α (say about 15) is reached. The part of the curve defined by the series (19) carried only to a finite number of terms,[28] and applied to values of $t < \alpha$, is therefore simply rejected.[29] It may save many words of explanation if the reader is simply referred to Fig. 4 on p. 897 of my previous publication "The Progeny of a Population Element," which illustrates the point, the minimum age of reproduction being just short of 15.

A major disadvantage of the method by differentiation is that it demands that the frequency distribution function $\varphi(a)$ be given in the form of a suitable analytic expression, or if it is not so given, that a *suitable* function or curve be fitted to it. The Hertz-Herglotz method, on the contrary, is directly applicable to the *raw data, regardless of their form.* Incidentally, curve fitting as practiced by Kurtz may produce a singular result. In 6 out of 7 of his types, the fitted frequency curve extends into negative field, implying that there are some replacements even before the actual installation. This may not be a very serious defect if the area of the curve in the negative field is negligible, but it should not pass unnoticed.

One of the principal merits of the Hertz-Herglotz expansion is that it renders the course of events over their whole extent, and, in particular, makes clear the mode of approach to the ultimate state represented by the aperiodic term. Because the method by differentiation requires a separate expression for each cycle, it is at best ill adapted to present to the eye or to the mind a comprehensive view of the evolution of the aggregate as a whole.

In the introductory paragraphs it was pointed out that the problems of population growth and those of industrial replacement were closely analogous, though there were certain points of difference. It is of interest here to give consideration to these differences.

One of these has already been noted. Replacement of industrial equipment may begin from the very moment of first installation, since accident as well as wear and tear must be provided for. Organic reproduction, on the other hand, does not occur immediately after birth. One result of this is that for any finite value of t, the number of generations contributing to the total births is itself finite; on the contrary, in the case of industrial replacement, if we interpret the equation (7) literally, there are at any moment an infinite number of generations contributing. In practice this, of course, does not occur, and the equation

[28] There are, of course, limitations to the application of the solution (19). No one with any experience in the treatment of practical problems by mathematical analysis would think of fitting, by means of a reasonably limited number of terms, the first phases of the processes here discussed, in the case of a rectangular distribution of the first generation, for example. But the distributions with which we are actually concerned in practice are far from rectangular. Such as they are, they are well adapted to the method, as is seen in the two examples illustrated.

[29] There is nothing unusual in this rejection of negative values of the frequency function where it falls outside the range of actual values. It is what we all do in using such a frequency curve as Pearson's type I, defined by a function which becomes negative outside the range of actual interest.

does not truly represent the facts in that a continuous distribution is assumed throughout, whereas for the higher order replacements ultimately the early frequencies are so thinned out that the discreteness of the units can no longer be disregarded.

Nevertheless, from the very start we must be prepared to consider several generations of replacement as contributing to the total; this lends a certain special interest, in dealing with the first cycle of replacements, to the method of solution by differentiation, as used by Herbelot, Risser, Zwinggi, Schulthess, and lately Preinreich. It is true that this interest is much diminished by the limitations in the applicability of the method.

On the other hand, in the case of organic reproduction, for the early part of the first cycle, the progeny of a population element belongs exclusively to a single ("first") generation. Between $t = 15$ and $t = 30$, in our example, only first generation births are taking place, and here the solution (19) is of more theoretical than practical interest, since the distribution of births is simply that of the first generation births.

Another point of difference is that the curve of $\varphi(a)$ in the case of industrial replacement, if we may judge by Kurtz's data, is a comparatively well behaved Pearson type curve. On the contrary, the corresponding curve of organic reproduction is a very inconvenient type to fit by any of the standard methods. In view of this it is all the more remarkable that the solution (19) gives as good a fit as it does with only four components, as will be seen on referring to my original publication, "The Progeny of a Population Element," p. 897, Fig. 4, already referred to.

Lastly, while the analogy is exact so long as we are dealing with industrial or organic aggregates maintained at a constant level, an essential difference arises when the case of a growing aggregate is considered. Organic growth takes place by what might be called "multiple replacement," that is, one individual in the course of life gives rise, on the average, to n individuals, where n may exceed unity. Analytically this finds expression in that

$$\int_0^\infty p(a)m(a)\,da > 1$$

and the fact is automatically taken care of in the solution (19) by the fact that in such a case the single real root $r > 0$.

Growth of industrial equipment, on the other hand, takes place by new units being installed *in addition to* replacement of disused units. The fundamental equations must be altered accordingly to take care of this case.

In conclusion I want to make a remark regarding the function of such analyses as the one here presented. In this connection I can do no better than to quote a sentence from Cournot:[30] "Those skilled in mathematical analysis know that

[30] A. Cournot, *Researches into the Mathematical Principles of the Theory of Wealth*, translated by N. Bacon, Macmillan Co., 1897, p. 3.

its object is not simply to calculate numbers, but that it is also employed to find the relations between magnitudes . . . ."

It is essentially in this sense that the analysis of a problem of industrial replacement is here offered. If we are merely interested in numbers, the direct arithmetical approach as practiced by Kurtz may be as good as any. But if an insight into the anatomy of the processes involved, and into their evolution from an initial condition to a final state is desired, then the setting up of the fundamental equations, and their solution in exponential series or in other suitable analytical form, and a concise expression of the relation between the distributions in time of successive generations, or orders of replacements, have greatly superior merit as compared with brute attacks by arithmetic without regard to mathematical form. Nor are the systematic relations (in terms of certain seminvariants) that have been shown to exist between the distribution of successive generations to be regarded merely as "short cuts" for their computation, though sometimes they may be found convenient in that way. Their real significance lies in that they serve to complete for us the analytical picture of the process of evolution of the system under consideration.

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BIBLIOGRAPHY

The following is a partial list of publications bearing on the general problem of self-renewing aggregates. The list is not the result of an exhaustive search, but merely of notes made in passing. Items marked with an asterisk are those referred to in the text. References are arranged essentially in chronological order.

1907 LOTKA, A. J. "Relation between Birth Rates and Death Rates," *Science*, vol. 26, p. 21.

"Studies of the Mode of Growth of Material Aggregates," *American Journal of Science*, vol. 24, pp. 199, 375.

*1908 HERTZ, P. "Die Bewegung eines Elektrons unter dem Einflusse einer stets gleich gerichteten Kraft," *Mathematische Annalen*, vol. 65, p. 1.

HERGLOTZ, G. "Über die Integralgleichungen der Elektronentheorie," *Mathematische Annalen*, vol. 65, p. 87.

*1909 HERBELOT, L. "Application d'un Théorème d'analyse à l'étude de la repartition par âge," *Bulletin Trimestriel de l'Institut des Actuaires Français*, vol. 19, p. 293.

*1911 SHARPE, F. R. AND LOTKA, A. J. "A Problem in Age Distribution," *Philosophical Magazine*, vol. 21, p. 435.

*1912 RISSER, R. "Sur une application d'une équation fonctionnelle à un problème d'assurance," *Bulletin Trimestriel de l'Institut des Actuaires Français*, vol. 23, p. 298.

1913 LOTKA, A. J. "A Natural Population Norm," *Journal of the Washington Academy of Sciences*, vol. 3, pp. 241, 289.

1918 LOTKA, A. J. "The Relation between Birth Rate and Death Rate in a Normal Population and the Rational Basis of an Empirical Formula for the Mean Length of Life, given by William Farr," *Quarterly Publications of the American Statistical Association*, vol. 16, p. 121.

*1920 RISSER, R. "Sur une application de l'équation de Volterra au problème de la répartition par âge dans les milieux à effectif constant," *Comptes Rendus de l'Académie des Sciences*, p. 845.

BATEMAN, H. "An Integral Equation Occurring in a Mathematical Theory of Retail Trade," *Messenger of Mathematics*, vol. 49, p. 134.

1921 MOSER, C. "Über Gleichungen für eine sich erneuernde Gesellschaft mit Anwendung auf Sozialversicherungskassen," *Verhandlungen der Schweizerischen Naturforschenden Gesellschaft*, p. 9.

LOTKA, A. J. "A Simple Graphic Construction for Farr's Relation between Birth Rate, Death Rate and Mean Length of Life," *Journal of the American Statistical Association*, vol. 17, p. 998.

1922 LOTKA, A. J. "The Stability of the Normal Age Distribution," *Proceedings of the National Academy of Sciences*, vol. 8, p. 339.

1923 DOETSCH, G. "Die Integrodifferentialgleichungen vom Faltungstypus," *Mathematische Annalen*, vol. 89, p. 192.

1924 SCHOENBAUM, E. "Anwendung der Volterra'schen Integralgleichungen in der mathematischen Statistik" I, II, III, *Skandinavisk Aktuarietidskrift*, p. 241; 1925, p. 1.

FOCK, V. "Über eine Klasse von Integralgleichungen," *Mathematische Zeitschrift*, vol. 21, p. 161.

1925 LOTKA, A. J. *Elements of Physical Biology*, Williams & Wilkins Company, Baltimore, pp. 110, 117, 118 and 128.

DUBLIN, L. I. AND LOTKA, A. J. "On the True Rate of Natural Increase of a Population," *Journal of the American Statistical Association*, vol. 20, p. 569.

1926 LOTKA, A. J. "The Progressive Adjustment of Age Distribution to Fecundity," *Journal of the Washington Academy of Sciences*, vol. 16, p. 505.

NORTON, H. T. J. "Natural Selection and Mendelian Variation," *Proceedings of the London Mathematical Society*, Ser. 2, Part 1, vol. 28, p. 1.

MOSER, C. "Beiträge zur Darstellung von Vorgängen und des Beharrungszustandes bei einer sich erneuernden Gesamtheit," *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker*, Heft 21, p. 1.

MEIDELL, B. Über periodische und angenäherte Beharrungszustände," *Skandinavisk Aktuarietidskrift*, No. 3–4, p. 172.

WICKSELL, S. D. "Sveriges framtida befolkning under olika förutsättningar," *Ekonomisk Tidskrift*, p. 91.

CONNOR, L. R. "Fertility of Marriage and Population Growth," *Journal of the Royal Statistical Society*, Part III, vol. 89, p. 553.

DOETSCH, G. "Bemerkung zu der Arbeit von V. Fock: Über eine Klasse von Integralgleichungen," *Mathematische Zeitschrift*, vol. 24, p. 785.

1927 LOTKA, A. J. "The Size of American Families in the Eighteenth Century," *Journal of the American Statistical Association*, vol. 22, p. 154.

"The Actuarial Treatment of Official Birth Records," *Eugenics Review*, vol. 19, p. 257.

TRICOMI, F. "Risoluzione di un problema demografico," *Atti della Reale Academia delle Scienze di Torino*, vol. 62, p. 22.

INSOLERA, F. "Sul calcolo della probabilità annuale di morte in gruppi aperti di popolazione, "*Atti della Reale Academia delle Scienze di Torino*, vol. 62, p. 611.

"Su particulari equazioni di Volterra e loro applicazione finanziaria e demografica," *Rendiconti del R. Instituto Lombardo di Scienze e Lettere*, p. 247.

FRIEDLI, W. "Über die Stabilität der gegenseitigen Hilfskassen," *Zeitschrift für Schweizerische Statistik und Volkswirtschaft*, No. 3, vol. 63, p. 322.

HALDANE, J. B. S. "A Mathematical Theory of Natural and Artificial Selection," *Cambridge Philosophical Society Proceedings*, vol. 23, p. 607.

FRISCH, R. "Sammenhengen mellem primaerinvestering og reinvestering," *Statsekonomisk Tidskrift*, p. 117.

DOETSCH, G. "Überblick über Gegenstand und Methode der Funktionalanalysis," *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 36, p. 1.

CANTELLI, F. P. "Intorno alla risoluzione di un problema demografico," *Atti della Reale Academia delle Scienze di Torino*, vol. 62, p. 449.

1928 MEIDELL, B. "Mathematical Statistics—On damping effects and approach to equilibrium in certain general phenomena," *Journal of the Washington Academy of Sciences*, No. 16, vol. 18, p. 437.

* LOTKA, A. J. "The Progeny of a Population Element," *The American Journal of Hygiene*, No. 6, vol. 8, p. 875.

1929 LOTKA, A. J. "The Spread of Generations," *Human Biology*, No. 3, vol. 1, p. 305.

"Biometric Functions in a Population Growing in Accordance with a Prescribed Law," *Proceedings of the National Academy of Sciences*, No. 10, vol. 15, p. 793.

WYSS, H. "Lage, Entwicklung und Beharrungszustand der eidgenössischen Versicherungskasse," *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker*, Heft 24, p. 39.

1930 JENZER, H. "Hauptgrössen der Witwenversicherung bei Einführung eines veränderlichen, exponentiellen Parameters für Witwensterblichkeit," *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker*, Heft 25, p. 203.

WICKSELL, S. D. "Individens fruktsamhet och släktets förökning," *Ekonomisk Tidskrift*, Häft 2, p. 29.

KURTZ, E. B. *Life Expectancy of Physical Property*, Ronald Press Company, New York.

1931 ZWINGGI, E. "Die Witwenversicherung als Teil der allgemeinen Alters—und Hinterlassenenversicherung," *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, Heft 26, p. 79.

"Das Problem der Erneuerung," *Festgabe Moser, Versicherungswissenschaftliche, juristische und mathematische Untersuchungen*, p. 127.

HUSSON, R. "Natalité et accroissement de la Population en France et à l'Étranger avant et après la guerre," *Bulletin de la Statistique générale de la France*, No. 2, vol. 20, p. 245.

WINFREY, R. AND KURTZ, E. B. "Life Characteristics of Physical Property," *Iowa State College, Bulletin* 103, vol. 30, No. 3.

ZWINGGI, E. "Zum Problem der Erneuerung," *Blätter für Versicherungs-Mathematik und verwandte Gebiete*, No. 1, vol. 31, p. 18.

WYSS, H. "Über die Prämiensysteme der Gruppenversicherung," *Festgabe Moser*, p. 175.

WICKSELL, S. D. "Nuptiality, Fertility, and Reproductivity," *Skandinavisk Aktuarietidskrift*, p. 125.

BONZ, F. AND HILBURG, F. "Die voraussichtliche Bevölkerungsentwicklung in Deutschland," *Zeitschrift für angewandte Mathematik und Mechanik*, No. 3, vol. 11, p. 237.

GOLDSTEIN, S. "Note on the Operational Solution of an Integral Equation," *Journal of the London Mathematical Society*, Part 4, vol. 6, p. 262.

LOTKA, A. J. "The Structure of a Growing Population," Paper presented before the *Second General Assembly of the International Union for the Scientific Investigation of Population Problems*, London, 1931 and published in the *Proceedings* (Problems of Population) George Allen & Unwin, 1932. Also published in *Human Biology*, vol. 3, p. 459.

*1933 LOTKA, A. J. "Industrial Replacement," *Skandinavisk Aktuarietidskrift*, 1933, p. 51.

"Applications de l'analyse au phénomène démographique," *Journal de la Société de Statistique de Paris*, vol. 74, p. 336.

ZWINGGI, E. "Methodischer Beitrag zur Deckungs-Kapitalberechnung in der Sozialversicherung," *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker*, Heft 28, p. 77.

v. MISES, R. Über die Vorausberechnung von Umfang und Altersschichtung der Bevölkerung Deutschlands," *Blätter für Versicherungs-Mathematik und verwandte Gebiete*, No. 10, vol. 2, p. 359.

1934 MARET, A. *Untersuchungen über diskontinuierlich sich erneuernde Gesamtheiten.* Inaugural Dissertation, Bern.

WICKSELL, S. D. "Bidrag till den formella befolkningsteorien," *Statsokonomisk Tidsskrift*, Heft 1–2, p. 1.

*1935 SCHULTHESS, H. *Volterra'sche Integralgleichungen in der Versicherungsmathematik.* Inaugural Dissertation der Philosophischen Fakultät II der Universität Bern.

1936 LOTKA, A. J. "The Geographic Distribution of Intrinsic Natural Increase in the United States, and an Examination of the Relation between Several Measures of Net Reproductivity." *Journal of the American Statistical Association*, vol. 31, p. 273.

LOTKA, A. J. "Modern Trends in Birth Rate," *The Annals of the American Academy of Political and Social Science*, vol. 188, November, p. 1.

SCHULTHESS, H. "Über die Verwendung von Integralgleichungen zur Berechnung der mathematischen Reserve (Deckungskapital)," *Blätter für Versicherungs Mathematik und verwandte Gebiete*, vol. 3, p. 412.

KERMACK, W. O. AND MCKENDRIK, A. G. "The Solution of Sets of Simultaneous Integral Equations Related to the Equation of Volterra," *Proceedings of the London Mathematical Society*, Ser. 2, vol. 41, p. 462; 472–477.

KREIS, H. "Stabilität einer sich jährlich erneuernden Gesamtheit," *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker*, Heft 32, p. 17.

1937 LOTKA, A. J. "A Historical Error Corrected," *Human Biology*, vol. 9, No. 1, p. 104.

"Population Analysis: A Theorem Regarding the Stable Age Distribution," *Journal of the Washington Academy of Sciences*, vol. 27, p. 299.

SCHULTHESS, H. "Über das Erneuerungsproblem bei Verwendung eines analytischen Sterbegesetzes," *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, Heft 33, p. 69.

HADWIGER, H. "Zur Berechnung der Erneuerungsfunktion nach einer Formel von V. A. Kostitzin," *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker*, Heft 34, p. 37.

LOTKA, A. J. "Quelques résultats récents de l'analyse démographique," *Congrès International de la population*, Paris. I. Théorie générale de la population, p. 96.

HUSSON, R. "La démographie mathématique, Équations générales Propriétés générales des trajectoires démographiques," *Congrès International de la Population*, Paris. I. Théorie générale de la population, p. 108.

1938 LOTKA, A. J. "Some Recent Results in Population Analysis," *Journal of the American Statistical Association*, vol. 33, p. 164.

*1938 PREINREICH, G. A. D. "Annual Survey of Economic Theory: The Theory of Depreciation," *Econometrica*, vol. 6, p. 219.

ZWINGGI, E. "Bemerkungen zum Erneuerungsproblem," *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker*, Heft 36, p. 69.

HADWIGER, H. "Über die Entwicklung der Erneuerungsfunktion in eine Potenzreihe," *Skandinavisk Aktuarietidskrift*, Häft 1–2, p. 31.