the test? For alternative hypothesis H_4 ($p_1 = .75$), and for $P > .90$, referring to Table I, it is seen that 42 paired samples must be employed and not more than 14 may be placed incorrectly. Under the same alternative hypothesis, if it be required merely that $P > .50$ (i.e., an expert with an ability of .75 have better than an even chance of passing), then only 18 paired samples are necessary and not more than 4 may be arranged incorrectly.

Thus, before conducting an experiment in which the Sign Test is to be employed, if the experimenter first decides what power the test must have relative to a certain alternative hypothesis; then from the accompanying table he may learn the minimum number of paired samples that are necessary; and the related maximum value of r.

If this procedure is not followed, and an experimenter employs, say 6 paired samples, he may (as can be seen from the table) discover, to his dismay, that "experts" of ability .75 will be unrecognized more than 80% of the time.

MOMENTS OF THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE MEAN SQUARE DIFFERENCE IN SAMPLES FROM A NORMAL UNIVERSE

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The following result may have considerable application to trend analysis. The specific problem was proposed to me by R. H. Kent.

Consider a sample $0_n : X_1, X_2, \cdots, X_n$ from a normal population with zero mean and variance σ^2, the variates being arranged in temporal order. We seek the moments of the ratio of δ^2 to S^2, where

$$(1) \qquad (n-1)\delta^2 = \sum_{j=1}^{n-1} (X_j - X_{j+1})^2$$

and

$$(2) \qquad nS^2 = \sum_{j=1}^{n} (X_j - \bar{X})^2.$$

Here $\bar{X}$ is the mean of the X_j. In order to simplify the algebra, we will work with quantities A and B defined by

$$(3) \qquad \begin{aligned} 2\sigma^2 A &= (n-1)\delta^2, \\ 2\sigma^2 B &= nS^2. \end{aligned}$$

The characteristic function for the joint distribution of A and B is

$$(4) \qquad \begin{aligned} \varphi(t_1, t_2) &= E(e^{At_1 + Bt_2}) \\ &= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \int\int_{-\infty}^{\infty} \cdots \int \exp\left(At_1 + Bt_2 - \frac{1}{2\sigma^2}\sum_{j=1}^{n} X_j^2\right) \prod_{j=1}^{n} dX_j, \end{aligned}$$

where t_1 and t_2 are pure imaginaries. For the method of analysis which will be used here t_1 and t_2 will be considered as real variables. By straight forward methods we have

$$(5) \qquad \varphi^{-2}(t_1, t_2) = \begin{vmatrix} a & b & d & \cdot & \cdot & \cdot & \cdot & \cdot & d \\ b & c & b & d & & & & & \cdot \\ d & b & c & b & d & & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & & & d & b & c & b & d \\ \cdot & & & & & d & b & c & b \\ d & \cdot & \cdot & \cdot & \cdot & \cdot & d & b & a \end{vmatrix}$$

where the determinant is of nth order and its elements are

$$(6) \qquad \begin{aligned} a &= 1 - t_1 - (n - 1)T \\ b &= t_1 + T \\ c &= 1 - 2t_1 - (n - 1)T \\ d &= T = t_2/n. \end{aligned}$$

It can be verified that the determinant has the value

$$(7) \qquad \varphi^{-2}(t_1, t_2) = \sum_{j=0}^{n-1} \binom{2n - j - 1}{j} (-t_1)^j (1 - t_2)^{n-j-1},$$

where the symbol $\binom{2n - j - 1}{j}$ represents a binomial coefficient. From (7) we find the moments m_j of A/B as follows: Setting

$$(8) \qquad t_2 = \sum_{k=1}^{j} t_{2k},$$

we have

$$(9) \qquad \begin{aligned} m_j &= \int\int_{-\infty}^{0} \cdots \int \left. \frac{\partial^j \varphi(t_1, t_2)}{\partial t_1^j} \right|_{t_1=0} \prod_{k=1}^{j} dt_{2k} \\ &= \frac{2^j \left[\dfrac{d^j}{dt_1^j} \varphi(t_1, 0) \right]_{t_1=0}}{(n - 1)(n + 1) \cdots (n + 2j - 3)}. \end{aligned}$$

The result is rather unexpected, for we have established that the moments of A/B are equal to the moments of A divided by the moments of B.

We find the following explicit values for the first few moments m_j:

$$
\begin{aligned}
&m_0 = 1 \\
&m_1 = 2 \\
&(n-1)(n+1)m_2 = 4(n^2+n-3) \qquad (10)\\
&(n-1)(n+1)(n+3)m_3 = 8(n^3+6n^2+2n-21) \\
&(n-1)(n+1)(n+3)(n+5)m_4 = 16(n^4+14n^3+53n^2-8n-231).
\end{aligned}
$$

These are valid subject to the restriction $2n - 1 \geq j$, because in arriving at the explicit forms we have treated the binomial coefficient $\binom{k}{j}$ as if it were identically equal to $k(k-1) \cdots (k-j+1)/j!$.

From (10) it is easy to pass to the moments of $R = \delta^2/S^2$. For example, we find the mean value and variance of R to be

$$\frac{2n}{n-1}$$

and

$$\frac{4n^2(n-2)}{(n+1)(n-1)^2}$$

respectively.