Values of $P(t^2 > t_0^2)$

$N_1 = N_2 = 10$

α \ μ/σ	0.1	0.2	0.5
0.05	0.0708	0.1355	0.5621
0.01	0.0165	0.0396	0.2940

$N_1 = N_2 = 20$

α \ μ/σ	0.1	0.2	0.5
0.05	0.0947	0.2345	0.8691
0.01	0.0251	0.0862	0.6730

In only one case was it necessary to calculate as many as ten terms of the corresponding series to obtain these values.

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[3] J. Neyman and B. Tokarska, "Errors of the second kind in testing 'Student's' hypothesis," *Amer. Stat. Assn. Jour.*, Vol. 31 (1936), pp. 318–326.
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NOTE ON AN APPLICATION OF RUNS TO QUALITY CONTROL CHARTS

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In the application of statistical methods to quality control work, a customary procedure is to construct a control chart with control limits spaced about the mean such that under conditions of statistical control, or random sampling, the probability of an observation falling outside these limits is a given α (e.g., .05). The occurrence of a point outside these limits is taken as an indication of the presence of assignable causes of variation in the production line. Such a form of chart has been found to be of particular value in the detection of the presence of assignable causes of variability in the quality of manufactured product. As recently pointed out, however, the statistician may not only help to detect the presence of assignable causes, but also help to discover the causes themselves in the course of further research and development. For this purpose, runs of different kinds and of different lengths have been found useful by industrial statisticians.[1] Quality control engineers have found, at least in research and development work, that a convenient indication of lack of control is the occurrence of long runs of observations whose values lie above or below that of the median of the sample. For example (as will be shown below), at least one succession of 9 or more observations above or below the median in a sample of 40 would be taken as evidence of lack of control at the .05 level; meaning that under conditions of control such a run would occur in approximately 5 per cent of the samples. Since this type of test has been found useful by quality control engineers, it is perhaps desirable to discuss the mathematical basis of such tests of control and provide a brief table for samples of various sizes at the significance levels .05 and .01.

The general distribution theory of runs of k kinds of elements, and in particular that of two kinds has been thoroughly investigated by A. M. Mood.[2] The purpose of this note is to give an application of the general method to quality control.

Let us consider a sample of size $2n$ drawn from a continuous distribution function $f(x)$. These are then arranged in the order in which they were drawn. We now separate the sample into two sets by considering the nth and $(n + 1)$st elements in order of magnitude, then if $x_i \leqq x_n$, x_i will be called an a, and if $x_i \geqq x_{n+1}$, x_i will be called a b. A run of a's will be defined as usual as a succession of a's terminated at each end by the occurrence of a b (with the obvious exceptions where the run includes the first or last element of the sample), and

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[1] The use of "runs up" and "runs down" as well as runs above and below the arithmetic mean of a sample were briefly described in a paper by W. A. Shewhart, "Contribution of statistics to the science of engineering," before the Bicentennial Celebration of the University of Pennsylvania, September 17, 1940, to be published in the proceedings of that meeting. In a paper, "Mathematical statistics in mass production," presented before the American Mathematical Society in February, 1941, Shewhart discussed some of the advantages of using runs above and below the median and showed how by comparing runs of different types in a given problem it is often possible to fix rather definitely the source of trouble. The present note considers only the frequency of occurrence of "long" runs which are often used by research and development engineers to indicate the presence of assignable causes of variation. The occurrence of more than one such run in a given sequence, if distributed above and below the median value may also constitute valid evidence of the presence of more than one state of statistical control between which the phenomena may oscillate. The interpretation of long runs in this sense, however, is not considered in the present note.

[2] A. M. Mood, "The distribution theory of runs," *Annals of Math. Stat.*, Vol. 11 (1940), pp. 367–392.

runs of b's are defined similarly. A run of a's may conveniently be called a run "below the median," and a run of b's a run "above the median."

We shall use Mood's notation throughout, i.e., r_{1i}, r_{2i}, $(i = 1, 2, \cdots, n)$ are the number of runs of a's and b's respectively of length i, and r_1, r_2 are the total number of runs of a's and b's; $\begin{bmatrix} m \\ m_i \end{bmatrix}$ will indicate a multinomial coefficient, and $\binom{n}{k}$ a binomial coefficient. Also we define

$$\begin{aligned} F(r_1, r_2) &= 0, & |r_1 - r_2| &> 1, \\ F(r_1, r_2) &= 1, & |r_1 - r_2| &= 1, \\ F(r_1, r_2) &= 2, & |r_1 - r_2| &= 0. \end{aligned}$$

Then the distribution of runs of a's for our case is

$$(1) \qquad P(r_{1i}) = \frac{\begin{bmatrix} r_1 \\ r_{1i} \end{bmatrix} \binom{n+1}{r_1}}{\binom{2n}{n}}.$$

We would like to find the probability of at least one run of s or more a's. The coefficient of x^n in

$$(2) \qquad [x + x^2 + \cdots + x^{s-1}]^{r_1},$$

gives the number of ways of partitioning n elements into r_1 partitions such that no partition contains s or more elements, and none is void. Rewriting (2) we have

$$x^{r_1}[(1 - x^{s-1})]^{r_1} \sum_{t=0}^{\infty} \binom{r_1 - 1 + t}{r_1 - 1} x^t,$$

and the coefficient of x^n is just

$$(3) \qquad \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} \binom{n - j(s-1) - 1}{r_1 - 1}.$$

Then the probability that we desire, of getting at least one run of s or more a's is immediately given by

$$P(r_{1i} \geqq 1, i \geqq s)$$

$$= \frac{\sum_{r_1=1}^{n-s+1} \left[\binom{n-1}{r_1 - 1} - \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} \binom{n - 1 - j(s-1)}{r_1 - 1} \right] \binom{n+1}{r_1}}{\binom{2n}{n}}.$$

Noting that when $j = 0$ in the inner summation we have just the total number of partitions, we get finally

$$(4)\qquad P(r_{1i} \geqq 1, i \geqq s) = \frac{\sum_{r_1=1}^{n-s+1} \binom{n+1}{r_1} \sum_{j=1}^{r_1} (-1)^{j+1} \binom{r_1}{j} \binom{n-1-j(s-1)}{r_1-1}}{\binom{2n}{n}}.$$

A similar result of course holds for the b's.

If we desire the probability of getting at least one run of s or more of either a's or b's, we compute the probability of getting no runs of this type and subtract from unity. Expression (3) multiplied by the total number of ways of getting no partitions of s or more b's for a given r_1, and then summed on r_1 gives exactly the number of ways of getting no runs of either a's or b's as great as s. This is

$$(5)\qquad \begin{aligned} A = \sum_{r_1 > n/s} &\left[\sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} \binom{n-1-j(s-1)}{r_1-1} \right] \\ &\cdot \left[\sum_{r_2=r_1-1}^{r_1+1} F(r_1, r_2) \sum_{i=0}^{r_2} (-1)^i \binom{r_2}{i} \binom{n-1-i(s-1)}{r_2-1} \right], \end{aligned}$$

and the probability desired is

$$(6)\qquad P(r_{1i} \geqq 1 \text{ or } r_{2i} \geqq 1 \text{ or both}; i \geqq s) = 1 - A \bigg/ \binom{2n}{n}.$$

In spite of the complex appearance of A, the sum can be rapidly calculated for any given s, n since the calculations for the sums on i and j need not be duplicated.

In the case of a quality control chart, we set a significance level α for a given n, this determines s the length of run of either type necessary for significance at the level chosen. Suppose we are interested only in runs occurring on *one side* of the median, say above, when $\alpha = .05$, $n = 20$ (i.e., sample size equal to 40). We determine the least value of s which will make the right hand side of equation (4) less than or equal to .05. It turns out that $s = 8$ for this case. This means that under conditions of statistical control, i.e., random sampling, one or more runs of length 8 or more, above the median will occur in approximately 5 per cent of samples of size 40. Naturally an identical result holds when we are considering only runs below the median.

On the other hand, if under the same conditions as given above ($n = 20$, $\alpha = .05$), we are using as our criterion of statistical control the occurrence of runs of length s or greater *either* above or below the median, we must determine the least value of s such that $1 - A/\binom{2n}{n} \leq .05$. This value turns out to be 9. In other words under conditions of statistical control at least one run of at least 9 will occur *either* above or below the median in less than 5 per cent of the cases on the average.

The following table gives smallest lengths of runs for .05 and .01 significance levels for samples of size 10, 20, 30, 40, 50.

	Runs on one side of median		Runs on either side of median	
$2n$	$\alpha = .05$	$\alpha = .01$	$\alpha = .05$	$\alpha = .01$
10	5	—	5	—
20	7	8	7	8
30	8	9	8	9
40	8	9	9	10
50	8	10	10	11

If there is an odd number of individuals, say $2n + 1$, in the sample, we would choose the value of the median as the dividing line for our sample and treat the data as if there were only $2n$ cases, thus ignoring the median completely.

The following table[3] gives the probabilities of getting at least one run of s or more on *one* side, *either* side, and *each* side of the median for samples of size 10, 20, and 40.

Length of Run (s)	$2n = 10$ One Side	$2n = 10$ Either Side	$2n = 10$ Each Side	$2n = 20$ One Side	$2n = 20$ Either Side	$2n = 20$ Each Side	$2n = 40$ One Side	$2n = 40$ Either Side	$2n = 40$ Each Side
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	.976	.992	.960	1.000	1.000	1.000	1.000	1.000	1.000
3	.500	.667	.333	.870	.956	.784	.992	.999	.986
4	.143	.230	.056	.457	.640	.274	.799	.930	.668
5	.024	.040	.008	.178	.293	.064	.450	.650	.249
6				.060	.106	.013	.207	.346	.068
7				.017	.032	.002	.087	.158	.016
8				.004	.007	.000	.034	.065	.004
9				.001	.001	.000	.013	.025	.001
10				.000	.000	.000	.005	.009	.000
11							.002	.003	.000
12							.000	.001	.000
13							.000	.000	.000

One method of computing such a table is to use expression (4) to obtain the probabilities on one side, and to use (6) to get probabilities for either side. Then the probabilities for runs on each side may be computed by using the relationship

$$2P \text{ (one side)} - P \text{ (either side)} = P \text{ (each side)}.$$

[3]The author is indebted to Dr. P. S. Olmstead of the Bell Telephone Laboratories for kindly placing this table at his disposal. Dr. Olmstead has pointed out that these probabilities have been found very useful in research and development work.