CONTINUED FRACTIONS FOR THE INCOMPLETE BETA FUNCTION[1]

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1. Introduction. Existing literature on the problem of calculating the incomplete Beta function

$$\text{B}_x(p, q) = \int_0^x x^{p-1}(1-x)^{q-1}\, dx, \qquad 0 < x < 1, p > 0, q > 0, \tag{1.1}$$

and the levels of significance of Fisher's z [1] leave further work to be done. Müller's continued fraction and a new continued fraction are shown to possess complementary features covering the range of $\text{B}_x(p, q)$ for all values of x, p, q. Previous methods of computing $I_x(p, q) = \text{B}_x(p, q)/\text{B}(p, q)$ are given in [2], [5], [6], [8], [10], [13], [14], [15].

Müller's continued fraction is

$$I_x(p, q) = C\left[\frac{b_1}{1+}\,\frac{b_2}{1+}\,\frac{b_3}{1+}\,\frac{b_4}{1+}\cdots\right], \tag{1.2}$$

where

$$C = \frac{\Gamma(p+q)}{\Gamma(p+1)\Gamma(q)}\, x^p(1-x)^{q-1}, \qquad b_1 = 1, \qquad \mu_s = \frac{q-s}{p+s},$$

$$b_{2s} = -\frac{(p+s-1)(p+s)}{(p+2s-2)(p+2s-1)}\,\mu_s\,\frac{x}{1-x},$$

$$b_{2s+1} = \frac{s(p+q+s)}{(p+2s-1)(p+2s)}\,\frac{x}{1-x}.$$

A convergent infinite series $1 + \sum_{n=1}^{\infty} d_n x^n$ can be converted into an infinite continued fraction of the form $\frac{1}{1+}\,\frac{c_1 x}{1+}\,\frac{c_2 x}{1+}\cdots$ where [4], [9] p. 304,

$$\begin{aligned} &c_1 = -\beta_1, \qquad c_2 = \frac{-\beta_2}{\beta_1}, \\ &c_{2s} = \frac{-\beta_{2s-3}\beta_{2s}}{\beta_{2s-2}\beta_{2s-1}}, \qquad c_{2s+1} = \frac{-\beta_{2s-2}\beta_{2s+1}}{\beta_{2s-1}\beta_{2s}}, \qquad s > 2 \end{aligned} \tag{1.3}$$

[1] Presented at a meeting of the American Mathematical Society, October 28, 1939, New York City.

where

$$\beta_{2s} = \begin{vmatrix} 1 & d_1 & d_2 & \cdots & d_s \\ d_1 & d_2 & d_3 & \cdots & d_{s+1} \\ \vdots & & & & \vdots \\ d_s & d_{s+1} & d_{s+2} & \cdots & d_{2s} \end{vmatrix}, \qquad \beta_{2s+1} = \begin{vmatrix} d_1 & d_2 & d_3 & \cdots & d_{s+1} \\ d_2 & d_3 & d_4 & \cdots & d_{s+2} \\ \vdots & & & & \vdots \\ d_{s+1} & d_{s+2} & d_{s+3} & \cdots & d_{2s+1} \end{vmatrix},$$

$$\beta_{2s} \neq 0, \qquad \beta_{2s+1} \neq 0.$$

The infinite continued fraction found in this manner is called the corresponding continued fraction and the power series is said to be semi-normal if $\beta_{2s} \neq 0$, $\beta_{2s+1} \neq 0$.

2. A new continued fraction. Müller found his continued fraction by converting in the manner of the preceding paragraph

$$(2.1) \qquad \begin{aligned} I_x(p, q) = {} & \frac{\Gamma(p+q)x^p(1-x)^{q-1}}{\Gamma(p+1)\Gamma(q)} \\ & \cdot\left\{1 + \sum_{r=0}^{\infty} \frac{(q-1)(q-2)\cdots(q-r-1)}{(p+1)(p+2)\cdots(p+r+1)}\left(\frac{x}{1-x}\right)^{r+1}\right\}, \\ & x < \tfrac{1}{2}. \end{aligned}$$

We convert

$$(2.2) \qquad \begin{aligned} I_x(p, q) = {} & \frac{\Gamma(p+q)x^p(1-x)^{q}}{\Gamma(p+1)\Gamma(q)} \\ & \cdot\left\{1 + \sum_{r=0}^{\infty} \frac{(p+q)(p+q+1)\cdots(p+q+r)}{(p+1)(p+2)\cdots(p+r+1)} x^{r+1}\right\}, \\ & 0 < x < 1. \end{aligned}$$

Consequently

$$\beta_1 = \frac{p+q}{p+1}, \qquad \beta_2 = \frac{(p+q)(1-q)}{(p+1)^2(p+2)}, \cdots,$$

$$\beta_{2s+1} = \frac{(p+q)(p+q+1)\cdots(p+q+s-1)(p+q+s)}{(p+s+1)(p+s+2)\cdots(p+2s)(p+2s+1)}\beta_{2s},$$

$$\beta_{2s+2} = \frac{(1-q)(2-q)\cdots(s-q)(s+1-q)(s+1)!}{(p+1)(p+2)\cdots(p+2s+1)(p+2s+2)}\beta_{2s+1},$$

$$c_{2s+1} = -\frac{(p+s)(p+q+s)}{(p+2s)(p+2s+1)}, \qquad c_{2s} = \frac{s(q-s)}{(p+2s-1)(p+2s)},$$

and

$$(2.3) \qquad I_x(p, q) = \frac{\Gamma(p+q)x^p(1-x)^q}{\Gamma(p+1)\Gamma(q)}\left\{\frac{1}{1+}\,\frac{C_1}{1+}\,\frac{C_2}{1+}\cdots\right\},$$

where $C_s = c_s x$. By well known theorems due to Van Vleck [12] and Perron [9] p. 347 we find (1.2) converges for $-1 < x < \infty$, and (2.3) converges for $-\infty < x < 1$, and in the neighborhood of zero (2.2) equals (2.3). The region of equivalence of the series and the fraction may be extended by the following argument. Let the infinite series be terminated at some arbitrary point which gives the desired accuracy. Then the continued fraction of the corresponding type represents this finite series, is finite and gives the result within the desired accuracy. The new continued fraction may also be derived by use of the hypergeometric series [9] p. 348. A special case of (2.3) was given by Markoff [3], pp. 135–41, [11] pp. 53–55, who applied the result only to the binomial distribution. The associated continued fraction provides more rapid convergence than the corresponding continued fraction. The associated continued fraction is found by means of the hypergeometric series [9] p. 331, p. 348:

$$I_x(p,q) = \frac{\Gamma(p+q)x^p(1-x)^q}{\Gamma(p+1)\Gamma(q)}\left\{1 + \frac{k_1 x}{1 + l_1 x+}\,\frac{k_2 x^2}{1 + l_2 x+}\,\frac{k_3 x^2}{1 + l_3 x+}\cdots\right\}$$

$$k_1 = \frac{p+q}{p+1}, \qquad l_1 = \frac{p+q+1}{p+2},$$

$$\text{(2.4)} \qquad k_{s+1} = \frac{s(s-q)(p+s)(p+q+s)}{(p+2s-1)(p+2s)^2(p+2s+1)},$$

$$l_{s+1} = \frac{s(q-s)}{(p+2s)(p+2s+1)} - \frac{(p+s+1)(p+q+s+2)}{(p+2s+2)(p+2s+1)}, \quad s \geqq 1.$$

The disadvantage of (2.4) lies in the unwieldy form of computation. For properties of an associated continued fraction and the corresponding continued fraction in connection with convergence and the Taylor series reference is made to [9] p. 331 and pp. 302–303.

3. Properties of the corresponding continued fraction. Müller and Soper [5], [10], pointed out the inadvisability of integration through the mode $x = \dfrac{p-1}{p+q-2}$. In such cases we change $I_x(p, q)$ to $I_{1-x}(q, p)$. Müller has shown for his continued fraction that if we do not integrate through the mode (we assume this in the remainder of the paragraph) that convergents 2, 3, 6, 7, etc., will be greater than the true value and the remaining convergents will be less than the true value provided q is an integer. However, if q is not an integer, and is small ($q < 20$), it may happen that all convergents are above the true value. In such cases we may consider whether Müller's continued fraction may apply by estimating the remainder $I(p + s, q - s)$, after s reductions by parts [10].

For the new continued fraction also

$$|C_{2s}| = \left|\frac{s(q-s)}{(p+2s-1)(p+2s)}\cdot\frac{p-1}{p+q-2}\right| < 1,$$

$$|C_{2s+1}| = \left|\frac{(p+s)(p+q+s)(p-1)}{(p+2s)(p+2s+1)(p+q-2)}\right| < 1,$$

and $C_{2s+1} < 0$; $C_{2s} > 0$ unless $s > q$ when $C_{2s} < 0$. If $C_{2s} > 0$ then the convergents 2, 3, 6, 7, 10, 11, etc., will be above the true value and the other convergents will be below the true value. If $C_{2s} < 0$, then all convergents will be above the true value. In such cases, since a remainder for the continued fraction has not been found, it seems best to estimate $I_x(p + s, q - s)$ to obtain an idea of the error.

4. $I_x(p + s, q - s)$ and the equivalent continued fraction. Soper [10] has given the remainder after s reductions by raising p. This will furnish an upper bound of the error in the corresponding continued fraction after s convergents. The remainder, when $q - s$ is a negative integer, is approximately

$$I_x(p + s, q - s) = \frac{2 \sin (q - s)\pi \sqrt{\xi(\xi - 1)/2\pi(p + q)}}{\xi - x} \left\{ \left(\frac{x}{\xi}\right)^{\xi} \left(\frac{1 - x}{\xi - 1}\right)^{1-\xi} \right\}^{p+q}, \tag{4.1}$$

$$\text{where } \xi = \frac{p + s}{p + q}.$$

Another approach is to use the equivalent continued fraction, for $s - 1$ convergents of the equivalent continued fraction reproduces exactly s terms of the infinite series. The infinite series and the equivalent continued fraction for the infinite series are alike in all respects except form. By [9] p. 210, we find that the equivalent continued fraction for (2.3) is

$$W_1 = \frac{\gamma_1}{1 + \gamma_1 -} \frac{\gamma_2}{1 + \gamma_2 -} \frac{\gamma_3}{1 + \gamma_3 -} \frac{\gamma_4}{1 + \gamma_4 -} \cdots$$

where

$$\gamma_1 = \frac{p + q}{p + 1} x, \qquad \gamma_2 = \frac{p + q + 1}{p + 2} x, \qquad \gamma_3 = \frac{p + q + 2}{p + 3} x, \cdots \tag{4.2}$$

$$\gamma_r = \frac{p + q + r - 1}{p + r} x,$$

and

$$I_x(p, q) = \frac{\Gamma(p + q)x^p(1 - x)^q}{\Gamma(p + 1)\Gamma(q)} \frac{1}{1 - W_1}.$$

The equivalent continued fraction for Müller's continued fraction is given in [5], p. 292.

5. Numerical illustration. If A_v and B_v represent the numerator and the denominator of the v-th convergent of a continued fraction $\frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \frac{a_4}{b_4 +} \cdots$ then

$$\begin{aligned} A_v &= b_v A_{v-1} + a_v A_{v-2} \\ B_v &= b_v B_{v-1} + a_v B_{v-2}, \qquad v > 2. \end{aligned} \tag{5.1}$$

As an example we calculate $I_{.5}$ (2.5, 1.5), which could not be done by Müller's continued fraction.

Convergent	A	B	A/B
1	1	1	1
2	1	.42857143	2.3333333
3	1.015873016	.44444444	2.2857142
4	.66233767	.29292929	2.2610838
5	.64812966	.28671329	2.2605498
6	.46471308	.20559441	2.2603391
7	.441837914	.195475117	2.2603281
8	.33105492	.14646345	2.2603245
9	.30890766	.13666520	2.2603242
10	.23762461	.10512856	2.2603240
11	.21882154	.096809808	2.2603240

Using the value of the eleventh convergent we have, $I_{.5}$ (2.5, 1.5) = .28779339. Pearson [7], p. 30, gives .2877934 and Soper [10], p. 32 gives .28779341.

6. Discussion of the various methods. Müller's continued fraction encounters difficulties when q is small due to the possible divergence of the series on which it is based. In such cases the new continued fraction works admirably. Where "reduction by parts" [10] is advisable it would seem Müller's results will be better, while if "integration raising p" is preferable, then the new continued fraction would be necessary. The other methods suggested in the past lacked in some cases remainder terms; were in other cases too long; were feasible only in a limited range; or were only approximations. I am particularly indebted to Professor C. C. Craig under whose guidance this study was completed.

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