$1 + \frac{c}{t}(P_{t\sigma} - 1)$ since $y_t > P_{t\sigma}$. Substitution of $1 + \frac{c}{t}(P_{t\sigma} - 1)$ for $P_{c\sigma}$ in (1) gives

$$(2) \qquad P_{t\sigma} \leqq \frac{\beta_{2r-2} - c^{2r}\left(1 - \frac{2rc}{t(2r+1)}\right)}{\left(\frac{t}{\theta}\right)^{2r} - c^{2r}\left(1 - \frac{2rc}{t(2r+1)}\right)}, \qquad t\sigma < \frac{c\sigma}{1 - P_{c\sigma}}, \qquad \theta \text{ as in (1).}$$

To indicate the amount of improvement let $c = r = 1$, and $t = 2$. From (1) $P_{2\sigma} \leqq .092$ while from (2) $P_{2\sigma} \leqq .056$. One may work from any origin other than the mean by letting $h = c\sigma$ in (2).

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CORRECTION OF SAMPLE MOMENT BIAS DUE TO LACK OF HIGH CONTACT AND TO HISTOGRAM GROUPING

By Dinsmore Alter

The first correction of sample moment bias was devised by W. F. Sheppard [1]. His method corrects for histogram grouping on the assumption of high contact at both ends of the frequency curve. Usually this is a sufficient correction. In some cases, however, of J-shaped curves the error remaining is even more serious than in the original histogram moments.

A method developed by E. Pairman and Karl Pearson [2] makes a complete correction for both of these sources of bias. The only advantage claimed for the method to be developed here over theirs lies in simplicity of mathematical theory.

A third correction is given by Elderton [3]. In his method he assumes that there is no error due to histogram grouping and he develops a correction for lack of high contact, in so far as the zero-th moment is concerned. The following work may be thought of largely as an extension of his method although it will have certain variations.

Let A_x and ν'_m be defined as follows,

$$A_x \equiv \int_{t=-\frac{1}{2}}^{+\frac{1}{2}} y_{x+t}\, dt$$

$$\nu'_m \Sigma A_x \equiv \Sigma x^m A_x$$

The definite integrals are the areas of the histogram rectangles if a scale of x be chosen to reduce their width to unity. Let μ'_m be defined by

$$\mu'_m \Sigma A_x \equiv \int_{l_1}^{l_2} x^m y_x\, dx$$

In the first equation the x's form a series of equally spaced constants. In the second, x is a continuous variable. The summations are to extend over the equally spaced values of x.

If the data form a histogram, l_1 and l_2 are respectively the values of x at the left edge of the left-hand rectangle and the right edge of the right-hand one. If the data are the values of y_x at isolated points, l_1 is the value of x one-half unit smaller than the smallest value given in the sample and l_2 is one-half greater than the largest. It would be perfectly satisfactory, of course, to define these limits differently. As defined, however, they parallel the histogram case. Distributions of this latter type will be called *point frequency distributions.*

As is customary, the primed moments denote those about an arbitrary origin. Moments corrected for lack of high contact and for grouping will be denoted by μ'_m or by μ_m if taken about the mean. Numerical raw moments will be denoted by ${}_n\mu'_m$. There are two entirely different methods of approach to this bias problem.

(a) The bias may be put into the algebraic form of the frequency curve and equated directly to the numerical raw moments. In the case of a point frequency distribution such a method forms the algebraic values of y_x for each point given in the sample and, therefore, puts the raw moments into algebraic form to be equated to the numerical ones. This is the simplest method of correction if the distribution is a power series. For most types the method leads into difficulties which complicate it beyond practical use.

(b) The raw moments given, whether ${}_n\mu'_m$'s or ν'_m's can be corrected to approximate very closely the desired μ'_m's as defined above.

A point frequency distribution gives ${}_n\mu'_m \equiv \Sigma x^m y_x$. If there is high contact ${}_n\mu'_m$ is an unbiased observed estimate of μ'_m. This second form of method will be developed here primarily as a correction to ${}_n\mu'_m$.

Only one assumption is involved. Fifth differences of y_x will be considered as negligible. Any interpolation formula is available but Stirling's will be employed.

$$y_{x+t} = y_x + t\Delta'_x + \frac{t^2}{2}\Delta''_x + \frac{t(t^2-1)}{6}\Delta'''_x + \frac{t^2(t^2-1)}{24}\Delta^{\text{iv}}_x$$

$$\Delta'_x \equiv \tfrac{1}{2}(\Delta'_{x-\frac{1}{2}} + \Delta'_{x+\frac{1}{2}}), \qquad \Delta'''_x \equiv \tfrac{1}{2}(\Delta'''_{x-\frac{1}{2}} + \Delta'''_{x+\frac{1}{2}})$$

$$\mu'_m \Sigma A_x \equiv \int_{l_1}^{l_2} x^m y_x\,dx = \Sigma \int_{-\frac{1}{2}}^{+\frac{1}{2}} (x+t)^m y_{x+t}\,dt$$

$$= \Sigma \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left[x^m + mx^{m-1}t + \frac{m(m-1)}{2}x^{m-2}t^2 + \cdots + t^m\right] y_{x+t}\,dt$$

$$\mu'_m = \nu'_m + \frac{1}{\Sigma A_x}\Sigma\left[mx^{m-1}\int_{-\frac{1}{2}}^{+\frac{1}{2}} t y_{x+t}\,dt + \frac{m(m-1)}{2}x^{m-2}\int_{-\frac{1}{2}}^{+\frac{1}{2}} t^2 y_{x+t}\,dt + \cdots\right]$$

Using Stirling's formula:

$$\mu'_m = \nu'_m + \frac{1}{\Sigma A_x}\Sigma\left\{y_x\left[\frac{m(m-1)}{24}x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{1920}x^{m-4}\right]\right.$$
$$\left. + \Delta'_x\left[\frac{mx^{m-1}}{12} + \frac{m(m-1)(m-2)x^{m-3}}{480}\right]\right.$$

$$+ \Delta_x'' \left[\frac{m(m-1)x^{m-2}}{320} + \frac{m(m-1)(m-2)(m-3)x^{m-4}}{21504} \right]$$

$$- \Delta_x''' \left[\frac{17\,mx^{m-1}}{1440} + \frac{23m(m-1)(m-2)x^{m-3}}{80640} \right]$$

$$- \Delta_x^{\text{iv}} \left[\frac{23m(m-1)x^{m-2}}{107520} + \frac{29m(m-1)(m-2)(m-3)x^{m-4}}{9289728} \right]$$

$$+ \text{ terms involving } (m-4) \Big\}$$

From this

$$\mu_1' = \nu_1' + \frac{1}{\Sigma A_x} \Sigma \left[\frac{\Delta_x'}{12} - \frac{17\Delta_x'''}{1440} \right]$$

$$\mu_2' = \nu_2' + \frac{1}{\Sigma A_x} \Sigma \left[\frac{y_x}{12} + \frac{x\Delta_x'}{6} + \frac{\Delta_x''}{160} - \frac{17x\Delta_x'''}{720} - \frac{23\Delta_x^{\text{iv}}}{53760} \right]$$

$$\mu_3' = \nu_3' + \frac{1}{\Sigma A_x} \Sigma \left[\frac{xy}{4} + \left(\frac{1}{80} + \frac{x^2}{4} \right) \Delta_x' + \frac{3x}{160} \Delta_x'' - \left(\frac{17x^2}{480} + \frac{23}{13440} \right) \Delta_x''' - \frac{23}{17920} \Delta_x^{\text{iv}} \right]$$

$$\mu_4' = \nu_4' + \frac{1}{\Sigma A_x} \Sigma \left[\left(\frac{x^2}{2} + \frac{1}{80} \right) y_x + \left(\frac{x^3}{3} + \frac{x}{20} \right) \Delta_x' + \left(\frac{3x^2}{80} + \frac{1}{896} \right) \Delta_x'' - \left(\frac{17x^3}{360} + \frac{23x}{3360} \right) \Delta_x''' - \left(\frac{23x^2}{8960} + \frac{29}{387072} \right) \Delta_x^{\text{iv}} \right].$$

Ordinarily it will not be necessary to use all of the corrective terms.

For point frequency distributions the application of these equations is direct. The ν_m''s may be computed from

$$A_x = y_x + \frac{\Delta_x''}{24} - \frac{17\Delta_x^{\text{iv}}}{5760},$$

and the definition of ν_m'. There is, however, a theoretical difficulty in a case for which the data have been given as a histogram. In such a case the values of A_x are all that have been known originally. The Δ's are not the ones demanded by the equation. The relationship to the proper ones is simple:

$$\Delta_{A_x}' = \Delta_x' + \frac{\Delta_{x+1}'' - \Delta_{x-1}''}{24} - \frac{15}{5760} (\Delta_{x+1}^{\text{iv}} - \Delta_{x-1}^{\text{iv}}), \text{ etc.}$$

It is possible to compute the Δ_x^i's from this equation but the discrepancy is small and moreover the Δ^i's are used only in corrective terms. Probably the error involved by use of the wrong Δ^i's is negligible in any actual case of data that ever will be studied. In the numerical example to follow, the very slight errors remaining in the μ_m''s are due, probably, to this neglect.

Pairman and Pearson gave a numerical example in which both the lack of high contact and the grouping introduced large errors. They started with $y_x = 100{,}000 \sqrt{x}$ and from this formed ten values of A_x. From these they computed the ν'_m's and corrected them to get the μ'_m's. The exact values of the latter were already known to them through integration of the original equation. The following table compares four values of moments from these data.

m	ν'_m	μ'_m by Sheppard's Formula	μ'_m with Pairman-Pearson Full Corrections	Method Developed Here	True Values
1	5.9880	5.9880	5.9994	5.9996	6.0000
2	42.6900	42.6067	42.8570	42.8576	42.8571
3	331.0854	329.5884	333.3349	333.3387	333.3333
4	2698.7735	2677.4576	2727.2757	2727.3555	2727.2727

Despite the use of the $\Delta^i_{A_x}$'s instead of Δ^i_x's, the results of this method are almost as good as by the older one. The method has the additional advantage of unifying the theories of the correction of moments from the two types of distribution.

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FREQUENCY DISTRIBUTION OF PRODUCT AND QUOTIENT

By E. V. Huntington

The main purpose of this note is to establish Theorems 1 and 2. For the sake of completeness, the more familiar Theorems 3 and 4 are appended. All four of these theorems have numerous applications in the theory of frequency distributions. While the proofs of Theorems 1 and 2 in the elementary forms here given (and used in my class-room notes since 1934) can hardly be new, they seem not to be readily accessible in the current text-books.

Theorem 1. *Suppose a variable x is distributed in accordance with a probability law $\int_0^\infty f(x)dx = 1$; and a variable y in accordance with a probability law $\int_0^\infty F(y)dy$*