THE DISTRIBUTION THEORY OF RUNS

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1. Introduction. In studying a particular sample, the order in which the elements of the sample were drawn is frequently available to the statistician. This important information is usually entirely neglected by him. Such disregard must be attributed, to a considerable extent, to the unsatisfactory state of mathematical devices for using the knowledge in question. One reasonable mathematical method for handling this information, the one to be used in this paper, is to make use of the distribution of runs. A run is defined as a succession of similar events preceeded and succeeded by different events; the number of elements in a run will be referred to as its length.

The distribution theory of runs has had a stormy career. The theory seems to have been started toward the end of the nineteenth century rather than in the days of Laplace when there was so much interest in games of chance. In 1897 Karl Pearson [1], in a discussion of data taken from the roulette tables at Monte Carlo, wrote "... the theory of runs is a very simple one." In this book he developed no theory but it is evident from his computations that he regarded the distribution of runs as a special case of the multinomial distribution. The multinomial method, besides evading the issue somewhat and raising questions of random sampling, also gives incorrect results when one is interested in runs of more than one kind of element. In 1899 Karl Marbe [2] derived an expression for the mean of the number of iterations of a given length from a binomial population. This result was incorrect because he neglected dependence between overlapping iterations. An iteration is defined as a sequence of similar events; a run of length t is counted as $t - s + 1$ iterations of length s for $s \leq t$. Marbe has assembled a great mass of data with the object of proving the popular hypothesis that a "head" becomes highly probable after a long succession of "tails" has appeared. Ordinary significance tests applied to his data do not support this contention, but Marbe continues to advocate it [3] and [5]. Of course, he has been severely criticised by many mathematical statisticians.

In 1904 Grünbaum [6] derived the mean of the number of runs of given length from a binomial population by the multinomial method. The first correct formulae were derived in 1906 by Bruns [7] who found the mean and variance of the number of iterations of given length in samples from a binomial population. In a book published in 1917 von Bortkiewicz correctly derived for the first time the mean and variance of runs from a binomial population using a method similar to that of Bruns. This book [8] contains a great many formulae for means and variances of runs and iterations under various special circumstances; a large portion of it is devoted to an exhaustive criticism of Marbe's work. In 1921 von

Mises [9] showed that the number of long runs of given length was approximately distributed according to the Poisson law for large samples.

It was not until 1925 (so far as the author has been able to ascertain) that an actual distribution function appeared when Ising [10] gave the number of ways of obtaining a given total number of runs (without regard to length) from arrangements of two kinds of elements. Stevens [12] in 1939 published the same distribution and described a χ^2 criterion for significance. Wald and Wolfowitz [13] in 1940 published the same distribution and showed that it was asymptotically normal. These papers are all concerned with random arrangements of a fixed number of elements of each of two kinds; the last mentioned paper describes a very interesting application of the distribution to the problem of testing the hypothesis that two samples have come from the same continuous distribution. Wishart and Hirshfeld [11] in 1936 derived the distribution of the total number of runs (again without regard to length) in samples from a binomial population and showed it was asymptotically normal.

In this paper we shall derive distributions of runs of given length both from random arrangements of fixed numbers of elements of two or more kinds, and from binomial and multinomial populations. Also we shall give the limiting form of these distributions as the sample size increases. These limiting distributions are all normal. The distribution problem is, of course, a combinatorial one, and the whole development depends on some identities in combinatory analysis,—some new and some well known to students of partition theory.

The paper will be divided into two parts. The first will deal with distributions obtained from random arrangements of a fixed number of each kind of element. The second will deal with distributions of elements from a binomial or multinomial population.

PART I

2. Distribution of runs of two kinds of elements. Consider random arrangements of n elements of two kinds, for example n_1 a's and n_2 b's with $n_1 + n_2 = n$. Let r_{1i} denote the number of runs of a's of length i, and let r_{2i} denote the number of runs of b's of length i. For example the arrangement

$$a\ b\ b\ a\ b\ a\ a\ a\ b\ b\ a\ a\ a$$

will be characterized by the numbers $r_{11} = 2$, $r_{13} = 2$, $r_{21} = 1$, $r_{22} = 2$, and all other $r_{ij} = 0$. Also we let $r_1 = \sum_i r_{1i}$ and $r_2 = \sum_i r_{2i}$ denote the total number of runs of a's and b's respectively. Throughout the paper a binomial coefficient will be denoted by

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} \tag{2.1}$$

and this is defined to be zero when $m < k$. A multinomial coefficient will often be denoted by

$$\left[\begin{matrix} m \\ m_i \end{matrix}\right] = \frac{m!}{m_1!m_2! \cdots m_s!} \tag{2.2}$$

$$\Sigma m_i = m, \qquad m_i \geq 0 \tag{2.3}$$

and when such a coefficient is to be summed over the indices m_i the two conditions (2.3) are always understood and will not be repeated; other conditions on the indices will be placed below the summation sign.

Given a set of numbers r_{ij} $(i = 1, 2; j = 1, 2, \cdots, n_i)$ such that $\sum_j jr_{ij} = n_i$, there are $\left[\begin{matrix} r_1 \\ r_{1j} \end{matrix}\right]$ and $\left[\begin{matrix} r_2 \\ r_{2j} \end{matrix}\right]$ different arrangements of the runs of a's and b's respectively. Hence the total number of ways of obtaining the set r_{ij} is

$$N(r_{ij}) = \left[\begin{matrix} r_1 \\ r_{1j} \end{matrix}\right]\left[\begin{matrix} r_2 \\ r_{2j} \end{matrix}\right] F(r_1, r_2) \tag{2.4}$$

where $F(r_1, r_2)$ is the number of ways of arranging r_1 objects of one kind and r_2 objects of another so that no two adjacent objects are of the same kind. Thus

$$\begin{aligned} F(r_1, r_2) &= 0 \quad \text{if} \quad \mid r_1 - r_2 \mid > 1, \\ &= 1 \quad \text{if} \quad \mid r_1 - r_2 \mid = 1, \\ &= 2 \quad \text{if} \quad r_1 = r_2 \end{aligned} \tag{2.5}$$

Since there are $\binom{n}{n_1}$ possible arrangements of the a's and b's, we have at once the distribution of the r_{ij}

$$P(r_{ij}) = \frac{\left[\begin{matrix} r_1 \\ r_{1j} \end{matrix}\right]\left[\begin{matrix} r_2 \\ r_{2j} \end{matrix}\right] F(r_1, r_2)}{\binom{n}{n_1}}. \tag{2.6}$$

Certain marginal distributions will also be of interest. To obtain, for example, the distribution of the r_{1j}, it is first necessary to sum $\left[\begin{matrix} r_2 \\ r_{2j} \end{matrix}\right]$ over all partitions of n_2. This is easily accomplished by finding the coefficient of x^{n_2} in

$$\begin{aligned} (x + x^2 + x^3 + \cdots)^{r_2} = x^{r_2}(1 + x + x^2 + \cdots)^{r_2} &= \frac{x^{r_2}}{(1 - x)^{r_2}} \\ &= x^{r_2} \sum_{t=0}^{\infty} \binom{r_2 - 1 + t}{r_2 - 1} x^t. \end{aligned}$$

The term corresponding to $t = n_2 - r_2$ gives the desired result:

$$\sum_{\Sigma jr_{2j} = n_2} \left[\begin{matrix} r_2 \\ r_{2j} \end{matrix}\right] = \binom{n_2 - 1}{r_2 - 1}. \tag{2.7}$$

We have then

$$(2.8)\qquad P(r_{1j}, r_2) = \frac{\begin{bmatrix} r_1 \\ r_{1j} \end{bmatrix}\binom{n_2 - 1}{r_2 - 1} F(r_1, r_2)}{\binom{n}{n_1}}$$

and summing this over r_2, a slight simplification gives

$$(2.9)\qquad P(r_{1j}) = \frac{\begin{bmatrix} r_1 \\ r_{1j} \end{bmatrix}\binom{n_2 + 1}{r_1}}{\binom{n}{n_1}}.$$

The distribution (2.6) summed over r_{1j} and r_{2j} gives by means of (2.7)

$$(2.10)\qquad P(r_1, r_2) = \frac{\binom{n_1 - 1}{r_1 - 1}\binom{n_2 - 1}{r_2 - 1} F(r_1, r_2)}{\binom{n}{n_1}}$$

which is essentially the distribution derived by Wald and Wolfowitz [13], and summing this over r_2 we get the distribution discussed by Stevens [12]

$$(2.11)\qquad P(r_1) = \frac{\binom{n_1 - 1}{r_1 - 1}\binom{n_2 + 1}{r_1}}{\binom{n}{n_1}}.$$

Another marginal distribution which will be useful is obtained by summing (2.9) over r_{1i} for $i \geq k$. If we let

$$s_{1j} = r_{1j}, \qquad j < k,$$

$$s_{1k} = \sum_{k}^{n_1} r_{1j}, \qquad A = \sum_{1}^{k-1} j r_{1j},$$

we must then sum the multinomial coefficient

$$\frac{s_{1k}!}{r_{1k}! \cdots r_{1n_1}!}$$

over all partitions of $n_1 - A$ such that every part is greater than $k - 1$. This is given by the coefficient of $x^{n_1 - A}$ in

$$(x^k + x^{k+1} + \cdots)^{s_{1k}} = x^{ks_{1k}} \sum_{t=0}^{\infty} \binom{s_{1k} - 1 + t}{s_{1k} - 1} x^t$$

thus we have

$$(2.12)\qquad \sum\nolimits_{(k)} \frac{s_{1k}!}{r_{1k}! \cdots r_{1n_1}} = \binom{n_1 - A - (k-1)s_{1k} - 1}{s_{1k} - 1}$$

where $\sum_{(k)}$ denotes summation over all positive integers $r_{1k}, r_{1\,k+1}, \cdots, r_{1n_1}$ such that $\sum_k^{n_1} jr_{1j} = n_1 - A$. This identity with (2.9) gives

$$(2.13) \qquad P(s_{1i}) = \frac{\begin{bmatrix} s_1 \\ s_{1i} \end{bmatrix} \binom{n_2 + 1}{s_1} \binom{n_1 - A - (k-1)s_{1k} - 1}{s_{1k} - 1}}{\binom{n}{n_1}}, \quad i = 1, 2, \cdots, k.$$

Another useful distribution analogous to (2.13) is derived by considering runs of both kinds of elements. If we define s_{2j} $(j = 1, 2, \cdots, h)$ and B in terms of r_{2j} just as s_{1i} and A were defined above, it follows at once from (2.6) and (2.12) that

$$(2.14) \quad P(s_{1i}, s_{2j}) =$$
$$\frac{\begin{bmatrix} s_1 \\ s_{1i} \end{bmatrix} \begin{bmatrix} s_2 \\ s_{2j} \end{bmatrix} \binom{n_1 - A - (k-1)s_{1k} - 1}{s_{1k} - 1} \cdot \binom{n_2 - B - (h-1)s_{2h} - 1}{s_{2h} - 1} F(s_1, s_2)}{\binom{n}{n_1}},$$
$$i = 1, 2, \cdots, k; j = 1, 2, \cdots, h.$$

These last two distributions should be the most useful for applications. The long runs have been added together to form the new variables s_{1k} and s_{2h} thus decreasing materially the number of variables as compared with (2.6) and (2.9) while at the same time little information is lost. One is free to choose k and h so that the number of variables is appropriate for the data at hand. Moreover, it is shown in Section 5 that these variables are asymptotically normally distributed so that one may apply a simple χ^2 test of significance for "randomness of elements with respect to order" when dealing with large samples. We shall then be able to test whether a sample has been "randomly" drawn in a certain sense.

3. Moments for runs of two kinds of elements. Instead of dealing with the ordinary moments we shall obtain formulae for the factorial moments because the expressions are much more compact. As is customary, a factorial will be denoted by

$$(3.1) \qquad x^{(a)} = x(x-1)(x-2) \cdots (x - a + 1),$$

and $x^{(0)}$ is defined to be 1. Of course the ordinary moments are determined by the factorial moments by means of relations of the type

$$x^a = \sum_{i=0}^{a} C_i^a x^{(i)}.$$

A recent discussion of the coefficients C_i^a has been given by Joseph [14]. The mathematical expectation of a function $f(r)$ will be denoted by

$$E(f(r)) = \sum_r f(r)P(r). \tag{3.2}$$

Of course E is a linear operator. We shall require the following identity

$$\sum_{(1)} \prod_i r_{1i}^{(a_i)} \begin{bmatrix} r_1 \\ r_{1i} \end{bmatrix} = r_1^{(\Sigma a_i)} \binom{n_1 - \sum i a_i - 1}{r_1 - \sum a_i - 1} \tag{3.3}$$

where $\Sigma_{(1)}$ denotes summation over all positive integers $r_{11}, r_{12}, \cdots, r_{1n_1}$ such that $\sum_1^{n_1} i r_{1i} = n_1$. (3.3) may be verified by differentiating

$$\varphi(t_i) = (t_1 x + t_2 x^2 + \cdots)^{r_1}$$

a_i times with respect to t_i ($i = 1, 2, \cdots, n_1$), then finding the coefficient of x^{n_1} after putting $t_i = 1$. The identity (3.3) enables us to find the factorial moments of the variables in the distribution (2.9) for we have

$$E\left(\prod_i r_{1i}^{(a_i)}\right) = \sum_{r_{1i}} \prod r_{1i}^{(a_i)} \begin{bmatrix} r_1 \\ r_{1i} \end{bmatrix} \binom{n_2 + 1}{r_1} \Big/ \binom{n}{n_1}$$

$$= \sum r_1^{(\Sigma a_i)} \binom{n_1 - \sum i a_i - 1}{r_1 - \sum a_i - 1} \binom{n_2 + 1}{r_1} \Big/ \binom{n}{n_1}$$

$$= \sum (n_2 + 1)^{(\Sigma a_i)} \binom{n_1 - \sum i a_i - 1}{r_1 - \sum a_i - 1} \cdot \binom{n_2 - \sum a_i + 1}{r_1 - \sum a_i} \Big/ \binom{n}{n_1} \tag{3.4}$$

$$= (n_2 + 1)^{(\Sigma a_i)} \binom{n - \sum (i+1) a_i}{n_1 - \sum i a_i} \Big/ \binom{n}{n_1}.$$

The sum on r_1 involved in the last step is given by the identity

$$\sum_{i=0}^{B} \binom{A}{C + i} \binom{B}{i} = \binom{A + B}{C + B}, \tag{3.5}$$

which is readily obtained by equating coefficients of x^C in

$$(1 + x)^A \left(1 + \frac{1}{x}\right)^B = \frac{(1 + x)^{A+B}}{x^B}.$$

We shall give here the means, variances and covariances obtained from (3.4)

$$E(r_{1i}) = (n_2 + 1)^{(2)} n_1^{(i)} / n^{(i+1)}, \tag{3.6}$$

$$\sigma_{ij} = \frac{n_2^{(2)} (n_2 + 1)^{(2)} n_1^{(i+j)}}{n^{(i+j+2)}} - \frac{n_2^2 (n_2 + 1)^2 n_1^{(i)} n_1^{(j)}}{n^{(i+1)} n^{(j+1)}}, \tag{3.7}$$

$$\sigma_{ii} = \frac{n_2^{(2)} (n_2 + 1)^{(2)} n_1^{(2i)}}{n^{(2i+2)}} + \frac{(n_2 + 1)^{(2)} n_1^{(i)}}{n^{(i+1)}} \left(1 - \frac{(n_2 + 1)^{(2)} n_1^{(i)}}{n^{(i+1)}}\right). \tag{3.8}$$

These will be needed in the section dealing with asymptotic distributions. The moments for the distribution (2.6) follow at once from (3.3) as

$$E\left(\prod_{ij} r_{1i}^{(a_i)} r_{2j}^{(b_j)}\right) = \sum_{r_1, r_2} r_1^{(\Sigma a_i)} r_2^{(\Sigma b_j)} \cdot \binom{n_1 - \sum i a_i - 1}{r_1 - \sum a_i - 1} \binom{n_2 - \sum j b_j - 1}{r_2 - \sum b_j - 1} F(r_1, r_2) \Big/ \binom{n}{n_1}. \tag{3.9}$$

The summation on r_2 is accomplished by putting $r_2 = r_1 - 1, r_1$, and $r_1 + 1$, but after that has been done it is necessary to expand the product of the two factorial factors in factorial powers of the lower index of one of the binomial coefficients. This is easily done for the first few moments, but there appears to be no simple expression for the general case. The means, variances and covariances of r_{1i} are given by (3.6), (3.7) and (3.8) and those of r_{2i} are obtained from these equations by interchanging n_1 and n_2. The other covariances are

$$\sigma_{r_{1i} r_{2j}} = \frac{n_1^{(i+2)} n_2^{(j+2)}}{n^{(i+j+2)}} + 4 \frac{n_1^{(i+1)} n_2^{(j+1)}}{n^{(i+j+1)}} + 2 \frac{n_1^{(i)} n_2^{(j)}}{n^{(i+j)}} - \frac{(n_1 + 1)^{(2)} (n_2 + 1)^{(2)} n_1^{(i)} n_2^{(j)}}{n^{(i+1)} n^{(j+1)}}. \tag{3.10}$$

A slight variation of the method above will give the moments of the s_{1i} in the distribution (2.13). An accent on a summation sign will indicate that the term corresponding to $i = k$ is to be omitted. Differentiating

$$\varphi(t_i) = [t_1 x + t_2 x^2 + \cdots + t_{k-1} x^{k-1} + t_k(x^k + x^{k+1} + \cdots)]^{s_1}$$

a_i times with respect to t_i and finding the coefficient of x^{n_1} after putting $t_i = 1$, we obtain

$$\sum_{\Sigma' i s_{1i} = A} \prod_1^k s_{1i}^{(a_i)} \begin{bmatrix} s_1 \\ s_{1i} \end{bmatrix} \binom{n_1 - A - (k-1)s_{1k} - 1}{s_{1k} - 1} = s_1^{(\Sigma a_i)} \binom{n_1 - \sum i a_i + a_k - 1}{s_1 - \sum' a_i - 1}. \tag{3.11}$$

This with (2.13) gives by the same steps as used in obtaining (3.4)

$$E\left(\prod_1^k s_{1i}^{(a_i)}\right) = (n_2 + 1)^{(\Sigma a_i)} \binom{n - \sum i a_i - \sum' a_i}{n_1 - \sum i a_i} \Big/ \binom{n}{n_1}. \tag{3.12}$$

The first two moments are

$$E(s_{1k}) = \frac{(n_2 + 1) n_1^{(k)}}{n^{(k)}}, \tag{3.13}$$

$$\sigma_{ik} = \frac{n_2^2 (n_2 + 1) n_1^{(i+k)}}{n^{(i+k+1)}} - \frac{n_2 (n_2 + 1)^2 n_1^{(i)} n_1^{(k)}}{n^{(i+1)} n^{(k)}}, \tag{3.14}$$

$$(3.15) \qquad \sigma_{kk} = \frac{(n_2+1)^{(2)} n_1^{(2k)}}{n^{(2k)}} + \frac{(n_2+1)n_1^{(k)}}{n^{(k)}}\left(1 - \frac{(n_2+1)n_1^{(k)}}{n^{(k)}}\right).$$

The others are, of course, given by (3.6), (3.7) and (3.8).

The joint moments of the variables in (2.14) as obtained from (3.11) are

$$(3.16) \qquad E\left(\prod_{ij} s_{1i}^{(a_i)} s_{2j}^{(b_j)}\right) = \sum_{s_1, s_2} s_1^{(\Sigma a_i)} s_2^{(\Sigma b_j)} \binom{n_1 - \sum i a_i + a_k - 1}{s_1 - \sum' a_i - 1} \cdot \binom{n_2 - \sum j b_j + b_h - 1}{s_2 - \sum' b_j - 1} F(s_1, s_2) \Big/ \binom{n}{n_1}.$$

In addition to the covariances (3.10) we shall need

$$(3.17) \qquad \sigma_{s_{1k}s_{2j}} = \frac{n_1^{(k+2)} n_2^{(j+1)} + 2n_1^{(k+1)} n_2^{(j+1)}}{n^{(k+j+1)}} + 2\,\frac{n_1^{(k+1)} n_2^{(j)} + n_1^{(k)} n_2^{(j)}}{n^{(k+j)}} - \frac{(n_1+1)^{(2)}(n_2+1)^{(2)} n_1^{(k)} n_2^{(j)}}{n^{(k)} n^{(j+1)}},$$

$$(3.18) \qquad \sigma_{s_{1k}s_{2h}} = \frac{n_1^{(k+1)} n_2^{(h+1)}}{n^{(k+h)}} + 2\,\frac{n_1^{(k)} n_2^{(h)}}{n^{(k+h-1)}} - \frac{(n_1+1)(n_2+1) n_1^{(k)} n_2^{(h)}}{n^{(k)} n^{(h)}}.$$

The moments of r in the distribution (2.11) may be derived easily by means of (3.5) as

$$(3.19) \qquad E(r_1^{(a)}) = (n_2+1)^{(a)} \binom{n-a}{n_1-a} \Big/ \binom{n}{n_1}.$$

From which

$$(3.20) \qquad E(r_1) = \frac{(n_2+1)n_1}{n},$$

$$(3.21) \qquad \sigma_{r_1}^2 = \frac{(n_2+1)^{(2)} n_1^{(2)}}{n n^{(2)}}.$$

4. Distribution and moments of runs of k kinds of elements. This section is a generalization of the preceeding two sections to several kinds of elements. The case $k = 2$ was treated separately because the special character of the function $F(r_1, r_2)$ in this instance made the distribution comparatively simple. Now we shall be interested in k kinds of elements denoted by $a_1, \cdots, a_k$ and we shall suppose there are n_i elements of the ith kind. We let r_{ij} denote the number of runs of elements of the ith kind of length j, and put

$$n = \sum_1^k n_i, \qquad r_i = \sum_{j=1}^{n_i} r_{ij}.$$

The same argument as was used in deriving (2.6) gives

$$P(r_{ij}) = \frac{\prod_{i=1}^{k} \begin{bmatrix} r_i \\ r_{ij} \end{bmatrix} F(r_1, r_2, \cdots, r_k)}{\begin{bmatrix} n \\ n_i \end{bmatrix}} \tag{4.1}$$

where the function $F(r_1, r_2, \cdots, r_k)$, which will be referred to hereafter simply as $F(r_i)$, represents the number of different arrangements of r_1 objects of one kind, r_2 objects of a second kind, and so forth, such that no two adjacent objects are of the same kind. We shall be able to give the explicit expression for $F(r_i)$ after examining the marginal distribution $P(r_i)$. This is obtained by summing (4.1) over r_i with r_{ij} fixed by means of the identity (2.7) giving

$$P(r_i) = \frac{\prod_{i=1}^{k} \binom{n_i - 1}{r_i - 1} F(r_i)}{\begin{bmatrix} n \\ n_i \end{bmatrix}}. \tag{4.2}$$

Despite our present meager knowledge of $F(r_i)$ it is possible to find the moments of the r_i as distributed by (4.2). Since $\sum_{r_i} P(r_i) = 1$, we have the identity

$$\sum_{r_i} \Pi \binom{n_i - 1}{r_i - 1} F(r_i) = \begin{bmatrix} n \\ n_i \end{bmatrix}. \tag{4.3}$$

From this the moments are easily derived. If we put

$$u_i = n_i - r_i \tag{4.4}$$

we have

$$\begin{aligned} \sum_{r_i} \prod u_i^{(a_i)} \prod \binom{n_i - 1}{r_i - 1} F(r_i) &= \sum_{r_i} \prod (n_i - r_i)^{(a_i)} \prod \binom{n_i - 1}{r_i - 1} F(r_i) \\ &= \sum_{r_i} \prod (n_i - 1)^{(a_i)} \prod \binom{n_i - a_i - 1}{r_i - 1} F(r_i) \\ &= \prod (n_i - 1)^{(a_i)} \sum_{r_i} \prod \binom{n_i - a_i - 1}{r_i - 1} F(r_i) \\ &= \prod_{i=1}^{k} (n_i - 1)^{(a_i)} \begin{bmatrix} n - \sum a_i \\ n_i - a_i \end{bmatrix}. \end{aligned}$$

The summation involved in the last step is given by (4.3). On dividing the last equation by $\begin{bmatrix} n \\ n_i \end{bmatrix}$ we get the factorial moments of the u_i

$$E\left(\prod_{1}^{k} u_i^{(a_i)}\right) = \prod_{1}^{k} (n_i - 1)^{(a_i)} \begin{bmatrix} n - \Sigma a_i \\ n - a_i \end{bmatrix} \Big/ \begin{bmatrix} n \\ n_i \end{bmatrix}. \tag{4.5}$$

From these equations the moments of the r_i may be found; the means, variances and covariances are

$$E(r_i) = \frac{n_i(n - n_i + 1)}{n}, \tag{4.6}$$

$$\sigma_{ij} = \frac{n_i^{(2)} n_j^{(2)}}{nn^{(2)}}, \tag{4.7}$$

$$\sigma_{ii} = \frac{n_i^{(2)}(n - n_i + 1)^{(2)}}{nn^{(2)}}. \tag{4.8}$$

It is clear that

$$\begin{aligned} &\varphi(t_i) = \text{Coefficient of } \prod_1^k x_i^{n_i} \text{ in} \\ &(x_1 + \cdots + x_k)^k \prod_1^k (x_1 + \cdots + x_{i-1} + t_i x_i + x_{i+1} + \cdots + x_k)^{n_i - 1} \Big/ \begin{bmatrix} n \\ n_i \end{bmatrix} \end{aligned} \tag{4.9}$$

is a generating function for the moments of the variables u_i. This generating function will enable us to find the exact expression for $F(r_i)$ for we have

$$\begin{aligned} P(u_i = n_{ii}) &= \text{Coefficient of } \prod_1^k t_i^{n_{ii}} \text{ in } \varphi(t_i) \\ &= \sum_{\substack{\alpha_i,\, n_{ij} \\ \Sigma_i n_{ij} = n_j - \alpha_j}} \begin{bmatrix} k \\ \alpha_i \end{bmatrix} \prod_{i=1}^k \begin{bmatrix} n_i - 1 \\ n_{ij} \end{bmatrix} \Big/ \begin{bmatrix} n \\ n_i \end{bmatrix}. \end{aligned}$$

Also

$$P(u_i) = \prod_1^k \binom{n_i - 1}{r_i - 1} F(r_i) \Big/ \begin{bmatrix} n \\ n_i \end{bmatrix}$$

and equating the expressions on the right of the last two equations we have

$$F(r_i) = \frac{\displaystyle\sum_{\alpha_i,\, n_{ij}} \begin{bmatrix} k \\ \alpha_i \end{bmatrix} \prod_{i=1}^k \begin{bmatrix} n_i - 1 \\ n_{ij} \end{bmatrix}}{\displaystyle\prod_1^k \binom{n_i - 1}{r_i - 1}} \tag{4.10}$$

$$= \sum_{\substack{\alpha_i,\, n'_{ij} \\ \Sigma_i n'_{ij} = r_j - \alpha_j}} \begin{bmatrix} k \\ \alpha_i \end{bmatrix} \prod_{i=1}^k \begin{bmatrix} r_i - 1 \\ n'_{ij} \end{bmatrix} \tag{4.11}$$

in which the prime on the n'_{ij} indicates that the indices corresponding to $j = i$ are to be omitted; hence i takes all the values $1, 2, \cdots, k$ and j takes all values $1, 2, \cdots, k$ except i because the index n_{ii} has been cancelled with $n_i - r_i$ in the binomial coefficient in the denominator of (4.10). It is clear from (4.11) that $F(r_i)$ may be expressed as follows

$$\begin{aligned} F(r_i) = CT \prod_1^k x_i^{-r_i} (x_1 + \cdots + x_k)^k (x_2 + x_3 + \cdots + x_k)^{r_1 - 1} \\ (x_1 + x_3 + \cdots + x_k)^{r_2 - 1} \cdots (x_1 + \cdots + x_{k-1})^{r_k - 1} \end{aligned} \tag{4.12}$$

in which "CT" is an abbreviation for "constant term of."

We are now in a position to obtain moments of the variables r_{ij} in the distribution (4.1) by means of identities similiar to (4.3). As an illustration we compute

$$\sum_{r_i} \binom{n_1 - a - 1}{r_1 - a - 1} \prod_2^k \binom{n_i - 1}{r_i - 1} F(r_i) = \sum_{r_i} \binom{n_1 - a - 1}{r_1 - a - 1} \prod_2^k \binom{n_i - 1}{r_i - 1}$$

$$\cdot CT \prod_1^k x_i^{-r_i} \prod_1^k (x_1 + \cdots + t_i x_i + \cdots + x_k)^{r_i - 1} \Big]_{t_i = 0}$$

$$= CT \prod_1^k x_i^{-n_i} (x_1 + \cdots + x_k)^{n-a} (x_2 + \cdots + x_k)^a$$

$$= \begin{bmatrix} n \\ n_i \end{bmatrix} \frac{(n - n_1)^{(a)}}{n^{(a)}}$$

or

$$\sum_{r_i} \binom{n_1 - a - 1}{r_1 - a - 1} \prod_2^k \binom{n_i - 1}{r_i - 1} F(r_i) = \frac{(n-a)!(n - n_1)^{(a)}}{\prod_1^k n_i!}. \tag{4.13}$$

The moments of r_{ij} may be computed from identities of this type together with (3.3). The first two moments are

$$E(r_{ij}) = (n - n_i + 1)^{(2)} n_i^{(j)} / n^{(j+i)} \tag{4.14}$$

$$E(r_{ij}^{(2)}) = n_i^{(2j)} (n - n_i)^{(2)} (n - n_i + 1)^{(2)} / n^{(2j+2)} \tag{4.15}$$

$$E(r_{ij} r_{it}) = n_i^{(j+t)} (n - n_i)^{(2)} (n - n_i + 1)^{(2)} / n^{(j+t+2)} \qquad j \neq t \tag{4.16}$$

$$\begin{aligned} E(r_{ij} r_{st}) = {} & (n_i - j - 1)(n_s - t - 1) \frac{n_i^{(j-1)} n_s^{(t-1)}}{n^{(j+t+2)}} \{(n_i - j + 1)^{(2)} (n_s - t + 1)^{(2)} \\ & + 2(n - n_i - n_s)(n_i - j + 1)(n_s - t + 1)(n_s - t + n_i - j) \\ & + (n - n_i - n_s)^{(2)} [(n_s - t + 1)^{(2)} + 2(n_i - j + 1)(n_s - t + 1) \\ & + (n_i - j + 1)^{(2)}] + 2(n - n_i - n_s)^{(3)} (n_i - j + n_s - t + 2) \\ & + (n - n_i - n_s)^{(4)}\} + 2(n_i - j - 1) \frac{n_i^{(j-1)} n_s^{(t-1)}}{n^{(j+t+1)}} \{(n_i - j + 1) \\ & \cdot (n_s - t + 1)^{(2)} + (n - n_i - n_s)[2(n_i - j + 1)(n_s - t + 1) \\ & + (n_s - t + 1)^{(2)}] + (n - n_i - n_s)^{(2)} [2(n_s - t + 1) + (n_i - j + 1)] \\ & + (n - n_i - n_s)^{(3)}\} + 2(n_s - t - 1) \frac{n_i^{(j-1)} n_s^{(t-1)}}{n^{(j+t+1)}} \{(n_s - t + 1) \\ & \cdot (n_i - j + 1)^{(2)} + (n - n_i - n_s)[2(n_i - j + 1)(n_s - t + 1) \\ & + (n_i - j + 1)^{(2)}] + (n - n_i - n_s)^{(2)} [2(n_i - j + 1) + (n_s - t + 1)] \\ & + (n - n_i - n_s)^{(3)}\} + 4 \frac{n_i^{(j-1)} n_s^{(t-1)}}{n^{(j+t)}} \{(n_i - j + 1)(n_s - t + 1) \\ & + (n - n_i - n_s)(n_i - j + n_s - t + 2) + (n - n_i - n_s)^{(2)}\}. \end{aligned} \tag{4.17}$$

Such a lengthy expression as this last one can hardly be useful to the statistician, and for this reason we shall not define variables s_{ij} analogous to the s_{1i} and s_{2j} of Section 2 and take the time and space to find their moments.

5. Asymptotic distributions. We shall show that some of the distributions obtained previously are asymptotically normal when the n_i become large in such a way that the ratios n_i/n remain fixed. The description "asymptotically normal" means that the distribution approaches the normal distribution uniformly over any finite region as $n_i \to \infty$. The ratios n_i/n will be denoted by e_i, hence $\Sigma\, e_i = 1$. The symbol $O(1/n^a)$ will represent any function such that

$$\operatorname*{Lim}_{n\to\infty} n^a O\left(\frac{1}{n^a}\right) = L < \infty.$$

We shall not, of course, be able to get any limit theorems for distributions like (2.6) or (2.9) because the number of independent variables increases with n. We shall consider first the distribution (2.13) whose asymptotic character is given in the following theorem.

THEOREM 1. *The variables*

$$(5.1)\qquad \begin{aligned} x_i &= \frac{s_{1i} - ne_1^i e_2^2}{\sqrt{n}} \qquad i < k \\ x_k &= \frac{s_{1k} - ne_1^k e_2}{\sqrt{n}} \end{aligned}$$

are asymptotically normally distributed with zero means and variances and covariances

$$(5.2)\qquad \begin{aligned} \sigma_{ij} &= e_1^{i+j-1}e_2^3[(i+1)(j+1)e_1e_2 - ije_2 - 2e_1],\ i, j < k,\ i \neq j \\ \sigma_{ii} &= e_1^{2i-1}e_2^3[(i+1)^2e_1e_2 - i^2e_2 - 2e_1] + e_1^ie_2^2,\ i < k \\ \sigma_{ik} &= e_1^{i+k-1}e_2^2[(i+1)ke_1e_2 - ike_2 - e_1],\ i < k \\ \sigma_{kk} &= e_1^{2k-1}e_2[k^2(e_1 - 1)e_2 - e_1] + e_1^ke_2. \end{aligned}$$

The limiting means, variances and covariances are obtained from the relations (3.6), (3.7), (3.8), (3.13), (3.14) and (3.15).

To demonstrate this theorem we make the substitutions

$$(5.3)\qquad \begin{aligned} n_i &= ne_i & i &= 1, 2 \\ s_{1i} &= ne_1^ie_2^2 + \sqrt{n}x_i & i &= 1, 2, \cdots, k-1 \\ s_{1k} &= ne_1^ke_2 + \sqrt{n}\, x_k \\ s_1 &= ne_1e_2 + \sqrt{n}\sum_1^k x_i \\ A &= n(e_1 - e_1^k - ke_1^ke_2) + \sqrt{n}\sum_1^{k-1} ix_i \end{aligned}$$

in (2.13), and estimate the factorials by means of Stirling's formula

$$(5.4)\qquad m! = \sqrt{2\pi}\, m^{m+\frac{1}{2}} e^{-m}\left(1 + O\left(\frac{1}{m}\right)\right).$$

The result is an unwieldy expression which we shall not present at the moment. First we note that the exponential factors cancel out because the sum of the lower indices of a binomial or multinomial coefficient is equal to the upper index. Also we simplify the expression by considering in detail only terms which involve the x_i; the normalizing constant can be determined from the final limit function. Any function of the parameters will be represented by the letter K. Thus in (5.4) we need consider only the factor $m^{m+\frac{1}{2}}$. All factorials will be of the form

$$(5.5)\qquad m = na + \sqrt{n}L(x) + b$$

where $L(x)$ is a linear function of the x_i, and a and b are independent of n and x_i. Now

$$\begin{aligned} m^{m+\frac{1}{2}} &= (na + \sqrt{n}L(x) + b)^{na+\sqrt{n}L(x)+b+\frac{1}{2}} \\ &= (na)^{na+\sqrt{n}L(x)+b+\frac{1}{2}}\left(1 + \frac{L(x)}{a\sqrt{n}} + \frac{b}{an}\right)^{na+\sqrt{n}L(x)+b+\frac{1}{2}} \\ &= K(na)^{\sqrt{n}L(x)}\left(1 + \frac{L(x)}{a\sqrt{n}} + \frac{b}{an}\right)^{na+\sqrt{n}L(x)+b+\frac{1}{2}} \end{aligned}$$

and
$$\begin{aligned} \log m^{m+\frac{1}{2}} &= K + \sqrt{n}L(x)\log na + (na + \sqrt{n}L(x) + b + \tfrac{1}{2}) \\ &\qquad \cdot\log\left(1 + \frac{L(x)}{a\sqrt{n}} + \frac{b}{an}\right) \\ &= K + \sqrt{n}L(x)\log na + (na + \sqrt{n}L(x) + b + \tfrac{1}{2}) \\ &\qquad \cdot\left(\frac{L(x)}{a\sqrt{n}} + \frac{b}{an} - \frac{L^2(x)}{a^2 n} + O\left(\frac{1}{n^{3/2}}\right)\right) \qquad (5.6) \\ &= K + \sqrt{n}L(x)(1 + \log na) + \frac{1}{2a}L^2(x) + O\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

so terms arising from b (and $b + \frac{1}{2}$ in the exponent) will be neglected as they give rise only to terms independent of the x_i or of order $1/n^{\frac{1}{2}}$. Of course $\log (1 + O(1/m)) = O(1/m)$. Thus, keeping significant terms only, the result of the substitutions (5.3) and (5.4) in (2.13) after taking logarithms and using (5.6) is

$$\begin{aligned} -\log P(r_i) &= K + \sqrt{n}\sum_1^{k-1} x_i (\log n e_1^i e_2^2 + 1) + \sum_1^{k-1} \frac{x_i^2}{2e_1^i e_2^2} \\ &\quad - \sqrt{n}\left(\sum_1^k x_i\right)(\log n e_2^2 + 1) + \frac{1}{2e_2^2}\left(\sum_1^k x_i\right)^2 \end{aligned}$$

$$(5.7)\quad + \sqrt{n}\left(\sum_1^{k-1} ix_i + (k-1)x_k\right)(\log ne_1^k + 1) - \frac{1}{2e_1^k}\left(\sum_1^k ix_i + (k-1)x_k\right)^2$$
$$+ 2\sqrt{n}x_k(\log ne_1^k e_2 + 1) + \frac{x_k^2}{e_1^k e_2} - \sqrt{n}\left(\sum_1^k ix_i\right)(\log ne_1^{k+1} + 1)$$
$$+ \frac{1}{2e_1^{k+1}}\left(\sum_1^k ix_i\right)^2 + O\left(\frac{1}{\sqrt{n}}\right).$$

The coefficients of $x_i (i < k)$ and x_k are

$$\sqrt{n}(\log ne_1^i e_2^2 + 1 - \log ne_2^2 - 1 + i\log ne_1^k + i - i\log ne_1^{k+1} - i) = 0,$$
$$\sqrt{n}(- \log ne_2^2 - 1 + k\log ne_1^k + k + 2\log ne_1^k e_2 + 2 - k\log ne_1^{k+1} - k) = 0.$$

Hence only the quadratic terms remain and we have

$$(5.8)\qquad -\log P = K + \tfrac{1}{2}\sum_{i,j}\sigma^{ij}x_i x_j + O\left(\frac{1}{\sqrt{n}}\right)$$

where

$$(5.9)\qquad \begin{aligned}
\sigma^{ij} &= \frac{1}{e_2^2} + \frac{ije_2}{e_1^{k+1}} && i, j < k,\ i \neq j,\\
\sigma^{ii} &= \frac{1}{e_2^2} + \frac{1}{e_1^i e_2^2} + \frac{i^2 e_2}{e_1^{k+1}} && i < k,\\
\sigma^{ik} &= \frac{1}{e_2^2} + \frac{i + i(k-1)e_2}{e_1^{k+1}} && i < k,\\
\sigma^{kk} &= \frac{1}{e_2^2} + \frac{2}{e_1^k e_2} + \frac{k^2}{e_1^{k+1}} - \frac{(k-1)^2}{e_1^k}.
\end{aligned}$$

It is merely a matter of straightforward multiplication of the two matrices to verify that $\| \sigma^{ij} \|$ is the inverse of $\| \sigma_{ij} \|$, hence is a positive definite matrix. The details of the verification will be omitted. We have then

$$(5.10)\qquad P = Ke^{-\frac{1}{2}\Sigma\sigma^{ij}x_i x_j}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

In this equation K must necessarily contain the factor $\left(\frac{1}{\sqrt{n}}\right)^k$ because there are $k + 5$ factorials in the denominator and 5 in the numerator of (2.13). Since $\Delta r_i = 1$, this factor, in view of (5.1), may be replaced by $\Pi\Delta x_i$, so

$$(5.11)\qquad P = Ke^{-\frac{1}{2}\Sigma\sigma^{ij}x_i x_j}\Pi\Delta x_i\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

If we restrict the x_i to any finite region R in the x-space, the function $O(1/n^{\frac{1}{2}})$ approaches zero uniformly as $n \to \infty$. Thus, if $A_i < B_i$ are any positive numbers such that the corresponding values of x_i, say a_i and b_i, obtained by substituting A_i and B_i for r_i in (5.1), determine a rectangular region $R'(a_i < x_i < b_i)$, which lies in R we have

$$\sum_{r_i=A_i}^{B_i} P(r_i) = \sum_{x_i=a_i}^{b_i} Ke^{-\frac{1}{2}\Sigma\sigma^{ij}x_ix_j}\Pi\Delta x_i\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \tag{5.12}$$

$$\xrightarrow[n\to\infty]{} \int_{R'} Ke^{-\frac{1}{2}\Sigma\sigma^{ij}x_ix_j}\Pi\, dx_i$$

by the definition of a definite integral and Riemann's fundamental theorem.

We have given some details of this proof in order that it may serve as a model for other theorems of a similiar nature which will appear later, and for which a complete proof will not be given. Two immediate consequences of Theorem 1 will now be stated as corollaries.

COROLLARY 1. *The variable*

$$x = \frac{r - ne_1e_2}{\sqrt{n}e_1e_2}$$

where r is the total number of runs of one kind of element, is asymptotically normally distributed with zero mean and unit variance. The limiting mean and variance were computed from (3.20) and (3.21).

COROLLARY 2. *The variable* $Q = \Sigma\sigma^{ij}x_ix_j$ *is asymptotically distributed according to the* χ^2*-law with k degrees of freedom.*

In exactly the same manner in which Theorem 1 was deduced from (2.13), we may prove the following theorem corresponding to the distribution (2.14).

THEOREM 2. *The variables*

$$x_i = \frac{s_{1i} - ne_1^ie_2^2}{\sqrt{n}} \qquad i < k,$$

$$x_k = \frac{s_{1k} - ne_1^ke_2}{\sqrt{n}}, \tag{5.13}$$

$$y_i = \frac{s_{2i} - ne_1^2e_2^i}{\sqrt{n}} \qquad i < h,$$

are asymptotically normally distributed with zero means and variances and covariances

$$\sigma_{x_ix_j} = e_1^{i+j-1}e_2^3[(i + 1)(j + 1)e_1e_2 - ije_2 - 2e_1] \qquad i, j < k,$$

$$\sigma_{x_ix_i} = e_1^{2i-1}e_2^3[(i + 1)^2e_1e_2 - i^2e_2 - 2e_1] + e_1^ie_2^2 \qquad i < k,$$

$$\sigma_{x_ix_k} = e_1^{i+k-1}e_2^2[(i + 1)ke_1e_2 - ike_2 - e_1] \qquad i < k,$$

$$\sigma_{x_kx_k} = e_1^{2k-1}e_2[- k^2e_2^2 - e_1] + e_1^ke_2,$$

$$
\begin{aligned}
(5.14)\quad \sigma_{y_i y_j} &= e_2^{i+j-1} e_1^3[(i+1)(j+1)e_1e_2 - ije_1 - 2e_2] && i, j < h,\\
\sigma_{y_i y_i} &= e_2^{2i-1} e_1^3[(i+1)^2 e_1e_2 - i^2 e_1 - 2e_2] + e_2^i e_1^2 && i < h,\\
\sigma_{x_i y_j} &= e_1^{i+1} e_2^{j+1}[(i+1)(j+1)e_1e_2 - 2ie_2 - 2je_1 + 4e_1e_2 + 2] && \\
& && i < k, j < h,\\
\sigma_{x_k y_j} &= e_1^{k+1} e_2^j[k(j+1)e_1e_2 - 2(k-1)e_2 - (j-1)e_1 + 2e_1e_2] && j < h.
\end{aligned}
$$

These limiting variances were computed from the variances and covariances given in Section 3. We have chosen the variable s_{2h} of (2.14) as the dependent variable. The proof of this theorem is omitted. From it the following corollaries are deduced immediately.

COROLLARY 3. *If $u_i = x_i$ and $u_{k+i} = y_i$ of* (5.13) *and $\| \sigma^{ij} \|$ $(i, j = 1, 2, \cdots, k + h - 1)$ denotes the inverse of* (5.14), *then the variable $Q = \Sigma\sigma^{ij}u_iu_j$ is asymptotically distributed according to the χ^2-law with $k + h - 1$ degrees of freedom.*

COROLLARY 4. *If $s_i = s_{1i} + s_{2i}$ denotes the total number of runs of both kinds of elements of length i, and s_k the total number of runs of length greater than $k - 1$, then the variables*

$$
(5.15)\qquad
\begin{aligned}
x_i &= \frac{s_i - n(e_1^i e_2^2 + e_2^i e_1^2)}{\sqrt{n}} \qquad i < k\\
x_k &= \frac{s_k - n(e_1^k e_2 + e_2^k e_1)}{\sqrt{n}}
\end{aligned}
$$

are asymptotically normally distributed with zero means and variances and covariances

$$
(5.16)\qquad \sigma_{ij} = \sigma_{x_i x_j} + \sigma_{x_i y_j} + \sigma_{x_j y_i} + \sigma_{y_i y_j}\,.
$$

We have put $h = k$ in Theorem 2 to obtain this result. The terms on the right of (5.16) are defined by (5.14); terms which do not appear there may be found by interchanging e_1 and e_2 in one of the relations. For example $\sigma_{y_k y_k}$ is given by interchanging e_1 and e_2 in the fourth equation of the set (5.14).

COROLLARY 5. *The variable $Q = \Sigma\sigma^{ij}x_ix_j$ where the x_i are defined by* (5.15) *and $\| \sigma^{ij} \|$ is the inverse of* (5.16), *is asymptotically distributed according to the χ^2-law with k degrees of freedom.*

COROLLARY 6. *If s denotes the total number of runs of both kinds of elements, then the variable*

$$
x = \frac{s - 2ne_1e_2}{2\sqrt{n}e_1e_2}
$$

is asymptotically normally distributed with zero mean and unit variance. This is the result derived by Wald and Wolfowitz [13].

6. Asymptotic distributions for k kinds of elements. We now investigate the asymptotic character of the distribution (4.2)

$$(6.1) \qquad P(r_i) = \frac{\prod_{i=1}^{k} \binom{n_i - 1}{r_i - 1} F(r_i)}{\begin{bmatrix} n \\ n_i \end{bmatrix}}$$

where r_i is the total number of runs of the ith kind of element.

THEOREM 1. *If $k > 2$, the variables*

$$(6.2) \qquad x_i = \frac{r_i - ne_i(1 - e_i)}{\sqrt{n}}$$

are asymptotically normally distributed with zero means and variances and covariances

$$(6.3) \qquad \sigma_{ij} = e_i^2 e_j^2, \qquad \sigma_{ii} = e_i^2(1 - e_i)^2.$$

The restriction $k > 2$ is made because in the case $k = 2$ the correlation between the two variables approaches one, and the numbers σ_{ij} are all equal. The result may be called a degenerate normal distribution and might be included in the theorem in this sense; we have chosen to omit it because this case is better taken care of by Corollary 1 of the previous section.

The proof of this theorem will be simplified if in the moments (4.5) we replace the numbers $n_i - 1$ by n_i. This substitution will not, of course, affect the limiting moments. Hence we consider the variables v_i with moments given by

$$(6.4) \qquad E\left(\prod_1^k v_i^{(a_i)}\right) = \frac{\prod_1^k n_i^{(a_i)} \begin{bmatrix} n - \Sigma a_i \\ n_i - a_i \end{bmatrix}}{\begin{bmatrix} n \\ n_i \end{bmatrix}}$$

and shall show that

$$(6.5) \qquad y_i = \frac{v_i - ne_i^2}{\sqrt{n}}$$

are asymptotically normally distributed with zero means and variances and covariance (6.3). It is possible to prove this statement by showing that the characteristic function (Fourier transform) obtained by substituting $i\theta_i$ for t_i in the moment generating function

$$(6.6) \qquad \begin{aligned} \varphi_n(t_i) &= \text{Coef. of } \prod_1^k x_i^{n_i} \text{ in} \\ &\prod_1^k (x_1 + \cdots + x_{i-1} + t_i x_i + x_{i+1} + \cdots + x_k)^{n_i} \Big/ \begin{bmatrix} n \\ n_i \end{bmatrix} \end{aligned}$$

approaches

$$\varphi(\theta_i) = e^{-\frac{1}{2}\Sigma\sigma_{ij}\theta_i\theta_j}$$

as $n \to \infty$. This method is not appropriate for proving a similiar theorem which appears in Part II, and we prefer to give here a demonstration that will suffice for both theorems.

In order to prove our theorem we consider the general term in the coefficient of $\Pi x_i^{n_i}$ in (6.6)

$$C(m_{ij}) = \prod_{i=1}^{k} \begin{bmatrix} n_i \\ m_{ij} \end{bmatrix} \prod t_i^{m_{ii}} \Big/ \begin{bmatrix} n \\ n_i \end{bmatrix} \tag{6.7}$$

in which

$$\sum_{i=1}^{k} m_{ij} = n_j \tag{6.8}$$

must be required as well as the usual restriction on indices of a multinomial coefficient, $\sum_{j=1}^{k} m_{ij} = n_i$. Therefore only $(k-1)^2$ of the indices are independent. Clearly $m_{ii} = v_i$. Now without concerning ourselves about the statistical significance of the variables m_{ij}, let us consider their distribution

$$D(m_{ij}) = \prod_{i=1}^{k} \begin{bmatrix} n_i \\ m_{ij} \end{bmatrix} \Big/ \begin{bmatrix} n \\ n_i \end{bmatrix} \tag{6.9}$$

in which the variables corresponding to the values $i, j = 1, 2, \cdots, k-1$ will be chosen as the independent ones. We shall now prove a theorem from which Theorem 1 follows immediately.

THEOREM 2. *The variables*

$$x_{ij} = \frac{m_{ij} - ne_ie_j}{\sqrt{n}} \qquad i, j = 1, 2, \cdots, k-1 \tag{6.10}$$

are asymptotically normally distributed with zero means and variances and covariances given by

$$\begin{aligned} \sigma_{ij,pq} &= e_ie_je_pe_q, \\ \sigma_{ij,ip} &= -e_i(1-e_i)e_je_p, \\ \sigma_{ij,ij} &= e_ie_j(1-e_i)(1-e_j). \end{aligned} \tag{6.11}$$

First it is to be noted that the moments of the m_{ij} are easily obtained from the identity

$$\sum_{\Sigma_i m_{ij} = n_j} \prod_{i=1}^{k} \begin{bmatrix} n_i \\ m_{ij} \end{bmatrix} = \begin{bmatrix} n \\ n_i \end{bmatrix} \tag{6.12}$$

as follows

$$\sum \prod_{ij} m_{ij}^{(a_{ij})} \prod_i \begin{bmatrix} n_i \\ m_{ij} \end{bmatrix} = \sum \prod_i n_i^{(\Sigma_j a_{ij})} \prod_i \begin{bmatrix} n_i - \sum_j a_{ij} \\ m_{ij} - a_{ij} \end{bmatrix}$$

$$= \prod_i n_i^{(\Sigma_j a_{ij})} \begin{bmatrix} n - \sum_{ij} a_{ij} \\ n_j - \sum_i a_{ij} \end{bmatrix}$$

and on dividing this last relation by $\begin{bmatrix} n \\ n_i \end{bmatrix}$ we obtain

$$E\left(\prod_{i,j} m_{ij}^{(a_{ij})}\right) = \prod_i n_i^{(\Sigma_j a_{ij})} \prod_j n_i^{(\Sigma_i a_{ij})} / n^{(\Sigma_{ij} a_{ij})} \tag{6.13}$$

from which the moments (6.11) and the means in (6.10) were computed.

The proof of the theorem is similar to that of Theorem 1 in Section 5. We make the substitutions

$$n_i = ne_i, \qquad m_{kj} = n_j - \sum_{i=1}^{k-1} m_{ij},$$

$$m_{ik} = n_i - \sum_{j=1}^{k-1} m_{ij}, \qquad m_{kk} = 2n_k + \sum_{i,j=1}^{k-1} m_{ij} - n,$$

$$m_{ij} = ne_ie_j + \sqrt{n}x_{ij},$$

in (6.9) and employ Stirling's formula exactly as before. The details are too similiar to warrant repetition. The final result is

$$D(m_{ij}) = Ke^{-\frac{1}{2}\Sigma\sigma^{ij,pq}x_{ij}x_{pq}} \Pi \, dx_{ij}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right). \tag{6.14}$$

Where $\| \sigma^{ij,pq} \|$ is the inverse of (6.11) and is defined by

$$\sigma^{ij,pq} = \frac{1}{e_k^2}, \qquad \sigma^{ij,ij} = \frac{1}{e_k^2} + \frac{1}{e_ie_k} + \frac{1}{e_je_k} + \frac{1}{e_ie_j},$$

$$\sigma^{ij,ip} = \frac{1}{e_1e_k} + \frac{1}{e_k^2}, \qquad \sigma^{ij,pj} = \frac{1}{e_1e_k} + \frac{1}{e_k^2}.$$

Theorem 1 is a corollary of Theorem 2. Also we may state these additional results:

COROLLARY 1. *If k (≥ 3) kinds of elements are arranged at random and r denotes the total number of runs of all kinds of elements, then the variable*

$$x = \frac{r - n(1 - \Sigma e_i^2)}{\sqrt{n}}$$

is asymptotically normally distributed with zero mean and variance

$$\sigma^2 = \Sigma e_i^2 - 2\Sigma e_i^3 + (\Sigma e_i^2)^2$$

where e_i is the proportion of elements of the i-th kind.

COROLLARY 2. *The variable $Q = \Sigma\sigma^{ij}x_ix_j$ where the x_i are defined by (6.2) and $\| \sigma^{ij} \|$ is the inverse of (6.3), is asymptotically distributed according to the χ^2-law with k degrees of freedom.*

As was mentioned in Section 4, we could define variables s_{ij} ($i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, h_i$, the h_i being a set of k arbitrary integers) with a distribution similiar to (2.14). If one worked through the details he would find, no doubt, that these variables are asymptotically normal. The matrix of variances and covariances is so complicated, however, that such a theorem would hardly be useful to the statistician, and the author does not feel that it would be worthwhile to go through the long and tedious details merely for the sake of completeness.

PART II

Instead of having the number of elements of each kind fixed, we now suppose that they are randomly drawn from a binomial or multinomial population. The numbers n_i thus become random variables subject only to the restriction that $\Sigma n_i = n$, the sample number. The development will be entirely analogous to that of Part I, and the same notation will be used. The probability associated with the ith kind of element will be denoted by p_i.

7. Distributions and moments. The major part of the derivation of the various distribution functions has already been done in Sections 2 and 3. With the distributions of these sections we need only employ the fundamental relation

$$P(X, Y) = P_1(X \mid Y)P_2(Y) \tag{7.1}$$

in order to obtain the distributions required here. X will represent the set of variables r_{ij} or r_i, and Y the variables n_i. For the binomial population $P_2(Y)$ will be

$$P(n_1, n_2) = \binom{n}{n_1} p_1^{n_1} p_2^{n_2}. \tag{7.2}$$

Therefore we may write down at once the distributions

$$P(r_{ij}, n_i) = \begin{bmatrix} r_1 \\ r_{1j} \end{bmatrix} \begin{bmatrix} r_2 \\ r_{2j} \end{bmatrix} F(r_1, r_2) p_1^{n_1} p_2^{n_2}, \tag{7.3}$$

$$P(r_{1i}, n_i) = \begin{bmatrix} r_1 \\ r_{1j} \end{bmatrix} \binom{n_2 + 1}{r_1} p_1^{n_1} p_2^{n_2}, \tag{7.4}$$

$$P(r_1, n_i) = \binom{n_1 - 1}{r_1 - 1} \binom{n_2 + 1}{r_1} p_1^{n_1} p_2^{n_2}, \tag{7.5}$$

$$P(s_{1j}, n_i) = \begin{bmatrix} s_1 \\ s_{1j} \end{bmatrix} \binom{n_1 - A - (k-1)s_{1k} - 1}{s_{1k} - 1} \binom{n_2 + 1}{s_1} p_1^{n_1} p_2^{n_2}, \tag{7.6}$$

$$\begin{aligned} P(s_{1i}, s_{2j}, n_i) = \begin{bmatrix} s_1 \\ s_{1i} \end{bmatrix} \begin{bmatrix} s_2 \\ s_{2j} \end{bmatrix} \binom{n_1 - A - (k-1)s_{1k} - 1}{s_{1k} - 1} \\ \cdot \binom{n_2 - B - (h-1)s_{2h} - 1}{s_{2h} - 1} F(s_1, s_2) p_1^{n_1} p_2^{n_2}, \\ i = 1, \cdots, k, j = 1, \cdots, h, \end{aligned} \tag{7.7}$$

corresponding to the distributions (2.6), (2.9), (2.11), (2.13) and (2.14) respectively. Of course there is some dependence among the arguments. In (7.4), for example, n_1 is determined by $\Sigma i r_{1i} = n_1$, and n_2 by $n - n_1 = n_2$. In the last three distributions one of the n_i is independent and one may sum these with respect to n_1 from zero to n and obtain the distributions of the r's alone. The results of such summations are quite cumbersome and in some cases can only be indicated, so we shall retain the n_i as relevant variables. This remark applies also to the multinomial distribution.

We shall obtain expressions for the joint moments of the variables in these distributions. It is clear that the moments in Section 3 will be of considerable aid; for, using the notation of (7.1), we have

$$(7.8) \quad E(f(X)g(Y)) = \sum_{XY} f(X)g(Y)P(X, Y) = \sum_{Y} g(Y)P_2(Y)[\sum_{X} f(X)P_1(X/Y)]$$

and the sum in the bracket on the right has been computed in Section 3. It remains only for us to multiply the previous moments by $g(Y)P_2(Y)$ and sum on Y. Corresponding to (3.4), (3.12), (3.9) and (3.19) we have

$$(7.9) \quad E\left(n_1^{(a)} \prod_1^{n_1} r_{1i}^{(a_i)}\right) = \sum_{n_1=0}^{n} n_1^{(a)}(n_2 + 1)^{(\Sigma a_i)} \binom{n - \Sigma i a_i - \Sigma a_i}{n_1 - \Sigma i a_i} p_1^{n_1} p_2^{n_2},$$

$$(7.10) \quad E\left(n_1^{(a)} \prod_1^{k} s_{1i}^{(a_i)}\right) = \sum_{n_1=0}^{n} n_1^{(a)}(n_2 + 1)^{(\Sigma a_i)} \binom{n - \Sigma i a_i - \Sigma' a_i}{n_1 - \Sigma i a_i} p_1^{n_1} p_2^{n_2},$$

$$(7.11) \quad E(n_1^{(a)} r_1^{(b)}) = \sum_{n_1=0}^{n} n_1^{(a)}(n_2 + 1)^{(b)} \binom{n - b}{n_1 - b} p_1^{n_1} p_2^{n_2},$$

$$(7.12) \quad \begin{aligned} E\left(n_1^{(a)} \prod_1^{k} s_{1i}^{(a_i)} \prod_1^{h} s_{2j}^{(b_j)}\right) &= \sum_{n_1, s_1, s_2} n_1^{(a)} s_1^{(\Sigma a_i)} s_2^{(\Sigma b_j)} \binom{n_1 - \Sigma i a_i + a_k - 1}{s_1 - \Sigma' a_i - 1} \\ &\quad \cdot \binom{n_2 - \Sigma j b_j + b_h - 1}{s_2 - \Sigma' b_j - 1} F(s_1, s_2) p_1^{n_1} p_1^{n_2}, \end{aligned}$$

for moments from (7.4), (7.6), (7.5) and (7.7) respectively. In order to perform the summations indicated in these last relations it is necessary to expand the factors multiplying the binomial coefficient in factorial powers of its lower index. That is, we must write

$$(7.13) \quad n_1^{(a)}(n_2 + 1)^{(b)} = \sum_{i=0}^{a+b} C_i(n, a, b)(n_1 - b)^{(i)}.$$

Again it is not possible to give a simple expression for the coefficients $C_i(n, a, b)$ in general, but for the first few moments they present no difficulty. For example from (7.9)

$$
\begin{aligned}
E(n_1 r_{1i}) &= \sum_{n_1=0}^{n} n_1(n - n_1 + 1)\binom{n-i-1}{n_1-i} p_1^{n_1} p_2^{n_2} \\
&= \sum_{n_1} [i(n-i+1) + (n-2i)(n_1-i) + (n_1-i)^{(2)}] \\
&\qquad \cdot \binom{n-i-1}{n_1-i} p_1^{n_1} p_2^{n_2} \\
&= \sum_{n_1} \left[i(n-i+1)\binom{n-i-1}{n_1-i} + (n-2i)(n-i-1) \right. \\
&\qquad \left. \cdot \binom{n-i-2}{n_1-i-1} - (n-i-1)^{(2)} \binom{n-i-3}{n_1-i-2} \right] p_1^{n_1} p_2^{n_2} \\
&= [i(n-i+1) + (n-2i)(n-i-1)p_1 - (n-i-1)^{(2)} p_1^2] p_1^i p_2 .
\end{aligned} \tag{7.14}
$$

We give below some means, variances and covariances which will be required later.

$$
\begin{aligned}
E(r_{1i}) &= p_1^i p_2[(n-i-1)p_2 + 2], \\
E(s_{1k}) &= p_1^k[(n-k)p_2 + 1], \\
\sigma_{r_{1i}r_{1j}} &= p_1^{i+j} p_2^2\{(n-i-j)^{(2)} p_2^2 + (n-i-j)p_2(1+5p_1) + 6p_1^2 \\
&\qquad - [(n-i-1)p_2 + 2][(n-j-1)p_2 + 2]\}, \\
\sigma_{r_{1i}r_{1i}} &= p_1^{2i} p_2^2\{(n-2i)^{(2)} p_2^2 + (n-2i)p_2(1+5p_1) + 6p_1^2 \\
&\qquad - [(n-i-1)p_2 + 2]^2\} + p_1^i p_2[(n-i-1)p_2 + 2], \\
\sigma_{r_{1i}r_{2j}} &= p_1^i p_2^j\{(n-i-j-2)^{(2)} p_1^2 p_2^2 + 4(n-i-j-1)p_1 p_2 + 2 \\
&\qquad - [(n-i-1)p_2 + 2][(n-j-1)p_1 + 2], \\
\sigma_{s_{1i}s_{1k}} &= p_1^{i+k} p_2\{(n-i-k+1)^{(2)} - 2(n-i-k)^{(2)} p_1 \\
&\qquad + (n-i-k-1)^{(2)} p_1^2 - [(n-i-1)p_2 + 2][(n-k)p_2 + 1]\}, \\
\sigma_{s_{1k}s_{1k}} &= p_1^{2k}\{(n-2k+1)^{(2)} - 2(n-2k)^{(2)} p_1 + (n-2k)^{(2)} p_1^2 \\
&\qquad - [(n-k)p_2 + 1]^2\} + p_1^k[(n-k)p_2 + 1], \\
\sigma_{s_{1k}s_{2j}} &= p_1^k p_2^j\{(n-k-j-2)^{(2)} p_1^2 p_2 + 2(n-k-j-1)p_1(1+p_2) \\
&\qquad + 2(1+p_1) - p_1[(n-k)p_2 + 1][(n-j-1)p_1 + 2]\}.
\end{aligned} \tag{7.15}
$$

In order to obtain the distribution of runs in samples from a multinomial population, we multiply the distributions of Section 4 by

$$
P(n_i) = \begin{bmatrix} n \\ n_i \end{bmatrix} \prod_1^k p_i^{n_i}. \tag{7.16}
$$

Corresponding to (4.1) and (4.2) then, we have

$$
P(r_{ij}, n_i) = \prod_{i=1}^k \begin{bmatrix} r_i \\ r_{ij} \end{bmatrix} F(r_i) \prod_1^k p_i^{n_i} \tag{7.17}
$$

$$
P(r_i, n_i) = \prod_1^k \binom{n_i - 1}{r_i - 1} F(r_i) \prod_1^k p_i^{n_i}. \tag{7.18}
$$

In (7.17) r_{ij} is the number of runs of length j of elements with probability p_i. In (7.18) r_i is the total number of runs of elements with probability p_i. As before, we shall investigate in detail only the distribution (7.18). The moments of n_i and r_i follow at once from (7.8) and (4.5)

$$E\left(\prod_1^k (n_i^{(a_i)} u_i^{(b_i)})\right) = \sum_{n_i} \prod_1^k (n_i^{(a_i)}(n_i - 1)^{(b_i)}) \begin{bmatrix} n - \Sigma b_i \\ n_i - b_i \end{bmatrix} \prod_1^k p_i^{n_i} \tag{7.19}$$

where $u_i = n_i - r_i$. The means, variances and covariances of the r_i are

$$\begin{aligned} E(r_i) &= np_i(1 - p_i) + p_i^2, \\ \sigma_{r_i r_j} &= -np_i p_j(1 - 2p_i - 2p_j + 3p_i p_j) - p_i p_j(2p_i + 2p_j - 5p_i p_j), \\ \sigma_{r_i r_i} &= np_i(1 - 4p_i + 6p_i^2 - 3p_i^3) + p_i^2(3 - 8p_i + 5p_i^2). \end{aligned} \tag{7.20}$$

8. **Asymptotic distributions from binomial population.** We turn our attention first to the distribution (7.7) and state a theorem analogous to Theorem 2 of Section 5.

THEOREM 1. *The variables*

$$\begin{aligned} u_i = x_i &= \frac{s_{1i} - np_1^i p_2^2}{\sqrt{n}}, & i &= 1, \cdots, k - 1, \\ u_k = x_k &= \frac{s_{1k} - np_1^k p_2}{\sqrt{n}}, \\ u_{k+i} = y_i &= \frac{s_{2i} - np_1^2 p_2^i}{\sqrt{n}}, & i &= 1, \cdots, h - 1, \\ u_{k+h} = z &= \frac{n_1 - np_1}{\sqrt{n}}, \end{aligned} \tag{8.1}$$

are asymptotically normally distributed with zero means and variances and covariances

$$\begin{aligned} \sigma_{x_i x_i} &= p_1^i p_2^2 - (2i + 1)p_1^{2i} p_2^4 + 2p_1^{2i+1} p_2^3, \\ \sigma_{x_i x_j} &= -(i + j + 1)p_1^{i+j} p_2^4 + 2p_1^{i+j+1} p_2^3, \\ \sigma_{x_i x_k} &= -(i + k + 1)p_1^{i+k} p_2^3 + p_1^{i+k+1} p_2^2, \\ \sigma_{x_k x_k} &= p_1^k p_2 - (2k + 1)p_1^{2k} p_2^2, \\ \sigma_{y_i y_j} &= -(i + j + 1)p_1^4 p_2^{i+j} + 2p_1^3 p_2^{i+j+1}, \\ \sigma_{y_i y_i} &= p_1^2 p_2^i - (2i + 1)p_1^4 p_2^{2i} + 2p_1^3 p_2^{2i+1}, \\ \sigma_{x_i y_j} &= -(i + j + 3)p_2^{i+2} p_2^{j+2} + 2p_1^{i+1} p_2^{j+1}, \\ \sigma_{x_k y_j} &= -(k + j + 2)p_1^{k+2} p_2^{j+1} + p_1^{k+1} p_2^j(1 + p_2), \\ \sigma_{x_i z} &= ip_1^i p_2^3 + p_1^{i+1} p_2(1 - 4p_2), \\ \sigma_{x_k z} &= (k + 1)p_1^k p_2^2 - p_1^k(1 + p_2), \\ \sigma_{y_i z} &= ip_1^3 p_2^i + p_1 p_2^{i+1}(1 - 4p_1), \\ \sigma_{zz} &= p_1 p_2. \end{aligned} \tag{8.2}$$

We have taken s_{2k} and n_2 to be the dependent variables of (7.7). The method of proof of this theorem is the same as that of Theorem 1 in Section 5, and will be omitted. As consequences of the theorem we have

COROLLARY 1. *The variable*

$$Q = \sum_{1}^{k+h} \sigma^{ij} u_i u_j$$

is asymptotically distributed according to the χ^2*-law with* $k + h$ *degrees of freedom.*

COROLLARY 2. *Any subset* $u_{i_1}, u_{i_2}, \cdots, u_{i_m}$ *of the variables* (8.1) *is asymptotically normally distributed with zero means and variances and covariances* $\| \sigma_{i_j i_k} \|$, *and*

$$Q = \sum_{j,k=1}^{m} \sigma^{i_j i_k} u_{i_j} u_{i_k}$$

is asymptotically distributed according to the χ^2*-law with* m *degrees of freedom.* $\| \sigma^{i_j i_k} \|$ is the inverse of $\| \sigma_{i_j i_k} \|$.

COROLLARY 3. *If* $s_i = s_{1i} + s_{2i}$ *represents the total number of runs of length* i *of both kinds of elements, and* s_k *the number of runs of length greater than* $k - 1$, *then the variables*

$$\text{(8.3)} \qquad \begin{aligned} x_i &= \frac{s_i - n(p_1^i p_2^2 + p_1^2 p_2^i)}{\sqrt{n}}, \qquad i = 1, \cdots, k-1, \\ x_k &= \frac{s_k - n(p_1^k p_2 + p_1 p_2^k)}{\sqrt{n}}, \end{aligned}$$

are asymptotically normally distributed with zero means and variances and covariances

$$\text{(8.4)} \qquad \sigma_{ij} = \sigma_{x_i x_j} + \sigma_{x_i y_j} + \sigma_{x_j y_i} + \sigma_{y_i y_j}$$

where the terms on the right of (8.4) *are defined by* (8.2). We have put $h = k$ in Theorem 1 to obtain this result.

COROLLARY 4. *The variable*

$$\text{(8.5)} \qquad Q = \sum_{1}^{k} \sigma^{ij} x_i x_j$$

where the x_i *are defined by* (8.3) *and* $\| \sigma^{ij} \|$ *is the inverse of* (8.4), *is asymptotically distributed according to the* χ^2*-law with* k *degrees of freedom.*

COROLLARY 5. *If* r *denotes the total number of runs of both kinds of elements, then*

$$\text{(8.6)} \qquad x = \frac{r - 2np_1p_2}{2\sqrt{np_1p_2(1 - 3p_1p_2)}}$$

is asymptotically normally distributed with zero mean and unit variance. This is the result obtained by Wishart and Hirshfeld [11].

9. Asymptotic distributions from the multinomial population. In this section we assume $k > 2$ to avoid degenerate distributions. Because of the function $F(r_i)$ in (7.18) we do not investigate this distribution directly, but derive a more general asymptotic distribution as was done in Section 6. We consider the distribution

$$D(m_{ij}, n_i) = \prod_{i=1}^{k} \left(\begin{bmatrix} n_i \\ m_{ij} \end{bmatrix} p_i^{n_i} \right) \tag{9.1}$$

corresponding to (6.9). This is derived from (7.19) in the same manner as (6.9) was from (4.5). As before, we have replaced the numbers $n_i - 1$ in (7.19) by n_i, an unessential change as far as the asymptotic theory is concerned. We recall that

$$r_i = n_i - m_{ii} \tag{9.2}$$

hence we need only show that the variables on the right are asymptotically normally distributed in order to have the same result for the r_i. Corresponding to Theorem 2 of Section 6, we state

THEOREM 1. *The variables*

$$\begin{aligned} x_{ij} - \frac{m_{ij} - np_i p_j}{\sqrt{n}} \qquad & i, j = 1, \cdots, k - 1, \\ x_i = \frac{n_i - np_i}{\sqrt{n}} \qquad & i = 1, \cdots, k - 1 \end{aligned} \tag{9.3}$$

are asymptotically normally distributed with zero means and variances and covariances

$$\begin{aligned} \sigma_{ij,st} &= -3p_i p_j p_s p_t, & \sigma_{ij,it} &= -3p_i^2 p_j p_t, \\ \sigma_{ii,st} &= -3p_i^2 p_s p_t, & \sigma_{ii,ij} &= p_i^2 p_j (1 - 3p_i), \\ \sigma_{ij,ij} &= p_i p_j (1 - 3p_i p_j), & \sigma_{ii,ii} &= p_i^2 (1 + 2p_i - 3p_i^2), \\ \sigma_{ii,jj} &= -3p_i^2 p_j^2, & \sigma_{ij,s} &= -2p_i p_j p_s, \\ \sigma_{ii,s} &= -2p_i^2 p_s, & \sigma_{ij,i} &= p_i p_j (1 - 2p_i), \\ \sigma_{ii,i} &= 2p_i^2 (1 - p_i), & \sigma_{i,j} &= -p_i p_j, \\ & & \sigma_{i,i} = p_i (1 - p_i). & \end{aligned} \tag{9.4}$$

In these relations the symbols are defined by

$$\sigma_{ij,st} = \sigma_{x_{ij} x_{st}}, \qquad \sigma_{ij,s} = \sigma_{x_{ij} x_s}, \qquad \sigma_{i,j} = \sigma_{x_i x_j}$$

and different literal subscripts represent different numerical subscripts. These moments have been computed by means of the identity (6.12). The proof of the theorem is like that of Theorem 2 of Section 6 and will be omitted. We can now give the limiting form of the distribution of the r_i in (7.18) as

COROLLARY 1. *The variables*

$$x_i = \frac{r_i - np_i(1 - p_i)}{\sqrt{n}} \qquad i = 1, 2, \cdots, k \tag{9.5}$$

are asymptotically normally distributed with zero means and variances and covariances

$$\begin{aligned} \sigma_{ii} &= p_i(1 - p_i) - 3p_i^2(1 - p_i)^2, \\ \sigma_{ij} &= -p_ip_j(1 - 2p_i - 2p_j + 3p_ip_j). \end{aligned} \tag{9.6}$$

These limiting moments follow at once from equations (7.20).

COROLLARY 2. *The variable*

$$Q = \sum_1^k \sigma^{ij} x_i x_j$$

where the x_i are defined by (9.5) *and* $\| \sigma^{ij} \|$ *is the inverse of* (9.6), *is asymptotically distributed according to the χ^2-law with k degrees of freedom.*

COROLLARY 3. *If $r = \Sigma r_i$ denotes the total number of runs, then*

$$x = \frac{r - n(1 - \Sigma p_i^2)}{\sqrt{n}}$$

is asymptotically normally distributed with zero mean and variance

$$\sigma^2 = \Sigma p_i^2 + 2\Sigma p_i^3 - 3(\Sigma p_i^2)^2.$$

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