of the result, but is easily shown to be the correct expression. Upon differentiating k times ($0 \leqq k \leqq m_r$) all the terms in the summation except the one corresponding to $i = r$ will contain the factor $(x - a_r)^{m_r - k + 1}$, and will therefore vanish for $x = a_r$. Moreover, the non-vanishing term, before differentiation, will agree, up to and including terms containing $(x - a_r)^{m_r}$, with the Taylor expansion of $f(x)$ in powers of $x - a_r$, since the product expression within the brackets will be exactly canceled, as far as terms of degree m_r, by the n binomial expansions. Hence the kth derivative of the non-vanishing term in the summation will be $f^{(k)}(a_r)$ for $x = a_r$. This establishes the formula.

This formula is clearly equivalent to the Newton divided difference interpolation formula with repeated arguments [1, p. 33], the argument a_i occurring $m_i + 1$ times. Therefore, if $f(x)$ is any function other than a polynomial of degree N or less, it is necessary to add a remainder term [1, pp. 22–23] of the form

$$f_N(x) \prod_{i=0}^{n} (x - a_i)^{m_i+1},$$

where $f_N(x)$ denotes the limiting value [1, pp. 20–21] of the divided difference of order N involving the arguments $x, a_0, a_1, \cdots, a_n$, with each argument a_i appearing $m_i + 1$ times. The existence of all the indicated derivatives is, of course, essential.

REFERENCES

[1] J. F. Steffensen, *Interpolation*, Baltimore, 1927.
[2] E. Waring, "Problems concerning interpolation," *Phil. Trans. Royal Soc.*, Vol. 69 (1779), pp. 59–67.

NOTE ON THE VARIANCE AND BEST ESTIMATES

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The purpose of this note is to point out a certain relation between the variances, σ_1^2 and σ_2^2, of the random variables, x_1 and x_2, and the probabilities,

$$P_1(t) = Pr[\,|x_1 - E(x_1)| < t]$$

$$P_2(t) = Pr[\,|x_2 - E(x_2)| < t].$$

This is, if $\sigma_1^2 < \sigma_2^2$, then $P_1(t) > P_2(t)$ in at least one interval, $t_1 < t < t_2$.

A note by A. T. Craig [1] gave an example for which it was stated that $\sigma_1^2 < \sigma_2^2$ and $P_1(t) \leq P_2(t)$ for every t; but, as was pointed by Neyman [2], calculation of the probabilities involved shows the statement to be incorrect.

The present result provides a certain justification for the use of minimum variance estimates by assuring that no other estimate with the same mean can have, for every value of t, a greater probability of a deviation from the mean less than t. If an estimate can be found which has a greater value of $P(t)$ for all t than does any other estimate, it is necessarily the minimum variance estimate.

The theorem below includes a similar relation for equal variances. This theorem can be obtained from known general results on inequalities for distributions determined by moments, [3] and [4]. The formulation given here with its significance for estimates does not appear to have been remarked.

THEOREM. *If the random variables, x_1 and x_2, have finite variances, σ_1^2 and σ_2^2, and*

$$\sigma_1^2 \leq \sigma_2^2,$$

then, either

$$Q(t) = P_1(t) - P_2(t),$$

is equal to zero at all points of continuity, which can occur only for $\sigma_1^2 = \sigma_2^2$, or there is an interval, $t_1 < t < t_2$, in which $Q(t)$ is positive.

PROOF. We write the variance as the Stieltjes integral,

$$\sigma_1^2 = \int_0^\infty t^2\, dP_1(t),$$

and similarly for σ_2^2.

Let

$$S(T) = \int_0^T t^2\, dP_1(t) - \int_0^T t^2\, dP_2(t) = \int_0^T t^2\, dQ(t)$$
$$= T^2 Q(T) - 2 \int_0^T tQ(t)\, dt,$$

integrating by parts.

Now

$$T^2[1 - P_1(T)] = T^2 \int_T^\infty dP_1(t) \leq \int_T^\infty t^2\, dP_1(t),$$

and since σ_1^2 is finite, $\int_T^\infty t^2\, dP_1(t) \to 0$ as $T \to \infty$, so that $\lim_{T\to\infty} T^2[1 - P_1(T)] = 0$, and similarly for $P_2(t)$.

Hence $T^2 Q(T) = T^2[1 - P_2(T)] - T^2[1 - P_1(T)] \to 0$ as $T \to \infty$, and since by definition $\lim_{T\to\infty} S(T) = \sigma_1^2 - \sigma_2^2$ it follows that

$$\sigma_1^2 - \sigma_2^2 = -2 \int_0^\infty tQ(t)\, dt.$$

From this it can be seen that either, $Q(t)$ vanishes at all points of continuity, in which case $\sigma_1^2 = \sigma_2^2$, or $Q(t)$ must be positive in some interval, since otherwise $\int_0^\infty tQ(t)\, dt$ must be negative and hence $\sigma_1^2 - \sigma_2^2 > 0$ contrary to the assumption, $\sigma_1^2 \leq \sigma_2^2$.

REFERENCES

[1] A. T. Craig, "A note on the best linear estimate," *Annals of Math. Stat.*, Vol. 14 (1943), pp. 88–90.
[2] J. Neyman, *Math. Reviews*, Vol. 4 (1943), p. 280.
[3] J. V. Uspensky, *Introduction to Mathematical Probability*, New York, McGraw-Hill (1937), pp. 373–380.
[4] A. Wald, "Limits of a distribution function determined by absolute moments and inequalities satisfied by absolute moments," *Trans. Amer. Math. Soc.*, Vol. 46 (1939), pp. 280–306.