NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

NOTE ON THE DISTRIBUTION OF NON-CENTRAL t WITH AN APPLICATION

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If we adopt the notation recently used by N. L. Johnson and B. L. Welch [1], non-central t is defined by

$$t = \frac{z + \delta}{\sqrt{w}},$$

in which δ is a constant and z and w are independent variables, z being distributed normally about zero with unit variance and w being distributed as χ^2/f in which f is the number of degrees of freedom for χ^2.

In the paper referred to Johnson and Welch discuss some applications of non-central t and give suitable tables calculated from the probability integral of the distribution of this variable. Previously tables of this probability integral for the purpose of calculating the power of the t test had been given by J. Neyman [2] and Neyman and B. Tokarska [3].

It is the purpose of this note to call attention to a series expansion for the probability integral of non-central t which is simple in form and in most cases convenient for direct calculation. As an application of some intrinsic interest this series is used to compute in several numerical cases the power of a test proposed by E. J. G. Pitman [4] based on the randomization principle.

If for convenience we write,

$$\sqrt{w} = \psi, \ (0 \leq \psi \leq \infty),$$

we have for the joint distribution of $z + \delta$ and ψ,

$$df(z + \delta, \psi) = \frac{2(f/2)^{f/2}}{\sqrt{2\pi}\,\Gamma(f/2)} e^{-\frac{1}{2}(f\psi^2 + z^2)} \psi^{f-1}\, d\psi\, dz. \tag{1}$$

From this

$$\begin{aligned} df(t, \psi) &= \frac{2(f/2)^{f/2} e^{-\delta^2/2}}{\sqrt{2\pi}\,\Gamma(f/2)} e^{-\psi^2(f+t^2)/2 + \delta\psi t} \psi^f\, d\psi\, dt \\ &= \frac{2(f/2)^{f/2} e^{-\delta^2/2}}{\sqrt{2\pi}\,\Gamma(f/2)} e^{-\psi^2(f+t^2)/2} \sum_{r=0}^{\infty} \frac{(\delta t)^r}{r!} \psi^{f+r}\, d\psi\, dt, \end{aligned} \tag{2}$$

Now this series can be integrated term by term with respect to ψ over its range and we have,

$$(3)\qquad df(t) = \frac{(f/2)^{f/2}e^{-\delta^2/2}}{\sqrt{2\pi}\,\Gamma(f/2)}\sum_{r=0}^{\infty}\frac{\Gamma[\frac{1}{2}(f+r+1)]}{r!}(\delta t)^r\left(\frac{2}{f+t^2}\right)^{\frac{1}{2}(f+r+1)}dt.$$

This series converges uniformly in any finite interval for t and it may be integrated term by term over the entire range for t or over any part of it. In particular, after some reduction, we get,

$$(4)\qquad \begin{aligned} P(0 \leq t \leq t_0 \mid f, \delta) &= \int_0^{t_0} df(t) \\ &= \frac{e^{-\delta^2/2}}{2}\sum_{r=0}^{\infty}\frac{(\delta^2/2)^{r/2}}{\Gamma(r/2+1)}I\left((r+1)/2, f/2; \frac{t_0^2}{f+t_0^2}\right), \end{aligned}$$

in which $I\left((r+1)/2, f/2; \frac{t_0^2}{f+t_0^2}\right)$ is the incomplete Beta-function in the notation of Karl Pearson. Often what is wanted is

$$(5)\qquad P(-t_0 \leq t \leq t_0) = e^{-\delta^2/2}\sum_{r=0}^{\infty}\frac{(\delta^2/2)^r}{r!}I\left((r+1)/2, f/2; \frac{t_0^2}{f+t_0^2}\right).$$

Since the incomplete Beta-function is numerically less than unity it is seen that the series (4) or (5) converges rapidly for moderate values of δ such as will ordinarily occur in applications for small samples. The use of Pearson's tables of $I(p, q; x)$ will be convenient since interpolation will be required for only one of the three arguments.

As an application let us consider the test proposed by Pitman in the paper referred to above. Two independent samples, $x_1, x_2, \cdots, x_{N_1}$, and $y_1, y_2, \cdots, y_{N_2}$, have been drawn and it is desired in the absence of any information about the two populations from which the samples came to test the hypothesis that they have equal means. A test based on what may be termed the principle of randomization for this situation has been discussed by R. A. Fisher [5] and by E. S. Pearson [6]. It is as follows: Let the combined sample of $N_1 + N_2$ observations be separated into sets of N_1 observations, $u_1, u_2, \cdots, u_{N_1}$, and N_2 observations, $v_1, v_2, \cdots, v_{N_2}$, in all possible ways. For each such separation let the numerical difference of the means, $|\bar{u} - \bar{v}|$, be the spread. Then for a suitably chosen $\delta > 0$, we will reject the hypothesis of equal means if fewer than $100\alpha\%$ of the ${}_{N_1+N_2}C_{N_1}$ spreads exceed $|\bar{x} - \bar{y}|$, and otherwise not. It is clear that this test is fiducially valid independently of the populations actually sampled in the sense that if it be consistently followed for all such samples, the proportion of cases when the hypothesis is rejected when it is true will statistically approach α.

For all but very small samples it is very tedious to calculate the ${}_{N_1+N_2}C_{N_1}$

spreads and Pitman in his discussion shows that for quite moderate values of N_1 and N_2 the quantity,

$$w = \frac{\frac{N_1N_2}{(N_1+N_2)^2}(\bar{u}-\bar{v})^2}{\frac{\Sigma(x-\bar{x})^2+\Sigma(y-\bar{y})^2}{N_1+N_2}+\frac{N_1N_2}{(N_1+N_2)^2}(\bar{u}-\bar{v})^2} = \frac{\zeta^2}{\xi^2+\zeta^2}$$

has a distribution which in all but very exceptional cases is quite well approximated by a $B(\frac{1}{2}, \frac{1}{2}(N_1 + N_2 - 2))$-function. That is, the distribution of w for the ${}_{N_1+N_2}C_{N_1}$ spreads may for practical purposes be found from that of t, by a simple transformation, with $N_1 + N_2 - 2$ degrees of freedom.

It seems pertinent to make some inquiry into the power of such a test, that is, to make an attempt to learn something about the probability that such a test will fail to reject the hypothesis of equal means when it is in fact false. To do this it is now necessary to specify the populations which have actually been sampled. If we suppose that these populations are normal with equal variances but with unequal means which, with no loss of generality, may be taken to be μ and $-\mu$ respectively, the probability integral of the distribution of non-central t will give our answer.

If we set

$$\frac{t^2}{f+t^2} = \frac{\zeta^2}{\xi^2+\zeta^2},$$

we have

$$t = \sqrt{f}\,\zeta/\xi.$$

Also,

$$\xi^2 = \frac{(N_1-1)s_1^2+(N_2-1)s_2^2}{N_1+N_2-2}\cdot\frac{N_1+N_2-2}{N_1+N_2} = \frac{f}{N_1+N_2}s^2,$$

in which s^2 is the usual estimate of the population variance σ^2 based on $f = N_1 + N_2 - 2$ degrees of freedom. Then

$$t = \frac{\bar{u}-\bar{v}}{s}\sqrt{\frac{N_1N_2}{N_1+N_2}}$$

and this is a central t if $\mu = -\mu = 0$, otherwise it is non-central. In the latter case we write (the test is made on $\bar{x} - \bar{y}$),

$$t = \frac{(\bar{x}-\mu)-(\bar{y}+\mu)+2\mu}{s}\sqrt{\frac{N_1N_2}{N_1+N_2}}$$

$$= \frac{z+\delta}{\psi},$$

in which,

$$z = \frac{(\bar{x} - \mu) - (\bar{y} + \mu)}{\sigma} \sqrt{\frac{N_1 N_2}{N_1 + N_2}},$$

$$\psi = s/\sigma,$$

and

$$\delta = \frac{2\mu}{\sigma} \sqrt{\frac{N_1 N_2}{N_1 + N_2}}.$$

In applying Pitman's test for a given significance level α, one determines whether or not

$$P(w > w_0) \gtrless \alpha,$$

w_0 being the value of w calculated from the sample. This is equivalent to finding

$$P(t^2 > t_0^2),$$

for the proper f, in which

$$\frac{t_0^2}{f + t_0^2} = w_0$$

and this can be found from an ordinary table of the probability integral of the t-distribution.

For a numerical example let $N_1 = N_2 = 10$ so that $f = 18$. If we adopt a 5% significance level we have $t_0^2 = 2.101^2$ for the critical value. Let us suppose that $\mu/\sigma = 0.1$, and calculate the probability that the hypothesis that $\mu = 0$ will be rejected. We have $\delta = 0.1$ and

$$\frac{t_0^2}{f + t_0^2} = 0.1969.$$

Then

$$P(t^2 \leq t_0^2) = e^{-0.1}[I(0.5, 9; 0.1969) + 0.1\, I(1.5, 9; 0.1969) + \frac{0.01}{2!} I(2.5, 9; 0.1969) + \cdots]$$

$$= 0.9292.$$

Four terms of the series were enough to give this result. The probability of rejecting the hypothesis in this case is thus 0.0708.

The following tables show results for $\alpha = 0.05$ and 0.01, $\mu/\sigma = 0.1, 0.2$, and 0.5, and $N_1 = N_2 = 10$ and 20.

Values of $P(t^2 > t_0^2)$

$N_1 = N_2 = 10$

α \ μ/σ	0.1	0.2	0.5
0.05	0.0708	0.1355	0.5621
0.01	0.0165	0.0396	0.2940

$N_1 = N_2 = 20$

α \ μ/σ	0.1	0.2	0.5
0.05	0.0947	0.2345	0.8691
0.01	0.0251	0.0862	0.6730

In only one case was it necessary to calculate as many as ten terms of the corresponding series to obtain these values.

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NOTE ON AN APPLICATION OF RUNS TO QUALITY CONTROL CHARTS

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In the application of statistical methods to quality control work, a customary procedure is to construct a control chart with control limits spaced about the mean such that under conditions of statistical control, or random sampling, the probability of an observation falling outside these limits is a given α (e.g., .05). The occurrence of a point outside these limits is taken as an indication of the presence of assignable causes of variation in the production line. Such a form