A CHARACTERIZATION OF THE NORMAL DISTRIBUTION

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1. In sampling from a normal population the distributions of the mean and of the variance are mutually independent. This well known property of the normal distribution is used in deriving the distribution of "Student's" ratio. The independence of the distributions of the mean and of the variance characterizes the normal distribution. To show this one has to prove the following statement:

A necessary and sufficient condition for the normality of the parent distribution is that the sampling distributions of the mean and of the variance be independent.

That this condition is necessary follows from the above mentioned property of the normal distribution; so there is only to prove that this condition is sufficient. This was first proved by R. C. Geary[1] by using some of R. A. Fisher's general formulae for the seminvariants. However, a different proof, using characteristic functions might be of some interest.

2. Let $f(x)$ be the density function of a continuous probability distribution and let $x_1, x_2, \cdots, x_n$ be n observations of the variate x. Denote by $\bar{x} = \sum_{x=1}^{n} x_\alpha/n$ the sample mean, and by

$$s^2 = \sum_{\alpha=1}^{n} (x_\alpha - \bar{x})^2/n = [(n-1)\sum_{1}^{n} x_\alpha^2 - 2\sum_{\alpha=1}^{n-1}\sum_{\beta=\alpha}^{n-1} x_\alpha x_{\beta+1}]/n^2$$

the sample variance of these observations. The characteristic function of the distribution is then given by

$$\psi(t) = \int e^{itx} f(x)\, dx. \tag{1}$$

The characteristic function of the joint distribution of the statistics $\bar{x}$ and s^2 is known to be

$$\varphi(t_1, t_2) = \int \cdots \int e^{it_1\bar{x}+it_2s^2} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n. \tag{2}$$

In the same way one obtains the characteristic function of the mean $\bar{x}$ as

$$\varphi_1(t_1) = \varphi(t_1, 0) = \int \cdots \int e^{it_1\bar{x}} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n, \tag{2a}$$

[1] R. C. Geary, "Distribution of Student's ratio for nonnormal samples," *Roy. Stat. Soc. Jour.*, Supp. Vol. 3, no. 2.

and the characteristic function of the distribution of the variance s^2

(2b) $$\varphi_2(t_2) = \varphi(0, t_2) = \int \cdots \int {}^{it_2s^2} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n .$$

The independence of the distributions of $\bar{x}$ and s^2 means in terms of the characteristic functions $\varphi(t_1, t_2) = \varphi_1(t_1)\varphi_2(t_2)$, or

(3) $$\left.\frac{\partial\varphi(t_1, t_2)}{\partial t_2}\right|_{t_2=0} = \varphi_1(t_1) \left.\frac{\partial\varphi_2(t_2)}{\partial t_2}\right|_{t_2=0}.$$

Substituting in (2a) $\bar{x} = \sum_1^n x_\alpha \Big/ n$, it is seen

$$\varphi_1(t_1) = \prod_{\alpha=1}^n \int e^{it_1x_\alpha/n} f(x_\alpha)\, dx_\alpha = \left[\int {}^{it_1x/n} f(x)\, dx\right]^n = [\psi(t_1/n)]^n,$$

therefore

(3′) $$\left.\frac{\partial\varphi}{\partial t_2}\right|_{t_2=0} = [\psi(t_1/n)]^n \left.\frac{\partial\varphi_2}{\partial t_2}\right|_{t_2=0}.$$

Differentiating (2) with respect to t_2

$$\frac{\partial\varphi(t_1 t_2)}{\partial t_2} = i \int \cdots \int s^2 e^{it_1\bar{x}+it_2s^2} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n .$$

Substituting $s^2 = [(n-1)\sum x_\alpha^2 - 2\sum\sum x_\alpha x_{\beta+1}]/n^2$ and $\bar{x} = \sum_1^n x_\alpha/n$ we obtain easily

(4a) $$\left.\frac{\partial\varphi(t_1, t_2)}{\partial t_2}\right|_{t_2=0} = \frac{(n-1)i}{n}\left\{[\psi(t_1/n)]^{n-1} \int x^2 e^{it_1x/n} f(x)\, dx - [\psi(t_1/n)]^{n-2}\left[\int x e^{it_1x/n} f(x)\, dx\right]^2\right\}.$$

In a similar way it is seen

(4b) $$\left.\frac{\partial\varphi_2(t_2)}{\partial t_2}\right|_{t_2=0} = \frac{i(n-1)}{n}\sigma^2.$$

Here σ^2 denotes the population variance of the parent distribution. Substituting (4a) and (4b) in the relation (3′) and writing $t = t_1/n$ one has

(5) $$\psi(t) \int x^2 e^{itx} f(x)\, dx - \left[\int x e^{itx} f(x)\, dx\right]^2 = [\psi(t)]^2 \sigma^2.$$

Considering the definition (1) of the characteristic function it is seen that

(6) $$\frac{d^l \psi(t)}{dt^l} = i^l \int x^l e^{itx} f(x)\, dx.$$

The integrals on the left side of relation (5) are of this form. So one may write the relation expressing statistical independence of the sample mean and the sample variance as a differential equation for the characteristic function $\psi(t)$, namely

$$-\psi(t)\frac{d^2\psi}{dt^2} + \left(\frac{d\psi}{dt}\right)^2 = \sigma^2[\psi(t)]^2. \tag{7}$$

The initial conditions to be satisfied are

$$\psi(0) = 1, \qquad \psi'(0) = i\mu, \tag{7a}$$

where μ is the population mean of the parent distribution. Integrating this equation it is seen that the characteristic function is

$$\psi(t) = e^{i\mu t}e^{-\frac{1}{2}\sigma^2 t^2}, \tag{8}$$

which is the characteristic function of the normal distribution.

3. This reasoning applies also to the multivariate case. Let $f(x_1, x_2, \cdots, x_p)$ be the density of the p variates $x_1, x_2, \cdots, x_p$. Denote by $x_{k\alpha}$ ($k = 1, 2, \cdots, p$; $\alpha = 1, 2, \cdots, n$) the α-th observation on the k-th variate, by $\bar{x}_k$ the sample mean of this variate and by s_{kl} the sample covariance between the k-th and l-th variates. Assuming that the distribution of s_{kl} is independent of the joint distribution of the p sample means $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_p)$ one obtains the equation

$$\frac{\psi_{lm}}{\psi} - \frac{\psi_l\psi_m}{\psi^2} = -\sigma_{lm}. \tag{9}$$

Here σ_{lm} is the population covariance of the variates x_l and x_m,

$$\psi = \psi(t_1, \cdots t_p) = \int \cdots \int e^{i(t_1x_1+\cdots t_px_p)} f(x_1, \cdots, x_p)\, dx_1 \cdots dx_p,$$

denotes the characteristic function of the parent distribution and

$$\psi_l = \frac{\partial\psi}{\partial t_l}, \qquad \psi_{lm} = \frac{\partial^2\psi}{\partial t_l\,\partial t_m}.$$

If (9) holds for $l, m = 1, 2, \cdots, p$ one has a system of partial differential equations which leads to the characteristic function of the multivariate normal distribution.