n	20		100		∞	
k	10	∞	10	∞	10	∞
$\frac{\lambda_1}{k-1}$	1.084	1.081	1.016	1.015	1	1
$\frac{\lambda_2}{2(k-1)}$	1.176	1.170	1.032	1.031	1	1
$\frac{\lambda_3}{8(k-1)}$	1.275	1.265	1.048	1.046	1	1
$\frac{\lambda_4}{48(k-1)}$	1.384	1.369	1.065	1.062	1	1

These results indicate that the degree of approximation of $-2 \log \lambda$ to the χ^2 law with $k - 1$ degrees of freedom is mainly dependent on n, and is for all practical purposes independent of k when n is moderately large.

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ON TCHEBYCHEFF APPROXIMATION FOR DECREASING FUNCTIONS

BY C. D. SMITH

The problem of estimating the value of a probability by means of moments of a distribution function has been studied by Tchebycheff, Pearson, Camp, Meidel, Narumi, Markoff, and others. Approximations without regard to the nature of the function have not been very close. However the closeness of the approximation has been materially improved by placing certain restrictions on the nature of the distribution function.[1] For example, when $y = f(x)$ is an increasing function from $x = 0$ to $x = c\sigma$ and a decreasing function beyond that point, the corresponding probability function $y = P_x$ is concave downward from $x = 0$ to $x = c\sigma$ and concave upward beyond that point. Here P_x is the probability that a variate taken at random from the distribution will fall at a distance at least as great as x from the origin. Beginning with these conditions I have established the inequality[1]

[1] B. H. Camp, "A New Generalization of Tchebycheff's Statistical Inequality", *Bulletin of the American Mathematical Society*, Vol. 28, (1922), pp. 427–32.
C. D. Smith, "On Generalized Tchebycheff Inequalities in Mathematical Statistics," *The American Journal of Mathematics*, Vol. 52, (1930), pp. 109–26.

(1)
$$P_x \leqq \frac{\beta_{2r-2} - c^{2r}[(2rP_{c\sigma} + 1)/(2r+1)]}{(t/\theta)^{2r} - c^{2r}}; \qquad x = t\sigma, \qquad \beta_{2r-2} = \frac{M_{2r}}{\sigma^{2r}}.$$
$$t = \frac{2r}{2r+1} \cdot \frac{t^{2r+1} - (c\theta)^{2r+1}}{\theta[t^{2r} - (c\theta)^{2r}]}.$$

The upper bound was obtained by substituting $P_{t\sigma}$ for $P_{c\sigma}$, $t > c$, and the special values $c = r = 1$, and $t = 2$, gave the result $P_{2\sigma} \leqq .092$.

The purpose of this paper is to give an estimate of $P_{c\sigma}$ which will substantially improve the approximation to the value of P_x obtained from (1). Let $y = f(x)$ be a monotonic increasing function from $x = 0$ to $x = c\sigma$ and a monotonic decreasing function from $x = c\sigma$ to the upper end of the range of x. With P_x as the probability that a variate taken at random from the distribution will deviate from the origin by an amount at least x we know that the curve of $y = P_x$ is concave downward from $x = 0$ to $x = c\sigma$ and concave upward beyond that point. When $y = f(x)$ is of finite range the probability curve and the curve of $y = f(x)$ will terminate at the same point on the x-axis. The probability curve will approach the x-axis when the range of the function is infinite. In either case we may take a distribution $y = g(x)$ to follow the curve of the given function from $x = 0$ to $x = c\sigma$ and to follow a horizontal line from $x = c\sigma$ to a finite distance A from the orign and such that the area under the curve is the same as that under the curve of $y = f(x)$. Obviously the probability curve for $y = g(x)$ will be a straight line from $(c\sigma, P_{c\sigma})$ to the point $(A, 0)$, and since the curve of $y = P_x$ is concave upward beyond $(c\sigma, P_{c\sigma})$ it will remain below a straight line to a point very near $(A, 0)$. Also it is evident that the straight line has a y-intercept greater than unity since a straight line beginning at $(0, y)$ and extending a distance A from the origin would give a probability function whose graph follows the straight line from $(0, 1)$ to $(A, 0)$. Obviously the ordinates of this probability graph for values of x in the interval from $x = 0$ to $x = c\sigma$ are less than the corresponding ordinates for the curve which increases for x in the same interval and then follows the horizontal line. Hence a line through points $(0, 1)$ and $(c\sigma, P_{c\sigma})$ is above the line $(c\sigma, P_{c\sigma})$ to $(A, 0)$ for all points beyond $c\sigma$.

We may use the line through points $(0, 1)$ and $(c\sigma, P_{c\sigma})$ as a basis for estimating $P_{c\sigma}$ in (1). The equation of the line is $y = \frac{P_{c\sigma} - 1}{c\sigma} x + 1$ with x-intercept $\frac{c\sigma}{1 - P_{c\sigma}}$. The line remains above the curve of $y = P_x$ from $x = c\sigma$ to a point very near the x-intercept and so we may use the line from $x = c\sigma$ to the crossing point. The range of validity seems to be sufficient for practical use since $P_{c\sigma}$ is usually near .9 and c is a fraction. For $P_{c\sigma} = .9$, $c = .5$, the intercept of the line is approximately 5σ. Let the ordinate under the line be y_t, $(t > c)$, and then $P_{c\sigma} = 1 + \frac{c}{t}(y_t - 1)$. For the probability curve $y = P_x$ we have $P_{c\sigma} >$

$1 + \frac{c}{t}(P_{t\sigma} - 1)$ since $y_t > P_{t\sigma}$. Substitution of $1 + \frac{c}{t}(P_{t\sigma} - 1)$ for $P_{c\sigma}$ in (1) gives

$$(2) \quad P_{t\sigma} \leqq \frac{\beta_{2r-2} - c^{2r}\left(1 - \dfrac{2rc}{t(2r+1)}\right)}{\left(\dfrac{t}{\theta}\right)^{2r} - c^{2r}\left(1 - \dfrac{2rc}{t(2r+1)}\right)}, \qquad t\sigma < \frac{c\sigma}{1 - P_{c\sigma}}, \qquad \theta \text{ as in (1).}$$

To indicate the amount of improvement let $c = r = 1$, and $t = 2$. From (1) $P_{2\sigma} \leqq .092$ while from (2) $P_{2\sigma} \leqq .056$. One may work from any origin other than the mean by letting $h = c\sigma$ in (2).

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CORRECTION OF SAMPLE MOMENT BIAS DUE TO LACK OF HIGH CONTACT AND TO HISTOGRAM GROUPING

BY DINSMORE ALTER

The first correction of sample moment bias was devised by W. F. Sheppard [1]. His method corrects for histogram grouping on the assumption of high contact at both ends of the frequency curve. Usually this is a sufficient correction. In some cases, however, of J-shaped curves the error remaining is even more serious than in the original histogram moments.

A method developed by E. Pairman and Karl Pearson [2] makes a complete correction for both of these sources of bias. The only advantage claimed for the method to be developed here over theirs lies in simplicity of mathematical theory.

A third correction is given by Elderton [3]. In his method he assumes that there is no error due to histogram grouping and he develops a correction for lack of high contact, in so far as the zero-th moment is concerned. The following work may be thought of largely as an extension of his method although it will have certain variations.

Let A_x and ν'_m be defined as follows,

$$A_x \equiv \int_{t=-\frac{1}{2}}^{+\frac{1}{2}} y_{x+t}\, dt$$

$$\nu'_m \Sigma A_x \equiv \Sigma x^m A_x$$

The definite integrals are the areas of the histogram rectangles if a scale of x be chosen to reduce their width to unity. Let μ'_m be defined by

$$\mu'_m \Sigma A_x \equiv \int_{l_1}^{l_2} x^m y_x\, dx$$

In the first equation the x's form a series of equally spaced constants. In the second, x is a continuous variable. The summations are to extend over the equally spaced values of x.