# FURTHER CONTRIBUTIONS TO THE PROBLEM OF SERIAL CORRELATION

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**1. Introduction.** Recently, there has been an increasing interest in the study of the serial correlation of observations. The development of the distribution theory and significance criteria was retarded by the fact that the successive differences or successive products of statistical variates are not independent. However, these difficulties have been overcome to a considerable extent by recent work of several authors. In order to indicate the nature of the contributions embodied in the present paper, it will be necessary to describe rather precisely the contributions of these authors.

Suppose $x_1, x_2, \cdots, x_n$ are $n$ independent observations of a random variable $x$ which is normally distributed with mean $a$ and variance $\sigma^2$. Let us define

$$\begin{aligned} \delta_{n-1}^2 &= \sum_{i=1}^{n-1}(x_{i+1}-x_i)^2 & \delta_n^2 &= \sum_{i=1}^{n}(x_{i+1}-x_i)^2 \\ C_{n-1} &= \sum_{i=1}^{n-1}(x_i-\bar{x})(x_{i+1}-\bar{x}) & C_n &= \sum_{i=1}^{n}(x_i-\bar{x})(x_{i+1}-\bar{x}) \\ {}_LC_n &= \sum_{i=1}^{n}(x_i-\bar{x})(x_{i+L}-\bar{x}) & V_n &= \sum_{i=1}^{n}(x_i-\bar{x})^2 \end{aligned} \tag{1.1}$$

in which $x_{n+i} = x_i$. The ratio of any of the first five values to $V_n$ will be a measure of the relation between the successive observations $x_i$.

Von Neumann [2] has studied the ratio $\eta = \delta_{n-1}^2/V_n$. He obtains an expression for the sampling distribution of the ratio $\eta$. He solves the equivalent problem of determining the distribution of $\sum_{i=1}^{n-1} A_i y_i^2$ where the point $(y_1, y_2, \cdots, y_{n-1})$ is uniformly distributed over the spherical surface $\sum_{i=1}^{n-1} y_i^2 = 1$ and the $A_i$ are the characteristic values of $\delta_{n-1}^2$. He obtains the distribution $\omega(\gamma)$ of $\gamma = \sum_{i=1}^{m} B_i x_i^2$ ($m$ even) where the point $(x_1, x_2, \cdots, x_n)$ is uniformly distributed over the spherical surface $\sum_{i=1}^{m} x_i^2 = 1$ and $B_1 \geq B_2 \geq \cdots \geq B_m$. $\omega(\gamma)$ is found by solving the equation

$$\int_{B_m}^{B_1} (\gamma - z)^{-\frac{1}{2}m}\omega(\gamma)\, d\gamma = \prod_{i=1}^{m} (B_i - z)^{-\frac{1}{2}}. \tag{1.2}$$

The distribution of $\eta$ is then a special case of this distribution. The first four moments were obtained by Williams [5] by the use of a generating function. In the present paper we shall study the ratio $\delta_n^2/V_n$. The moments of this ratio will be developed and the moments and approximate distribution of $[2 - \delta_n^2/V_n]^2$.

Von Neumann [4] in a paper which removed a restriction (that $m$ be even) on the distribution of $\eta$ indicates how to determine the distribution of $C_{n-1}/V_n$. Koopmans [9] considers the stochastical process $x_t = \rho x_{t-1} + z_t$ $(t = 1, 2, \cdots)$, $|\rho| < 1$. The $z_t$ are independent drawings from a normal distribution with zero mean and variance $\sigma^2$. To test the hypothesis that $\rho = 0$ he shows that it is sufficient to know the distribution of $C_{n-1}/V_n$. He finds the distribution of $C_{n-1}/V_n$ and $C_n/V_n$ but finds that the numerical computation of these functions is very cumbersome. This prompts him to obtain approximate formulas for these distributions. The approximate formula for the distribution of $\bar{r} = C_n/V_n$ is

$$(1.3) \qquad (\tfrac{1}{2}n - 1)2^{\frac{1}{2}n}\pi^{-1}\int_0^{\arccos \bar{r}} (\cos\alpha - \bar{r})^{\frac{1}{2}n-2} \sin \tfrac{1}{2}n\alpha \sin\alpha \, d\alpha.$$

A similar approximation will be used in this paper to find the moments of $C_n/V_n$. It will be shown how good the approximation is and how by using this approximation we may obtain a tabled function (Pearson Type I) which fits the distribution of $1 - (C_n/V_n)^2$ up to $\frac{1}{2}n$ moments.

The quantity $1 - (C_n/V_n)^2$, we shall find, is equivalent to a likelihood ratio function for testing the hypothesis that the serial correlation is zero.

Anderson [8] obtained the distribution of ${}_LC_n/V_n = {}_LR_n$. He proved that the distribution of ${}_LR_n$ is the same as that of ${}_1R_n$ when $L$ and $n$ are prime to each other. He has computed the 1 per cent and 5 per cent significance values $(L = 1)$ up to $n = 75$. For values of $n > 75$ he indicates that a normal distribution which is an asymptotic approximation may be used. He has also computed some significance values for the cases of $N/L = 2, 3, 4$.

In this paper we shall develop the moments of ${}_LR_n$.

The use of the ratio $\eta$ in the study of serial effects in ballistics at Aberdeen Proving Ground is given in references [1] and [2]. The use of the ratios $C_{n-1}/V_n$ and $C_n/V_n$ in the study of economic time series is discussed by Koopmans [9].

**2. Likelihood criteria.** Given a sample of $n$ observations $x_1, x_2, \cdots, x_n$ we shall assume that they are distributed according to the law:

$$(2.1) \qquad dP_n = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{\alpha=1}^{n}(x_\alpha - a - bx_{\alpha-l})^2} dx_1 \cdots dx_n, \qquad (1 \leq l \leq n).$$

It will be convenient to use the phraseology that "the variate at the time $\alpha$ has as its mean value a linear function of the variate at the time $\alpha - l$." We shall take $x_i = x_{n+i}$. Due to the symmetry we may use $\alpha + l$ in place of $\alpha - l$. This will be done to obtain agreement with previous work. We wish to test the hypothesis $H_1$ that each variate is independent of the other variates, that is, that $b = 0$. The Neyman-Pearson specification of $H_1$ may be written as follows, where $\Omega$ is the space of admissible values of $\sigma^2$, $a$ and $b$, and $\omega$ the subspace defining $H_1$:

$$\text{(2.2)} \qquad \begin{cases} \Omega: \sigma^2 > 0 & -\infty < a, \quad b < \infty \\ \omega: \sigma^2 > 0 & -\infty < a < \infty, \quad b = 0. \end{cases}$$

The likelihood criterion $\lambda_1$ suitable to this hypothesis is the ratio of the maximum ($\omega$ (max.)) of (2.1) with the restriction that $b = 0$ to the maximum ($\Omega$ (max.)) of (2.1) without this condition. Now,

$$\text{(2.3)} \qquad \lambda_1 = \frac{dP_n(\omega \max)}{dP_n(\Omega \max)}.$$

We see that the likelihood function is

$$\text{(2.4)} \qquad L = -n \log (\sqrt{2\pi}\,\sigma) - \frac{1}{2\sigma^2} \sum_{\alpha=1}^{n} (x_\alpha - a - bx_{\alpha+l})^2$$

and to maximize $L$ over the space $\Omega$ we compute the following derivatives

$$\text{(2.5)} \qquad \begin{aligned} \frac{\partial L}{\partial a} &= \frac{1}{\sigma^2} \Sigma(x_\alpha - a - bx_{\alpha+l}), \\ \frac{\partial L}{\partial b} &= \frac{1}{\sigma^2} \Sigma(x_\alpha - a - bx_{\alpha+l})x_{\alpha+l}, \\ \frac{\partial L}{\partial \sigma} &= -\frac{n}{\sigma^2} + \frac{1}{\sigma^3} \Sigma(x_\alpha - a - bx_{\alpha+l}). \end{aligned}$$

The solutions of the equations obtained by setting the above derivatives equal to zero are:

$$\text{(2.6)} \qquad \begin{aligned} \hat{\sigma}^2 &= \frac{1}{n} \Sigma(x_\alpha - \hat{a} - \hat{b}x_{\alpha+l})^2 \\ \hat{a} &= \frac{1}{n} \Sigma x_\alpha(1 - \hat{b}) \\ \hat{b} &= \frac{n\Sigma x_\alpha x_{\alpha+l} - (\Sigma x_\alpha)^2}{n\Sigma x_\alpha^2 - (\Sigma x_\alpha)^2}. \end{aligned}$$

If we now maximize $L$ over the space $\omega$ we obtain

$$\text{(2.7)} \qquad \begin{aligned} \hat{\sigma}^2 &= \frac{1}{n} \Sigma(x_\alpha - \hat{a})^2 \\ \hat{a} &= \frac{1}{n} \Sigma x_\alpha \end{aligned}$$

so that we will have

$$\text{(2.8)} \qquad dP_n(\Omega \max) = [2\pi(1 - \hat{b}^2)\Sigma(x_\alpha - \bar{x})^2]^{-\frac{1}{2}n} e^{-\frac{1}{2}n},$$

$$\text{(2.9)} \qquad dP_n(\omega \max) = [2\pi\Sigma(x_\alpha - \bar{x})^2]^{-\frac{1}{2}n} e^{-\frac{1}{2}n},$$

$$\text{(2.10)} \qquad \lambda_1 = (1 - \hat{b}^2)^{\frac{1}{2}n},$$

where $b$ is defined as in (2.6) above. If we set $a = 0$ in (2.1) we may follow a similar procedure and find the criterion ${}_0\lambda_1 = (1 - \hat{b}_0^2)^{\frac{1}{2}n}$ for testing the hypothesis that $b = 0$ if it is known that the population mean equal zero. Here

$$\hat{b}_0 = \frac{\Sigma x_\alpha x_{\alpha+l}}{\Sigma x_\alpha^2}. \tag{2.11}$$

We notice that $\hat{b}$ is the criterion chosen by R. L. Anderson as a measure of serial correlation. He has obtained the distribution of $\hat{b}$ and has computed a number of significance values from this distribution. The distribution is a function of $n$ and $l$, and Anderson points out that the larger the values of $n$ the smaller the extent to which the significance values depend on $l$.

In the next section we shall find a distribution which approximates very closely the exact distribution of $\hat{b}$ and which is independent of the lag $l$.

**3. Moments of the likelihood criteria.** We shall determine the moments of $\hat{b}_0$ and ${}_0\lambda_1$ when the hypothesis ${}_0H_1$ is true and the moments of $\hat{b}$ and $\lambda_1$ when the hypothesis $H_1$ is true. Let us first consider $\hat{b}_0 = \Sigma x_\alpha x_{\alpha+l}/\Sigma x_\alpha^2$, the criterion we obtained for testing the hypothesis ${}_0H_1$. The moment generating function for the joint distribution of $C_0 = \Sigma x_\alpha x_{\alpha+l}/\sigma^2$ and $V_0 = \Sigma x_\alpha^2/\sigma^2$ is

$$\begin{aligned} \varphi(t_1, t_2) &= E[\exp(C_0 t_1 + V_0 t_2)] \\ &= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \int_{-\infty}^{\infty}\cdots\int \exp\left(C_0 t_1 + V_0 t_2 - \frac{1}{2\sigma^2}\Sigma x_\alpha^2\right) \prod_{\alpha=1}^{n} dx_\alpha . \end{aligned} \tag{3.1}$$

By reference to this last expression it can be seen that

$$E(\hat{b}_0) = \int_{-\infty}^{0} \left.\frac{\partial\varphi(t_1, t_2)}{\partial t_1}\right]_{t_1=0} dt_2, \tag{3.2}$$

and further

$$E(\hat{b}_0^k) = \int_{-\infty}^{0}\cdots\int \left.\frac{\partial^k \varphi(t_1, t_2)}{\partial t_1^k}\right]_{t_1=0} \prod_{j=1}^{k} dt_{2j}, \tag{3.3}$$

in which we set $t_2 = \sum_{j=1}^{k} t_{2j}$.

Now if we write (3.1) as follows:

$$\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \int_{-\infty}^{\infty}\cdots\int e^{-\frac{1}{2\sigma^2}\sum_{ij} A_{ij}x_i x_j}\,\Pi\, dx_i, \tag{3.4}$$

we see that $\varphi(t_1, t_2) = |A_{ij}|^{-\frac{1}{2}}$ and $A_{ij} = A_{i+a,j+a}$; that is, this determinant is a circulant. Let us write $A_{ij} = a_h$ where $h$ equal $j - i + 1$ or $j - i + 1 + n$ taking that subscript which gives $1 \leq h \leq n$, so that we have

$$\varphi^{-2}(t_1, t_2) = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{vmatrix}, \tag{3.5}$$

which expanded by the method of circulants becomes

$$\prod_{k=1}^{n} \sum_{i=1}^{n} a_i \omega_k^{i-1},$$

where the $\omega_k$ are the $n$th roots of unity.

First we shall consider $\hat{b}_0$ (lag 1). Here $a_1 = 1 - 2t_2$, $a_2 = a_n = -t_1$ and $a_3$ to $a_{n-1}$ equal zero.

$$(3.6) \qquad \begin{aligned} {}_0\varphi_1^{-2}(t_1, t_2) &= \prod_{k=1}^{n} (1 - 2t_2 - t_1(\omega_k + \omega_k^{-1})) \\ &= \prod_{k=1}^{n} \left(1 - 2t_2 - 2t_1 \cos \frac{2\pi k}{n}\right). \end{aligned}$$

For lag $l$, $a_1 = 1 - 2t_2$, $a_{l+1} = a_{n-l+1} = -t_1$ and the remaining $a$'s $= 0$.

$$(3.7) \qquad \begin{aligned} {}_0\varphi_1^{-2}(t_1, t_2) &= \prod_{k=1}^{n} (1 - 2t_2 - t_1(\omega_k^{l} + \omega_k^{-l})) \\ &= \prod_{k=1}^{n} \left(1 - 2t_2 - 2t_1 \cos \frac{2\pi lk}{n}\right). \end{aligned}$$

We shall develop an approximation to these functions (3.6) and (3.7) as follows:

$$(3.8) \qquad {}_0\varphi_1(t_1, t_2) = \prod_{k=1}^{n} (A + B \cos \alpha_k)^{-\frac{1}{2}} = e^{-\frac{1}{2}\sum_{k=1}^{n} \log (A+B\cos\alpha_k)},$$

where in this case $A = 1 - 2t_2$, $B = -2t_1$ and $\alpha_k = 2\pi lk/n$. Let us now alter the form of this exponent and replace the sum of a finite number of terms involving $\alpha_k$ by an integral of a continuous variable $\alpha$.

$$(3.9) \qquad {}_0\varphi_1(t_1, t_2) = \exp\left(\frac{-n}{4\pi l} \frac{2\pi l}{n} \sum_{k=1}^{n} \log (A + B \cos \alpha_k)\right).$$

Let us write $2\pi l/n = \Delta\alpha_k$, then we shall have

$$(3.10) \qquad {}_0\varphi_1(t_1, t_2) = \exp\left(\frac{-n}{4\pi l} \sum_{k=1}^{n} \log (A + B \cos \alpha_k)\Delta\alpha_k\right).$$

If we take $B < A$ we see that $A + B \cos \alpha_k$ is never negative; therefore as we let $n$ increase the summation will approach the value of the integral: $\int_0^{2\pi l} \log (A + B \cos \alpha)\, d\alpha$. Let us then replace this summation by its limiting integral. The resulting function will then be an approximation for $n$ large enough. We shall obtain then

$$(3.11) \qquad {}_0\varphi_1(t_1, t_2) \sim \exp\left(\frac{-n}{4\pi l} \int_0^{2\pi l} \log (A + B \cos \alpha)\, d\alpha\right) \qquad B < A,$$

from which by the use of the integral[1] we obtain

$$(3.12) \qquad {}_0\varphi_1(t_1, t_2) \sim \exp\left(-\tfrac{1}{2}n \log \tfrac{1}{2}(A + \sqrt{A^2 - B^2})\right).$$

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[1] This integral may be verified by differentiation with respect to the parameter $a$. It may be found as formula 523 in Pierce's "Short Table of Integrals."

So that

$$(3.13)\qquad {}_0\varphi_1(t_1, t_2) \sim [\tfrac{1}{2}(A + \sqrt{A^2 - B^2})]^{-\frac{1}{2}n}.$$

We now have $\varphi(t_1, t_2)$ represented approximately by a power of a single quantity. The question of how good this approximation is will be discussed in a later paragraph. This is similar to the approximation used by T. Koopmans in the distribution of $\hat{b}_0$. He makes the substitution "to obtain what intuitively seems to be in some sense the closest approximation." He approximates $\prod_{t=1}^{T} (\kappa - \bar{\kappa}_t)^{-\frac{1}{2}}$ where $\bar{\kappa}_t = \cos \frac{2\pi t}{T}$ by the process followed in (3.8)–(3.13) in order to find the distribution given in (1.3).

To obtain the corresponding function for

$$(3.14)\qquad \hat{b} = \frac{\Sigma x_\alpha x_{\alpha+l} - (\Sigma x_\alpha)^2/n}{\Sigma x_\alpha^2 - (\Sigma x_\alpha)^2/n},$$

we follow the same procedure as above for $\hat{b}_0$. Here

$$C = [\Sigma x_\alpha x_{\alpha+l} - (\Sigma x_\alpha)^2/n]/\sigma^2,$$

$$V = [\Sigma x_\alpha^2 - (\Sigma x_\alpha)^2/n]/\sigma^2,$$

and in (3.5) $a_1 = 1 - 2t_2 + 2(t_1 + t_2)/n$; $a_2 = a_n = -t_1 + 2(t_1 + t_2)/n$; and all the other $a$'s $= 2(t_1 + t_2)/n$ so that the expansion of this circulant becomes

$$(3.15)\qquad \varphi_1^{-2}(t_1, t_2) = \prod_{k=1}^{n} \Big\{[1 - 2t_2 + 2(t_1 + t_2)/n] + [-t_1 + 2(t_1 + t_2)/n](\omega_k + \omega_k^{-1}) + [2(t_1 + t_2)/n] \sum_{i=3}^{n-1} \omega_k^{i-1}\Big\},$$

and since

$$\sum_{i=3}^{n-1} \omega_k^{i-1} = \begin{cases} -(\omega_k + \omega_k^{-1} + 1) & k \neq n, \\ n - 3 & k = n, \end{cases}$$

we get

$$(3.16)\qquad \prod_{k=1}^{n-1} \{1 - 2t_2 - t_1(\omega_k + \omega_k^{-1})\},$$

and for lag $l$ we get

$$(3.17)\qquad \prod_{k=1}^{n-1} \{1 - 2t_2 - t_1(\omega_k^{l} + \omega_k^{-l})\}.$$

These two results are the same as those obtained previously except that the final term, $1 - 2t_1 - 2t_2 = A + B$, of the products is missing. We will then obtain as an approximation to these finite products $\varphi_1^{-2} = {}_0\varphi_1^{-2}/(A + B)$ or

$$(3.18)\qquad \varphi_1(t_1, t_2) = [\tfrac{1}{2}(A + \sqrt{A^2 - B^2})]^{-\frac{1}{2}n}\sqrt{A + B}$$

where $A = 1 - 2t_2$ and $B = -2t_1$.

A method of finding the moments of $\hat{b}_0$ and $\hat{b}$ was outlined in (3.1), (3.2) and (3.3) above. If we perform these operations on ${}_0\varphi_1(t_1, t_2)$ as defined in (3.13) we find

$$
\begin{aligned}
{}_0\varphi_1 &= u^{-\frac{1}{2}n} \\
\frac{\partial \varphi}{\partial t_1} &= -\tfrac{1}{2}nu^{-\frac{1}{2}n-1}\frac{\partial u}{\partial t_1} \qquad (3.19)\\
\frac{\partial^2 \varphi}{\partial t_1^2} &= -\tfrac{1}{2}nu^{-\frac{1}{2}n-2}\left[\left(-\frac{n}{2}-1\right)\left(\frac{\partial u}{\partial t_1}\right)^2 + u\frac{\partial^2 u}{\partial t_1^2}\right]
\end{aligned}
$$

where $u\,|_0 = 1 - 2t_2$, $\left.\dfrac{\partial u}{\partial t_1}\right|_0 = 0$, $\left.\dfrac{\partial^2 u}{\partial t_1^2}\right|_0 = \dfrac{-2}{1 - 2t_2}$, etc., and the zero subscript indicates that $t_1$ has been set equal to zero after differentiation. If these values are substituted and the required number of integrations with respect to $t_2$ are performed, we find the moments of the criterion $\hat{b}_0$ when ${}_0H_1$ is true.

$$
\begin{aligned}
M_1 &= 0 \qquad M_2 = \frac{1}{n+2} \\
M_3 &= 0 \qquad M_4 = \frac{3}{(n+2)(n+4)} \\
M_5 &= 0 \qquad M_6 = \frac{15}{(n+2)(n+4)(n+6)} \qquad (3.20)
\end{aligned}
$$

etc., or

$$
\begin{aligned}
M_{2k-1} &= 0 \\
M_{2k} &= \frac{1\cdot3\cdot5\cdots(2k-1)}{(n+2)(n+4)\cdots(n+2k)}.
\end{aligned}
$$

$M_{2k}$ may be verified by the use of an expansion of the generating function (3.13) by a method of Laplace [10]. He gives the expansion of $u^{-i}$ where $u$ is given by the equation $u^2 - 2u + e^2 = 0$ as follows:

$$
u^{-i} = \frac{1}{2^i} + \frac{ie^2}{2^{i+2}} + \frac{i(i+3)e^4}{1.2.2^{i+4}} + \frac{i(i+4)(i+5)e^6}{1\cdot2\cdot3\cdot2^{i+6}} + \cdots + \frac{i(i+k+1)\cdots(i+2k-1)e^{2k}}{(k-1)!\cdot2^{i+2k}} + \cdots.
$$

We see that $u = 1 + \sqrt{1 - e^2}$, and if we set $e = t_1/(\frac{1}{2} - t_2)$ and $i = \frac{1}{2}n$. We obtain ${}_0\varphi_1 = u^{-\frac{1}{2}n}$ as a series in the even powers of $t_1$. From this we can see that the odd moments are zero and from the form of the coefficients we can verify $M_{2k}$.

These moments are not contained in the works of the other authors writing on this subject. Although these moments are obtained from an approximate generating function they are, as will be shown later, the exact, not approximate, moments for $k < n$, for lag 1 and are the exact moments for $k < n/\alpha$ for any lag $l$ where $\alpha$ is the largest common factor of $n$ and the lag $l$.

To obtain the moments of $\hat{b}$ we follow a similar procedure with $\varphi_1 = u^{-\frac{1}{2}n}(1 - 2t_1 - 2t_2)^{\frac{1}{2}}$. Differentiating $\varphi_1$ the required number of times with respect to $t_1$ and integrating an equal number of times with respect to $t_2$ gives the following moments:

$$M_1 = \frac{-1}{n-1} \qquad M_2 = \frac{1}{n+1}$$

$$M_3 = \frac{-3}{(n-1)(n+3)} \qquad M_4 = \frac{3}{(n+1)(n+3)}$$

$$\vdots \tag{3.21}$$

$$M_{2k-1} = \frac{-1\cdot3\cdot5\cdots(2k-1)}{(n-1)(n+3)(n+5)\cdots(n+2k-1)}$$

$$M_{2k} = \frac{1\cdot3\cdot5\cdots(2k-1)}{(n+1)(n+3)\cdots(n+2k-1)}.$$

Examination of the moments of $\hat{b}_0$ will show that $\hat{b}_0 = x$ is distributed according to the law

$$K_1(1 - x^2)^{\frac{1}{2}(n-1)} = K_1(1 + x)^{\frac{1}{2}(n-1)}(1 - x)^{\frac{1}{2}(n-1)} \tag{3.22}$$

up to $n$ moments. This distribution is symmetric and we may wish a normal approximation to this curve. The mean is zero and the variance is $1/(n + 2)$. The $\lambda$ criterion ${}_0\lambda_1^{2/n} = (1 - \hat{b}_0^2) = y$ is distributed according to the law

$$K_2(1 - y)^{-\frac{1}{2}}y^{\frac{1}{2}(n-1)} \tag{3.23}$$

up to $\frac{1}{2}n$ moments. Here the mean is $\frac{n+1}{n+2}$ and the variance is $\frac{2(n+1)}{(n+2)^2(n+4)}$. If we inspect the moments of $\hat{b}$ we see that the distribution of $\lambda_1^{2/n} = (1 - \hat{b}^2) = z$ follows the law

$$K_3(1 - z)^{-\frac{1}{2}}z^{\frac{1}{2}(n-2)} \tag{3.24}$$

up to $\frac{1}{2}n$ moments, which is the same as the distribution immediately preceding except that $n$ is replaced by $n - 1$. The distributions (3.22), (3.23), and (3.24) are the same for lag $l$ except that the fit is up to $n/\alpha$, $n/2\alpha$, and $n/2\alpha$ moments respectively where $\alpha$ is the largest common factor of $l$ and $n$. These restrictions are necessary since the moments as given in (3.20) and (3.21) are obtained from the approximate generating functions (3.13) and (3.18). The exact generating function is given in (4.8) for lag 1 and it is found that the $n$th or higher derivatives bring contributions from the part of the generating function which was neglected in approximating the generating function. The additional restriction for lag $l \neq 1$ will be seen in the last two paragraphs of section 4. The extra factor $\frac{1}{2}$ in the second and third case above is due to the fact that only the even moments of (3.20) and (3.21) are used.

We have in (3.23) and (3.24), then, very close approximations to the distributions for the two $\lambda$ criteria for testing serial effects.

The following table is a comparison of the exact and approximate 5 per cent and 1 per cent points for the distribution of $\hat{b}$. The exact values are taken from the table given by R. L. Anderson. The normal approximation as given by Anderson in his table does not show such close agreement since he used an asymptotic second moment. He indicated that the exact values would have to be used for values of $n < 75$ in place of the values from the normal approximation which he obtained. Here we see that the normal approximation may be used for $n$ somewhat less than 75. The Pearson Type I approximation was obtained by using the first two moments of $\hat{b}$. The curve obtained is:

$$\frac{(1 + x)^{p-1}(1 - x)^{q-1}}{B(p, q)2^{p+q-1}} \tag{3.25}$$

in which $p = \dfrac{(n-1)(n-2)}{2(n-3)}$ and $q = \dfrac{n(n-1)}{2(n-3)}$.

The exact values marked with an asterisk in the table differ slightly from those previously published. They are more precise values from the exact distribution which R. L. Anderson has made available to the author.

*Positive tail*

| | | *5%* | | | *1%* | |
|---|---|---|---|---|---|---|
| *N* | *Exact* | *Type I* | *Normal* | *Exact* | *Type I* | *Normal* |
| 5 | .253 | .317 | .281 | .297 | .527 | .501 |
| 10 | .360 | .362 | .350 | .525 | .533 | .541 |
| 15 | .328 | .329 | .323 | .475 | .477 | .486 |
| 20 | .299 | .299 | .296 | .432 | .433 | .440 |
| 25 | .276 | .276 | .274 | .398 | .398 | .404 |
| 30 | .257 | .257 | .255 | .370 | .371 | .375 |
| 45 | .218 | .218 | .217 | .313* | .313 | .316 |
| 75 | .174* | .174 | •.174 | .250 | .250 | .251 |

*Negative tail*

| | | *5%* | | | *1%* | |
|---|---|---|---|---|---|---|
| *N* | *Exact* | *Type I* | *Normal* | *Exact* | *Type I* | *Normal* |
| 5 | .753 | .742 | .781 | .798 | .858 | 1.000 |
| 10 | .564 | .562 | .572 | .705 | .702 | .763 |
| 15 | .462 | .461 | .466 | .597 | .596 | .629 |
| 20 | .399 | .399 | .401 | .524 | .524 | .545 |
| 25 | .356 | .356 | .357 | .473 | .473 | .487 |
| 30 | .324* | .324 | .324 | .433 | .433 | .444 |
| 45 | .262 | .262 | .262 | .356 | .356 | .362 |
| 75 | .201* | .201 | .201 | .276 | .276 | .278 |

**4. Alternative expansions of the generating functions.** In this section we shall determine the exact generating functions which were approximated in (3.12) and (3.16) and obtain these same approximations in another manner. This development will enable us to see how good the approximation is in the sense that it gives a certain number of exact moments. The determinant in (3.4) for lag $l$ and mean zero can be written

$$(4.1) \qquad A_n = \begin{vmatrix} a & b & & & & b \\ b & a & b & & & \\ & b & a & b & 0 & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & 0 & & b & a & b \\ b & & & & b & a \end{vmatrix}_n$$

where $a = 1 - 2t_2$, $b = -t_1$ and all the other elements are zero. The $b$ in the upper right corner and lower left corner indicate the value $b$ in the $a_{1n}$ and $a_{n1}$ positions. Let us define the following determinants:

$$B_n = \begin{vmatrix} a & b & & & & \\ b & a & b & & & \\ & b & a & b & 0 & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & 0 & & & & \\ & & & b & a & b \\ b & & & & b & a \end{vmatrix}_n \qquad C_n = \begin{vmatrix} b & a & b & & & & \\ & b & a & b & & & \\ & & b & a & b & 0 & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & 0 & & & & \\ & & & & & b & a \\ b & & & & & & b \end{vmatrix}_n$$

(4.2)

$$D_n = \begin{vmatrix} a & b & & & & \\ b & a & b & & & \\ & b & a & b & 0 & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ 0 & & & b & a & b \\ & & & & b & a \end{vmatrix}_n$$

We see that

$$(4.3) \qquad \begin{aligned} A_n &= B_n + (-1)^{n-1} b C_{n-1}, \\ B_n &= D_n + (-1)^{n-1} b^n, \\ C_n &= b^n + (-1)^{n-1} b D_{n-1}, \end{aligned}$$

and $A_n$ can be expressed in terms of $D_n$ by substituting for $B_n$ and $C_n$ in the equation for $A_n$.

$$(4.4) \qquad A_n = D_n - b^2 D_{n-2} + 2(-1)^{n-1} b^n.$$

We can obtain an expression for $D_n$ if we expand this determinant by the first row. This gives

$$(4.5) \qquad D_n = a D_{n-1} - b^2 D_{n-2}.$$

Since this is a second order difference equation, the solution may be written $D_n = k_1u^n + k_2v^n$ where $u$, $v$ are roots of the equation $x^2 - ax + b^2 = 0$. Now,

$$
(4.6)\qquad \begin{aligned} D_1 &= u + v = k_1u + k_2v, \\ D_2 &= u^2 + v^2 + uv = k_1u^2 + k_2v^2 \end{aligned}
$$

so that we can determine $k_1$ and $k_2$. We now see that

$$
(4.7)\qquad D_n = \frac{u^{n+1} - v^{n+1}}{u - v}
$$

which upon substitution in the equation for $A_n$ gives

$$
(4.8)\qquad \begin{aligned} A_n &= u^n + v^n + 2(-1)^{n-1}b^n, \\ &= u^n + v^n - 2t_1^n, \end{aligned}
$$

where $u, v = \frac{1}{2}(1 - 2t_2) \pm \sqrt{(1 - 2t_2)^2 - 4t_1^2}$. Now ${}_0\varphi_1(t_1, t_2) = A_n^{-\frac{1}{2}}$ and it is easily seen from the form of $A_n$ directly above that derivatives up to the $n$th order with respect to $t_1$ in which $t_1$ is then set equal to zero will be given by derivatives of $A_n = u^n$ and this is the approximation (3.13) found by other methods.

The determinant in (3.4) for lag 1 and mean not equal to zero can be written

$$
(4.9)\qquad A_n = \begin{vmatrix} a & b & & & & b \\ b & a & b & & c & \\ & b & a & b & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & c & & & & \\ & & & b & a & b \\ b & & & & b & a \end{vmatrix} \qquad \begin{aligned} a &= 1 - 2t_2 + 2(t_1 + t_2)/n, \\ b &= -t_1 + 2(t_1 + t_2)/n, \\ c &= 2(t_1 + t_2)/n. \end{aligned}
$$

Let us define the following determinants:

$$
(4.10)\qquad B_n = \begin{vmatrix} b & a & b & & & \\ & b & a & b & c & \\ & & b & a & b & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & c & & & & \\ & & & & b & a \\ b & & & & & b \end{vmatrix} \qquad C_n = \begin{vmatrix} a & b & & & & \\ b & a & b & & c & \\ & b & a & b & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & c & & & & \\ & & & b & a & b \\ b & & & & b & a \end{vmatrix}
$$

$$
D_n = \begin{vmatrix} b & a & b & & & \\ & b & a & b & c & \\ & & b & a & b & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & c & & & & \\ & & & & b & a \\ & & & & & b \end{vmatrix} \qquad E_n = \begin{vmatrix} a & b & & & & \\ b & a & b & & c & \\ & b & a & b & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & c & & & & \\ & & & b & a & b \\ & & & & b & a \end{vmatrix}
$$

If we replace the $b$ in the upper right corner of $A_n$ by $c + (b - c)$ we obtain

(4.11) $$A_n = C_n + (-1)^{n-1}(b - c)B_{n-1}.$$

If we replace the $b$ in the lower left corner of $B_n$ and $C_n$ by $c + (b - c)$ we obtain

(4.12) $$B_n = D_n + (-1)^{n-1}(b - c)E_{n-1}, \qquad C_n = E_n + (-1)^{n-1}(b - c)D_{n-1}.$$

We now have $A_n$ in terms of $D_n$ and $E_n$. We must now evaluate $D_n$ and $E_n$.

(4.13) $$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & & 1 \\ 0 & b & a & b & & \\ 0 & & b & a & b & c \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot\ \cdot & \cdot \\ & c & & & & \\ & & & & b & a \\ 0 & & & & & b \end{vmatrix}_{n+1} = \begin{vmatrix} 1 & 1 & 1 & \cdots & & 1 \\ -c & r & s & r & & \\ -c & & r & s & r & 0 \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot\ \cdot & \cdot \\ & & 0 & & & \\ & & & & r & s \\ -c & & & & & r \end{vmatrix}_{n+1}$$

where $r = b - c$ and $s = a - c$ and the second determinant above is obtained from the first by subtracting $c$ times the first row from all other rows. Writing this last determinant as the sum of two determinants by separating the first column we get

(4.14) $$D_n = r^n - cF_{n+1}.$$

Combining the above difference equations we obtain

(4.15) $$A_n = E_n - r^2E_{n-2} + 2(-1)^n rcF_n + 2(-1)^{n-1}r^n$$

and see that we must obtain $E_n$ and $F_n$.

Expansion of $F_{n+1}$ by the second column gives

(4.16) $$F_{n+1} = -G_n + rF_n$$

and expanding $G_n$ by the last row we get

(4.17) $$G_n = rG_{n-1} + (-1)^{n-1}H_{n-1}.$$

(4.18) $$F_{n+1} = \begin{vmatrix} 0 & 1 & 1 & \cdots & & 1 \\ 1 & r & s & r & & 0 \\ 1 & & r & s & r & \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot\ \cdot & \cdot \\ & & 0 & & & \\ & & & & r & s \\ 1 & & & & & r \end{vmatrix}_{n+1} \qquad G_n = \begin{vmatrix} 1 & s & r & & & \\ 1 & r & s & r & & 0 \\ 1 & & r & s & r & \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot\ \cdot & \cdot \\ & & 0 & & & \\ & & & & r & s \\ 1 & & & & & r \end{vmatrix}_n$$

$$H_n = \begin{vmatrix} s & r & & & & \\ r & s & r & & & 0 \\ & r & s & r & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ 0 & & & & & \\ & & & r & s & r \\ & & & & r & s \end{vmatrix}$$

$H_n$ is the same type as (4.1), therefore $H_n = \dfrac{u^{n+1} - v^{n+1}}{u - v}$ where $u$ and $v$ are the roots of the equation $x^2 - sx + r^2 = 0$, so that (4.17) becomes

$$G_n - rG_{n-1} = (-1)^{n-1} \frac{u^n - v^n}{u - v} \tag{4.19}$$

and the solution of this equation gives

$$G_n = \frac{r^n}{2r + s} + \frac{(-1)^{n-1}[r(u^n - v^n) + u^{n+1} - v^{n+1}]}{(u - v)(2r + s)}. \tag{4.20}$$

Introducing this expression into (4.16) we find

$$F_n = (-1)^{n-1} \frac{u^n - v^n}{(u - v)(2r + s)} - \frac{nr^{n-1}}{2r + s} \tag{4.21}$$

$$E_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & & 1 \\ 0 & a & b & & & \\ 0 & b & a & b & & c \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & & \cdot & \cdot & \cdot \\ & c & & & & \\ & & & b & a & b \\ 0 & & & & b & a \end{vmatrix}_{n+1} = \begin{vmatrix} 1 & 1 & 1 & \cdots & & 1 \\ -c & s & r & & & \\ -c & r & s & r & & 0 \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & & \cdot & \cdot & \cdot \\ & 0 & & & & \\ & & & r & s & r \\ -c & & & & r & s \end{vmatrix}_{n+1} \tag{4.22}$$

where the second determinant is obtained from the first by subtracting $c$ times the first row from all other rows. If we separate this last determinant on the first column we get

$$E_n = H_n - cI_{n+1} \tag{4.23}$$

$$I_n = \begin{vmatrix} 0 & 1 & 1 & \cdots & & 1 \\ 1 & s & r & & & \\ 1 & r & s & r & & 0 \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & & \cdot & \cdot & \cdot \\ & 0 & & & & \\ & & & r & s & r \\ 1 & & & & r & s \end{vmatrix} \qquad J_n = \begin{vmatrix} 0 & 1 & 1 & \cdots & & 1 & 1 \\ 1 & s & r & & & & \\ 1 & r & s & r & & 0 & \\ \cdot & & \cdot & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ & & 0 & & r & s & r \\ & & & & & r & s \\ 1 & & & & & & r \; r \end{vmatrix} \tag{4.24}$$

Expanding $I_n$ by the last row, we get

$$I_n = (-1)^{n-1}G_{n-1} - rJ_{n-1} + sI_{n-1}. \tag{4.25}$$

Expanding $J_n$ by the last column, we get

$$J_n = (-1)^{n-1}G_{n-1} + rI_{n-1}. \tag{4.26}$$

If we combine these last two equations we find

$$I_n - sI_{n-1} + r^2I_{n-2} = (-1)^{n-1}(G_{n-1} + rG_{n-2}). \tag{4.27}$$

If we now solve this difference equation for $I_n$, substitute this solution in the equation for $E_n$, and in turn substitute this result and the expression we obtained for $F_n$ in (4.15), we get

$$A_n = \frac{u^n + v^n + 2(-1)^{n-1} r^n}{2r + s} = \frac{u^n + v^n - 2t_1^n}{1 - 2t_1 - 2t_2}. \tag{4.28}$$

The final form results since $2r + s = 1 - 2t_1 - 2t_2$, $r = -t_1$. $u$ and $v$ have the same values as before. If we compare this result with that obtained in (4.8) for mean equal to zero we see that this is the same except that here we have the added factor, $1 - 2t_1 - 2t_2$, in the denominator. We have a similar result then for the approximation for derivatives of $\varphi_1(t_1, t_2) = A_n^{-\frac{1}{2}}$ for $t_1 = 0$. Here this approximation is $A_n = u^n/(1 - 2t_1 - 2t_2)$, the same result as that obtained in (3.18). This approximation will yield moments which are exact for $n > \alpha k$ for any lag $l$ where $\alpha$ is the largest common factor of $n$ and $l$. The reason for this restriction is easily seen if we consider the expansion obtained in (3.7), for

$$\frac{\partial \varphi(t_1, t_2)}{\partial t_1} = -\tfrac{1}{2}\varphi(t_1, t_2) \sum_{k=1}^{n} \frac{-2 \cos \frac{2\pi lk}{n}}{1 - 2t_2 - 2t_1 \cos \frac{2\pi lk}{n}} \tag{4.29}$$

with $t_1 = 0$,

$$\left.\frac{\partial \varphi(t_1, t_2)}{\partial t_1}\right]_{t_1=0} = -\tfrac{1}{2}\varphi(0, t_2) \sum_{k=1}^{n} \frac{-2 \cos \frac{2\pi lk}{n}}{1 - 2t_2} = \frac{\varphi(0, t_2)}{1 - 2t_2} \sum_{k=1}^{n} \cos \frac{2\pi lk}{n}. \tag{4.30}$$

Further

$$\left.\frac{\partial^2 \varphi(t_1, t_2)}{\partial t_1^2}\right]_{t_1=0} = \frac{\varphi(0, t_2)}{(1 - 2t_2)^2} \left[\left(\sum_{k=1}^{n} \cos \frac{2\pi lk}{n}\right)^2 + 2 \sum_{k=1}^{n} \cos^2 \frac{2\pi lk}{n}\right] \tag{4.31}$$

and the $m$th partial derivative will contain the sum of the $m$th powers of the cosines. These are the sums of the powers of the real parts of the roots of unity and it is easily seen that $\Sigma \cos^m \frac{2\pi lk}{n} = \Sigma \cos^m \frac{2\pi k}{n}$ only for $m < \alpha k$ where $\alpha$ is the largest common factor of $n$ and $l$.

To change the moment generating function of $\hat{b}_0$ to that of $\hat{b}$ we must drop the last term of the product. In the above expressions we then have $\sum_{k=1}^{n-1} \cos^m \frac{2\pi lk}{n}$ and the same conclusion will hold.

**5. Application to successive differences.** If we change slightly the function $\eta = \delta_{n-1}^2/V_n$ investigated by von Neumann and Williams we find the moments and distribution greatly simplified. Let us define

$$\delta_n^2 = \sum_{i=1}^{n} (x_i - x_{i+1})^2 \tag{5.1}$$

where $x_{n+1} = x_1$ and consider the ratio ${}_0\eta_1$ of $\delta_n^2$ to $\Sigma x_i^2$. Now,

$$\delta_n^2 = 2\Sigma x_i^2 - 2\Sigma x_i x_{i+1} \tag{5.2}$$

therefore

$${}_0\eta_1 = \frac{\delta_n^2}{\Sigma x_i^2} = 2(1 - \hat{b}_0) \tag{5.3}$$

and we may find the moments and distribution of ${}_0\eta_1$ directly from those of $\hat{b}_0$. We find the moments to be:

$$\begin{aligned} m_1 &= 2 & m_2 &= \frac{2^2(n+3)}{n+2} \\ m_3 &= \frac{2^3(n+5)}{n+2} & m_4 &= \frac{2^4(n+5)(n+7)}{(n+2)(n+4)} \\ m_k &= \frac{2^k(n+2k-1)!}{(n+k)!\,(n+2)(n+4)\cdots(n+2k-2)}, & & (k < n) \end{aligned} \tag{5.4}$$

and the ratio ${}_0\eta_1$ is distributed according to the law

$$C_1\,{}_0\eta_1^{\frac{1}{2}(n-1)}(4 - {}_0\eta_1)^{\frac{1}{2}(n-1)} \tag{5.5}$$

up to $n$ moments.

If we replace $x_i$ in the above ratios by $x_i - \bar{x}$ we find the moments of the ratio $\eta_1 = \delta_n^2/\Sigma(x_i - \bar{x})^2$ to be:

$$\begin{aligned} m_1 &= \frac{2n}{n-1} & m_2 &= \frac{2^2 n(n+3)}{(n-1)(n+1)} \\ m_3 &= \frac{2^3 n(n+4)(n+5)}{(n-1)(n+1)(n+3)} & m_4 &= \frac{2^4 n(n+5)(n+6)(n+7)}{(n-1)(n+1)(n+3)(n+5)} \\ m_k &= \frac{2^k n(n+2k-1)!}{(n+k)!\,(n-1)(n+1)(n+3)\cdots(n+2k-3)} \end{aligned} \tag{5.6}$$

and $(\eta_1 - 2)^2 = z$ has the distribution

$$C_2 z^{-\frac{1}{2}}(4 - z)^{\frac{1}{2}(n-2)} \tag{5.7}$$

up to $\frac{1}{2}n$ moments.

The ratio $\delta_n^2/\Sigma x_i^2$ compares the variation of the first differences to that of the original variates. We might wish to compare the variation of the second differences to that of the first differences. For this purpose let us form the ratio

$$(5.8) \qquad \eta_2 = \frac{\sum_{i=1}^{n} (x_i - 2x_{i+1} + x_{i+2})^2}{\sum_{i=1}^{n} (x_i - x_{i+1})^2} \qquad \begin{aligned} x_{n+1} &= x_1 \\ x_{n+2} &= x_2 \end{aligned}$$

to test the hypothesis $H_{\eta_2}$ that the variation of the second differences compared to the variation of the first differences is such as would occur by chance. Let $x_1, x_2, \cdots, x_n$ be normally distributed with mean $a$ and variance $\sigma^2$. The ratio $\eta_2$ is independent of the mean value of the variates, therefore we may consider a distribution with mean equal zero. We shall develop the mean and variance of $\eta_2$ when the hypothesis to be tested is true. The moment generating function for the joint distribution of $D_2 = \Sigma(x_i - 2x_{i+1} + x_{i+2})^2/2\sigma^2$ and $D_1 = \Sigma(x_i - x_{i+1})^2/2\sigma^2$ is

$$(5.9) \qquad \begin{aligned} \varphi(t_1, t_2) &= E[\exp (D_2 t_1 + D_1 t_2)] \\ &= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \int_{-\infty}^{\infty} \cdots \int \exp\left(D_2 t_1 + D_1 t_2 - \frac{1}{2\sigma^2}\Sigma x_i^2\right) \prod_{i=1}^{n} dx_i . \end{aligned}$$

We may find the moments then by a process similar to that outlined in (3.2) and (3.3). The next few steps are identical with (3.4) and (3.5). For the present problem, however, $a_1 = 1 - 6t_1 - 2t_2$, $a_2 = 4t_1 + t_2$, $a_3 = -t_1$ so that

$$(5.10) \qquad \begin{aligned} \varphi^{-2}(t_1, t_2) &= \prod_{k=1}^{n} [1 - 6t_1 - 2t_2 + (4t_1 + t_2)(\omega_k^1 + \omega_k^{-1}) - t_1(\omega_k^2 + \omega_k^{-2})], \\ &= \prod_{k=1}^{n} \left[1 - 6t_1 - 2t_2 + (8t_1 + 2t_2) \cos \frac{2\pi k}{n} - 2t_1 \cos \frac{4\pi k}{n}\right], \\ &= \prod_{k=1}^{n} \left[a + b \cos \frac{2\pi k}{n} + c \cos \frac{4\pi k}{n}\right]. \end{aligned}$$

If we follow the same procedure indicated in (3.8) to (3.13) we obtain successively

$$(5.11) \qquad \varphi(t_1, t_2) = \prod_{k=1}^{n} (a + b \cos \alpha_k + c \cos 2\alpha_k)^{-\frac{1}{2}}$$

$$(5.12) \qquad = e^{-\frac{1}{2}\sum_{k=1}^{n} \log (a+b \cos \alpha_k + c \cos 2\alpha_k)}$$

$$(5.13) \qquad = \exp\left(\frac{-n}{4\pi} \frac{2\pi}{n} \sum_{k=1}^{n} \log (a + b \cos \alpha_k + c \cos 2\alpha_k)\right)$$

and replace the summation by the integral which is the limit of the summation as $n \to \infty$.

$$(5.14) \qquad \varphi(t_1, t_2) \sim \exp\left(\frac{-n}{4\pi} \int_0^{2\pi} \log (a + b \cos \alpha + c \cos 2\alpha)\, d\alpha\right)$$

$$(5.15) \qquad \sim \exp\left(-\frac{n}{2} \log\left[\kappa \frac{1 + \sqrt{1 - \delta^2}}{2} \frac{1 + \sqrt{1 - \eta^2}}{2}\right]\right)$$

where $\kappa = a - c; \qquad \eta, \delta = \dfrac{b \pm \sqrt{b^2 + 8c^2 - 8ac}}{2(a-c)}.$

We then have approximately

(5.16) $$\varphi(t_1, t_2) \sim [\tfrac{1}{4}\kappa(1 + \sqrt{1 - \delta^2})(1 + \sqrt{1 - \eta^2})]^{-\frac{1}{2}n}.$$

(5.14) follows from (5.13) if we replace the summation by an integral, and (5.15) is obtained in the following manner: replace $\cos 2\alpha$ by $2 \cos^2 \alpha - 1$ and factor the resulting quadratic and integrate the factors separately.

$$\int_0^{2\pi} \log (a + b \cos \alpha + c \cos 2\alpha)\, d\alpha = \int_0^{2\pi} \log (a - c + b \cos \alpha + 2c \cos^2 \alpha)\, d\alpha$$

(5.17) $$= \int_0^{2\pi} \log \kappa \, d\alpha + \int_0^{2\pi} \log (1 + \delta \cos \alpha)\, d\alpha + \int_0^{2\pi} \log (1 + \eta \cos \alpha)\, d\alpha$$
$$= 2\pi \log \kappa + 2\pi \log \tfrac{1}{2}(1 + \sqrt{1 - \delta^2}) + 2\pi \log \tfrac{1}{2}(1 + \sqrt{1 - \eta^2}).$$

If we now expand (5.16) by multiplying the factors within the brackets and substitute for $\kappa$, $\eta$ and $\delta$ we find

(5.18) $$\varphi(t_1, t_2) \sim [A + B + C + D]^{-\frac{1}{2}n} = P^{-\frac{1}{2}n},$$

where

(5.19)
$$\begin{aligned}
a &= 1 - 6t_1 - 2t_2, & A &= \tfrac{1}{4}(1 - 4t_1 - 2t_2),\\
b &= 8t_1 + 2t_2, & B &= \tfrac{1}{4}[1 - 12t_1 - 4t_2 + 8t_1t_2 + 2t_2^2\\
& & &\quad - 2(4t_1 + t_2)\sqrt{t_2^2 + 4t_1}]^{\frac{1}{2}},\\
c &= -2t_1, & C &= \tfrac{1}{4}[1 - 12t_1 - 4t_2 + 8t_1t_2 + 2t_2^2\\
& & &\quad + 2(4t_1 + t_2)\sqrt{t_2^2 + 4t_1}]^{\frac{1}{2}},\\
& & D &= \tfrac{1}{4}(1 - 16t_1 - 4t_2)^{\frac{1}{2}}.
\end{aligned}$$

From (5.18) $P = A + B + C + D$ and at $t_1 = 0$

(5.20)
$$P = \tfrac{1}{4}(1 + \sqrt{1 - 4t_2})^2,$$
$$\frac{\partial P}{\partial t_1} = -2[1 + 2(1 - 4t_2)^{-\frac{1}{2}}],$$
$$\frac{\partial^2 P}{\partial t_1^2} = \frac{-32}{(1 - 4t_2)^{\frac{3}{2}}} - \frac{(1 - \sqrt{1 - 4t_2})^2}{2t_2^2}.$$

Now

(5.21)
$$\frac{\partial \varphi}{\partial t_1} = -\tfrac{1}{2}nP^{-\frac{1}{2}n-1}\frac{\partial P}{\partial t_1}$$
$$\frac{\partial^2 \varphi}{\partial t_1^2} = -\tfrac{1}{2}nP^{-\frac{1}{2}n-2}\left[(-\tfrac{1}{2}n - 1)\left(\frac{\partial P}{\partial t_1}\right)^2 + P\frac{\partial^2 P}{\partial t_1^2}\right].$$

If we substitute in this formula and integrate the first with respect to $t_2$ we shall obtain the first moment of the ratio $\eta_2$. If we integrate the second twice with respect to $t_2$, we shall obtain the second moment of the ratio $\eta_2$. We find these moments to be

$$(5.22)\qquad M_1 = \frac{3n+2}{n+1} \qquad M_2 = \frac{9n^2+23n+12}{(n+1)(n+2)}$$
$$\sigma^2 = \frac{2n^2+7n+4}{(n+1)^2(n+2)}.$$

**6. Likelihood criteria for multiple serial correlation.** Given a sample of $n$ observations, $x_1, x_2, \cdots, x_n$, we shall assume that they are distributed according to the law

$$(6.1)\qquad dP_n = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{\alpha=1}^{n}\left(x_\alpha - a - \sum_{i=1}^{r} b_i x_{\alpha-l_i}\right)^2} dx_1 \cdots dx_n \qquad (x_i = x_{n+i})$$

that is, that each variate, say at the time $t$, has as its mean value a linear function of the variates at time $t - l_1, t - l_2$, etc. Let us investigate the likelihood criteria for testing the hypothesis, $H_r$, that each variate is independent of the others; i.e. that the $b_i = 0$ $(i = 1, \cdots, r)$. For the hypothesis $H_r$ we define the space $\Omega$ and the space $\omega$, as follows:

$$(6.2)\qquad \begin{cases} \Omega: & \sigma^2 > 0 \quad -\infty < a, b_i < \infty \\ \omega: & \sigma^2 > 0 \quad -\infty < a < \infty, \quad b_i = 0, \end{cases}$$

we find the likelihood ratio criterion

$$(6.3)\qquad \lambda_r^{2/n} = \frac{|a_{ij}|}{a_{00}\,|a_{pq}|} \qquad \begin{matrix} i, j = 0, 1, \cdots, r \\ p, q = 1, \cdots, r \end{matrix}$$

in which

$$(6.4)\qquad \begin{aligned} a_{00} &= \sum_\alpha (x_\alpha - \bar{x})^2 \\ a_{0i} &= \sum_\alpha (x_\alpha - \bar{x})(x_{\alpha+l_i} - \bar{x}) \\ a_{ij} &= \sum_\alpha (x_{\alpha+l_i} - \bar{x})(x_{\alpha+l_j} - \bar{x}) \end{aligned}$$

and it is noted that $a_{ii} = a_{00}$ and if the $l_i$ are equispaced $a_{i,i+c} = a_{0c}$. $\lambda_r$ is a statistic which measures how completely each variate at time $t$ can be expressed as a linear function of variates spaced at time $t - l_1, t - l_2$, etc.

Next we shall develop a statistic for testing the hypothesis, $H_{r,m}$, that of the set of the values $b_i$ $(i = 1, \cdots, r)$ in (6.1) the subset $b_{m+1}, b_{m+2}, \cdots, b_r = 0$. Here we have the same likelihood function but for $H_{r,m}$, we define the spaces $\Omega$ and $\omega$ as follows:

$$(6.5)\qquad \begin{cases} \Omega: & \sigma^2 > 0 \quad -\infty < a, b_i < \infty; \\ \omega: & \sigma^2 > 0 \quad -\infty < a, b_u < \infty, \quad b_w = 0, \end{cases}$$
$$u = (1, \cdots, m), \quad w = (m+1, \cdots, r),$$

and obtain the criterion

$$(6.6)\qquad \lambda_{r,m}^{2/n} = \frac{|a_{ij}|\,|a_{uv}|}{|a_{pq}|\,|a_{st}|} \qquad \begin{array}{c} i, j = 0, 1, \cdots, r, \\ p, q = 1, \cdots, r, \\ s, t = 0, 1, \cdots, m, \\ u, v = 1, \cdots, m, \\ (m < r). \end{array}$$

The form and the derivation of these $\lambda$ criteria parallels very closely that of the likelihood ratio criteria obtained in multivariate analysis for testing significance of regression coefficients.

CASE I. If we set $r = 1$ in $\lambda_r$ we obtain

$$(6.7)\qquad \lambda_1^{2/n} = \frac{\begin{vmatrix} a_{00} & a_{01} \\ a_{01} & a_{00} \end{vmatrix}}{a_{00} \quad a_{00}} = 1 - \frac{a_{01}^2}{a_{00}^2} = 1 - \hat{b}^2,$$

for which the distribution is given in (3.24).

CASE II. If we set $r = 2$, we have

$$(6.8)\qquad \lambda_2^{2/n} = \frac{\begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{vmatrix}}{a_{00}\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}},$$

for which if we take $l_1 = 1$, $l_2 = 2$ we get

$$(6.9)\qquad \frac{\begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{00} & a_{01} \\ a_{02} & a_{01} & a_{00} \end{vmatrix}}{a_{00}\begin{vmatrix} a_{00} & a_{01} \\ a_{01} & a_{00} \end{vmatrix}}.$$

The expanded form of this numerator is $a_{00}^3 + 2a_{01}^2a_{02} - a_{00}a_{02}^2 - 2a_{01}^2a_{00}$.

Let us consider

$$(6.10)\qquad \varphi(\theta_0, \theta_1, \theta_2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \int_{-\infty}^{\infty}\cdots\int e^{-\frac{1}{2\sigma^2}\{\Sigma x_\alpha^2(1-\theta_0)-\theta_1\Sigma x_\alpha x_{\alpha+1}-\theta_2\Sigma x_\alpha x_{\alpha+2}\}}\,\Pi\, dx_\alpha.$$

We shall find the mean and variance of ${}_0\lambda_2^{2/n}$ (mean $= 0$) when the hypothesis ${}_0H_r$ ($r = 2$) is true. We can find the first moment of ${}_0\lambda_2^{2/n}$ then by performing the following operations: (*a*) compute

$$(6.11)\qquad \frac{\partial^3\varphi}{\partial\theta_0^3} + 2\frac{\partial^2\varphi}{\partial\theta_1^2}\frac{\partial\varphi}{\partial\theta_2} - \frac{\partial\varphi}{\partial\theta_0}\frac{\partial^2\varphi}{\partial\theta_2^2} - 2\frac{\partial^2\varphi}{\partial\theta_1^2}\frac{\partial\varphi}{\partial\theta_0},$$

(this will give the first moment of the numerator) and set $\theta_2 = 0$, (*b*) integrate from $-\infty$ to $\theta_0 + \theta_1$ with respect to $\theta_0 + \theta_1 = \zeta$, from $-\infty$ to $\theta_0 - \theta_1$, with respect to $\theta_0 - \theta_1 = \xi$, and set $\theta_1 = 0$ (at this point we will have the first moment of the third order determinant divided by the second order determinant), (*c*) integrate with respect to $\theta_0$ from $-\infty$ to 0. The reason for step (b) is easily seen since the second order determinant $a_{00}^2 - a_{01}^2$ may be written $(a_{00} - a_{01})(a_{00} + a_{01})$.

Further moments may be computed in a similar manner. ${}_0\varphi_2(\theta_0, \theta_1, \theta_2)$ may be written as a determinant in the manner indicated in (3.4) and (3.5). Here, $a_1 = 1 - \theta_0$, $a_2 = a_n = -\frac{1}{2}\theta_1$ and $a_3 = a_{n-1} = -\frac{1}{2}\theta_2$ and $a_4$ to $a_{n-2} = 0$, then

$$(6.12) \qquad {}_0\varphi_2^{-2}(\theta_0, \theta_1, \theta_2) = \prod_{k=1}^{n} \sum_{i=1}^{n} a_i \omega_k^{i-1} = \prod_{k=1}^{n} \left(a_1 + 2a_2 \cos\frac{2\pi k}{n} + 2a_3 \cos\frac{4\pi k}{n}\right).$$

We shall approximate ${}_0\varphi_2(\theta_0, \theta_1, \theta_2)$ by the method contained between (5.11) and (5.18). We set

$$(6.13) \qquad {}_0\varphi_2(\theta_0, \theta_1, \theta_2) = \prod_{k=1}^{n} \left(a + b \cos\frac{2\pi k}{n} + c \cos\frac{4\pi k}{n}\right)^{-\frac{1}{2}}$$

and obtained

$$(6.14) \qquad {}_0\varphi_2(\theta_0, \theta_1, \theta_2) \sim [A + B + C + D]^{-\frac{1}{2}n} = P^{-\frac{1}{2}n}$$

where

$$(6.15) \qquad \begin{aligned} A &= \tfrac{1}{4}(a - c) \\ B &= \tfrac{1}{8}(4a^2 - 4c^2 - 2b^2 - 2bE^{\frac{1}{2}})^{\frac{1}{2}} = \tfrac{1}{8}\beta^{\frac{1}{2}} \\ C &= \tfrac{1}{8}(4a^2 - 4c^2 - 2b^2 + 2bE^{\frac{1}{2}})^{\frac{1}{2}} = \tfrac{1}{8}\gamma^{\frac{1}{2}} \\ D &= \tfrac{1}{4}((a + c)^2 - b^2)^{\frac{1}{2}} = \tfrac{1}{4}\epsilon^{\frac{1}{2}} \\ E &= b^2 + 8c(c - a). \end{aligned}$$

It is easily seen that we may operate (differentiate and integrate) with $a$, $b$, $c$, in place of $\theta_0$, $\theta_1$, $\theta_2$ respectively. Therefore, we compute

$$(6.16) \qquad \begin{aligned} \frac{\partial \varphi}{\partial c} &= -\tfrac{1}{2}nP^{-\frac{1}{2}n-1}\frac{\partial P}{\partial c} \\ \frac{\partial^2 \varphi}{\partial c^2} &= -\tfrac{1}{2}nP^{-\frac{1}{2}n-2}\left[(-\tfrac{1}{2}n - 1)\left(\frac{\partial P}{\partial c}\right)^2 + P\frac{\partial^2 P}{\partial c^2}\right] \end{aligned}$$

and since $P = A + B + C + D$ we compute

$$(6.17) \qquad \begin{aligned} \frac{\partial A}{\partial c} &= -\frac{1}{4}; & \frac{\partial^2 A}{\partial c^2} &= 0 \\ \frac{\partial B}{\partial c} &= \tfrac{1}{16}\beta^{-\frac{1}{2}}\frac{\partial \beta}{\partial c}; & \frac{\partial^2 B}{\partial c^2} &= \frac{1}{16}\left[-\tfrac{1}{2}\beta^{-\frac{3}{2}}\left(\frac{\partial \beta}{\partial c}\right)^2 + \beta^{-\frac{1}{2}}\frac{\partial^2 \beta}{\partial c^2}\right] \\ \frac{\partial C}{\partial c} &= \tfrac{1}{16}\gamma^{-\frac{1}{2}}\frac{\partial \gamma}{\partial c}; & \frac{\partial^2 C}{\partial c^2} &= \frac{1}{16}\left[-\tfrac{1}{2}\gamma^{-\frac{3}{2}}\left(\frac{\partial \gamma}{\partial c}\right)^2 + \gamma^{-\frac{1}{2}}\frac{\partial^2 \gamma}{\partial c^2}\right] \\ \frac{\partial D}{\partial c} &= \tfrac{1}{8}\epsilon^{-\frac{1}{2}}\frac{\partial \epsilon}{\partial c}; & \frac{\partial^2 D}{\partial c^2} &= \frac{1}{8}\left[-\tfrac{1}{2}\epsilon^{-\frac{3}{2}}\left(\frac{\partial \epsilon}{\partial c}\right)^2 + \epsilon^{-\frac{1}{2}}\frac{\partial^2 \epsilon}{\partial c^2}\right] \\ \frac{\partial E}{\partial c} &= 16c - 8a; & \frac{\partial^2 E}{\partial c^2} &= 16. \end{aligned}$$

In order to evaluate the expressions in (6.17) we must find

$$\frac{\partial\beta}{\partial c} = -8c - bE^{-\frac{1}{2}}\frac{\partial E}{\partial c}; \qquad \frac{\partial^2\beta}{\partial c^2} = -8 - bE^{-\frac{3}{2}}\left[-\frac{1}{2}\left(\frac{\partial E}{\partial c}\right)^2 + E\frac{\partial^2 E}{\partial c^2}\right];$$

$$(6.18) \qquad \frac{\partial\gamma}{\partial c} = -8c + bE^{-\frac{1}{2}}\frac{\partial E}{\partial c}; \qquad \frac{\partial^2\gamma}{\partial c^2} = -8 + bE^{-\frac{3}{2}}\left[-\frac{1}{2}\left(\frac{\partial E}{\partial c}\right)^2 + E\frac{\partial^2 E}{\partial c^2}\right]$$

$$\frac{\partial\epsilon}{\partial c} = 2(a + c); \qquad \frac{\partial^2\epsilon}{\partial c^2} = 2.$$

If we now set $c = 0$, we obtain

$$P = \tfrac{1}{2}(a + (a^2 - b^2)^{\frac{1}{2}})$$

$$(6.19) \qquad \frac{\partial P}{\partial c} = \frac{a - (a^2 - b^2)^{\frac{1}{2}}}{2(a^2 - b^2)^{\frac{1}{2}}}$$

$$\frac{\partial^2 P}{\partial c^2} = \frac{2a^4 - 4a^2b^2 + b^4 + (-2a^3 + 2ab^2)(a^2 - b^2)^{\frac{1}{2}}}{2b^2(a^2 - b^2)^{\frac{3}{2}}}.$$

We may now substitute these values in (6.16) and then substitute the resulting expressions in (6.11). The remaining values that are required for (6.11) are easily computed since they may be obtained from $\varphi$ with $c = 0$, i.e.

$$(6.20) \qquad {}_0\varphi_2(\theta_0, \theta_1, 0) = [\tfrac{1}{2}(a + (a^2 - b^2)^{\frac{1}{2}}]^{-\frac{1}{2}n}.$$

The result of these substitutions gives

$$(6.21) \qquad \frac{-n^2(n + 3)P^{-\frac{1}{2}n-2}}{8(a^2 - b^2)^{\frac{1}{2}}},$$

in which we set $d = \frac{1}{2}(a - b)$ and $e = \frac{1}{2}(a + b)$ and integrate with respect to $d$ and $e$. We obtain

$$(6.22) \qquad \frac{-n^2}{2(n + 2)}[\tfrac{1}{2}(a + (a^2 - b^2)^{\frac{1}{2}})]^{-\frac{1}{2}n-1},$$

and if we set $b = 0$ and integrate with respect to $a$, setting $a = 1$, ($\theta_0 = 0$), we finally have

$$(6.23) \qquad E({}_0\lambda_2^{2/n}) = \frac{n}{n + 2}.$$

We shall now obtain the first moment of $\lambda_2$ without the restriction that the mean equal zero. For this purpose let us consider

$$(6.24) \qquad \varphi_2(\theta_0, \theta_1, \theta_2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \int_{-\infty}^{\infty}\cdots\int e^{-(1/2\sigma^2)[a_{00}(1-\theta_0)-a_{01}\theta_1-a_{02}\theta_2]}\,\Pi\, dx_\alpha.$$

Here $a_1 = 1 - \theta_0 + m$, $a_2 = a_n = -\frac{1}{2}\theta_1 + m$, $a_3 = a_{n-1} = -\frac{1}{2}\theta_2 + m$, $a_4$ to $a_{n-2} = m$ where $m = (\theta_0 + \theta_1 + \theta_2)/n$. Expanding the determinant as in (6.12) we find

$$(6.25) \qquad \varphi_1^{-2}(\theta_0, \theta_1, \theta_2) = \prod_{k=1}^{n}\sum_{i=1}^{n} a_i\omega_k^{i-1}$$

$$= \prod_{k=1}^{n}\left(a_1 + 2a_2\cos\frac{2\pi k}{n} + 2a_3\cos\frac{4\pi k}{n} + \sum_{i=4}^{n-2} a_i\omega_k^{i-1}\right).$$

Now

$$\sum_{i=4}^{n-2} a_i \omega_k^{i-1} = m \sum_{i=4}^{n-2} \omega_k^{i-1} \tag{6.26}$$

$$= \begin{cases} -m(1 + \omega^1 + \omega^{-1} + \omega^2 + \omega^{-2}), & k \neq n, \\ m(n - 5), & k = n, \end{cases}$$

so that

$$\varphi_2^{-2}(\theta_0, \theta_1, \theta_2) = \prod_{k=1}^{n-1} \left( a_1 + 2a_2 \cos \frac{2\pi k}{n} + 2a_3 \cos \frac{4\pi k}{n} \right). \tag{6.27}$$

We have obtained here a product which is the same as that in (6.12) except that the last factor is missing. The approximation corresponding to (6.14) will then be

$$\varphi_2(\theta_0, \theta_1, \theta_2) \cong \frac{[A + B + C + D]^{-\frac{1}{2}n}}{(a + b + c)^{-\frac{1}{2}}}, \tag{6.28}$$

since we may take the approximation for the product from 1 to $n$ and divide by the last factor, $(a + b + c)$. The procedure for finding the first moment for $\lambda_2$ (mean $= a$) is exactly the same as that outlined for finding the first moment of ${}_0\lambda_2$ (mean $= 0$). We obtain

$$E(\lambda_2^{2/n}) = \frac{n - 1}{n + 1}. \tag{6.29}$$

Case III. If we set $r = 2$, $m = 1$ in $\lambda_{r,m}$ we have, if we take $l_1 = 1$, $l_2 = 2$

$$\lambda_{2,1}^{2/n} = \frac{\begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{00} & a_{01} \\ a_{02} & a_{01} & a_{00} \end{vmatrix} a_{00}}{\begin{vmatrix} a_{00} & a_{01} \\ a_{01} & a_{00} \end{vmatrix}^2}. \tag{6.30}$$

To find the moments of $\lambda_{2,1}$ let us consider the following distribution,

$$dP_n = \left( \frac{1}{\sqrt{2\pi}\,\sigma} \right)^n e^{-\frac{1}{2\sigma^2} \sum_{\alpha=1}^{n} [x_\alpha - \bar{x} - \beta(x_{\alpha+1} - \bar{x})]^2} \Pi\, dx_\alpha, \tag{6.31}$$

in which $\beta$ represents the population value of the serial correlation coefficient. The moment generating function for the joint distribution of $a_{00}/2\sigma^2$, $a_{01}/2\sigma^2$ and $a_{02}/2\sigma^2$ will be

$$\varphi_{2,1}(\theta_0, \theta_1, \theta_2) = \left( \frac{1}{\sqrt{2\pi}\,\sigma} \right)^n \int_{-\infty}^{\infty} \cdots \int \exp\left( \frac{-1}{2\sigma^2} \{ \Sigma[(x_\alpha - \bar{x}) - \beta(x_{\alpha+1} - \bar{x})]^2 - a_{00}\theta_0 - a_{01}\theta_1 - a_{02}\theta_2 \} \right) \Pi\, dx_\alpha \tag{6.32}$$

$$= \left( \frac{1}{\sqrt{2\pi}\,\sigma} \right)^n \int_{-\infty}^{\infty} \cdots \int \exp\left( -\frac{1}{2\sigma^2} [a_{00}(1 + \beta^2 - \theta_0) + a_{01}(-2\beta - \theta_1) + a_{02}(-\theta_2)] \right) \Pi\, dx_\alpha.$$

This function is very similar to (6.24). The approximation to $\varphi_{2,1}(\theta_0, \theta_1, \theta_2)$ here will be exactly the same as that obtained in (6.28) for $\varphi_2(\theta_0, \theta_1, \theta_2)$ except that here $a = 1 + \beta^2 - \theta_0$, $b = -2\beta - \theta_1$, $c = -\theta_2$. For the case where the mean is zero, we find the approximation (6.14) in which $a$, $b$, and $c$ have the above values.

We may obtain the first moment of $\lambda_{2,1}$ by operating on the function $\varphi_{2,1}(\theta_0, \theta_1, \theta_2)$ proceeding as follows: ($a$) compute (6.11) as before and set $\theta_2 = 0$, ($b$) integrate from $-\infty$ to $\theta_0 + \theta_1$ with respect to $\theta_0 + \theta_1 = \zeta$ from $-\infty$ to $\theta_0 - \theta_1$ with respect to $\theta_0 - \theta_1 = \xi$ (at this point we will have the first moment of the third order determinant divided by the second order determinant), ($c$) differentiate with respect to $\theta_0$, ($d$) repeat step ($b$), and set $\theta_0$ and $\theta_1 = 0$.

The first two steps for obtaining the first moment of ${}_0\lambda_{2,1}$ (mean $= 0$) were performed for the first moment of ${}_0\lambda_2$ so that we may perform step ($c$) on (6.22). We obtain

$$\frac{n^2[\frac{1}{2}(a + (a^2 - b^2)^{\frac{1}{2}})]^{-\frac{1}{2}n-1}}{4(a^2 - b^2)^{\frac{1}{2}}}, \tag{6.33}$$

and finally by step ($d$) we have

$$E({}_0\lambda_{2,1}^{2/n}) = \frac{n}{n+1}\,[\tfrac{1}{2}(a + (a^2 - b^2)^{\frac{1}{2}})]^{-\frac{1}{2}n-1}, \tag{6.34}$$

in which $a = 1 + \beta^2$ and $b = -2\beta$ since $\theta_0$ and $\theta_1$ have been set equal to zero. Substitution of these values in (6.34) shows that it is independent of $\beta$, and we find

$$E({}_0\lambda_{2,1}^{2/n}) = \frac{n}{n+1}. \tag{6.35}$$

Using $\varphi_{2,1}(\theta_0, \theta_1, \theta_2)$, the generating function for $\lambda_{2,1}$ (mean $= a$), we find

$$E(\lambda_{2,1}^{2/n}) = \frac{n-1}{n}. \tag{6.36}$$

The procedure for obtaining the second moments of the above criteria consists essentially of performing twice the operations prescribed for obtaining the first moments. The details given in connection with the first moment are sufficient to indicate the procedure. The details for the second moments are too complicated algebraically to list here. Table I indicates the second moments obtained as well as other moments obtained in the earlier parts of the paper.

**7. Serial correlation in several variables.** Given a sample of $n$ observations on each of $k$ variables $x_{i\alpha}$, $i = 1, \cdots, k$, we shall assume they are distributed as follows:

$$dP_n = \frac{A^{\frac{1}{2}n}}{(2\pi)^{\frac{1}{2}nk}}\, e^{-\frac{1}{2}\Sigma A_{ij}(x_{i\alpha}-a_i-b_ix_{i,\alpha+k})(x_{j\alpha}-a_j-b_jx_{j,\alpha+k})}\,\Pi\, dx_{i\alpha}. \tag{7.1}$$

We wish to test the hypothesis $H_{kn}$ that there is no serial correlation, i.e., that $b_i = 0, i = 1, \cdots, k$. For this purpose let us define the space $\Omega$ and $\omega$ as follows:

$$\begin{cases} \Omega: & \| A_{ij} \| \text{ pos. def. } -\infty < a_i, b_i < \infty \\ \omega: & \| A_{ij} \| \text{ pos. def. } -\infty < a_i < \infty; \quad b_i = 0. \end{cases} \tag{7.2}$$

TABLE I

| $x$ | formula no. | $E(x)$ | $E(x^2)$ | $\sigma^2$ |
|---|---|---|---|---|
| $\hat{b}_0$ | (2.11) | $0$ | $\frac{1}{n+2}$ | $\frac{1}{n+2}$ |
| $\hat{b}$ | (2.6) | $\frac{-1}{n-1}$ | $\frac{1}{n+1}$ | $\frac{n(n-3)}{(n-1)^2(n+1)}$ |
| ${}_0\eta_1$ | (5.3) | $2$ | $\frac{4(n+3)}{n+2}$ | $\frac{4}{n+2}$ |
| $\eta_1$ | (5.6) | $\frac{2n}{n-1}$ | $\frac{4n(n+3)}{(n-1)(n+1)}$ | $\frac{4n(n-3)}{(n-1)^2(n+1)}$ |
| $\eta_2$ | (5.8) | $\frac{3n+2}{n+1}$ | $\frac{9n^2+23n+12}{(n+1)(n+2)}$ | $\frac{2n^2+7n+4}{(n+1)^2(n+2)}$ |
| ${}_0\lambda_1$ | (2.10)ff. | $\frac{n+1}{n+2}$ | $\frac{(n+1)(n+3)}{(n+2)(n+4)}$ | $\frac{2(n+1)}{(n+2)^2(n+4)}$ |
| $\lambda_1$ | (2.10) | $\frac{n}{n+1}$ | $\frac{n(n+2)}{(n+1)(n+3)}$ | $\frac{2n}{(n+1)^2(n+3)}$ |
| ${}_0\lambda_2$ | (6.9)ff. | $\frac{n}{n+2}$ | $\frac{n}{n+4}$ | $\frac{4n}{(n+2)^2(n+4)}$ |
| $\lambda_2$ | (6.9) | $\frac{n-1}{n+1}$ | $\frac{n-1}{n+3}$ | $\frac{4(n-1)}{(n+1)^2(n+3)}$ |
| ${}_0\lambda_{2,1}$ | (6.30)ff. | $\frac{n}{n+1}$ | $\frac{n(n+2)}{(n+1)(n+3)}$ | $\frac{2n}{(n+1)^2(n+3)}$ |
| $\lambda_{2,1}$ | (6.30) | $\frac{n-1}{n}$ | $\frac{(n-1)(n+1)}{n(n+2)}$ | $\frac{2(n-1)}{n^2(n+2)}$ |

The mean of $\hat{b}_0$ and $\hat{b}$ were also obtained by Anderson [8].

The development of the appropriate $\lambda$ criterion for this case parallels very closely the development of the $\lambda$ criteria in multiple regression analysis. The criterion obtained for testing the hypothesis $H_{kn}$ is

$$(7.3) \qquad \lambda_{kn}^{2/n} = \frac{\begin{vmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{vmatrix}}{|\, a_{ij}\,|^2},$$

where

$$a_{ij} = \sum_\alpha (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j),$$

$$b_{ij} = \tfrac{1}{2}\left[\sum (x_{i\alpha} - \bar{x}_i)(x_{j,\alpha+l} - \bar{x}_j) + \sum (x_{i,\alpha+l} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)\right].$$

The probability theory for the $\lambda$ criteria in (7.3) remains to be developed.

**8. Summary.** A problem in serial correlation which has received considerable attention is that of devising a statistic for indicating the presence of a relation between successive observations, i.e. a lack of independence of the order in which the observations were drawn. Von Neumann developed the distribution and moments of the ratio of the mean square successive difference to the variance. R. L. Anderson presented the distribution of a serial correlation coefficient which is the ratio $R = \Sigma x_\alpha x_{\alpha+l}/\Sigma x_\alpha^2$ ($l \geq 1$, subscripts reduced mod $n$).

The present investigation was undertaken with the object of developing the likelihood ratio functions for testing various hypotheses connected with serial correlation in one or more variables and determining the moments and in some cases the distributions of these likelihood ratios.

The variates are considered to be ordered by their subscripts $\alpha = 1, \cdots, n$. The introduction of $x_{n+1} = x_1$, $x_{n+2} = x_2$ etc. is made to obtain a symmetry which greatly simplifies the problem.

The likelihood ratio criteria were developed for testing the hypotheses

*a*) that $x_\alpha$ is independent of $x_{\alpha+l}$

*b*) that $x_\alpha$ is independent of $x_{\alpha+l_i}$, $i = 1, \cdots, r$;

*c*) that $x_\alpha$ is independent of some subset of the $x_{\alpha+l_i}$

*d*) that in the case of several variables $x_{i\alpha}$, $i = 1, \cdots, k$, $\alpha = 1, \cdots, n$ the $x_{i\alpha}$, $i = 1, \cdots, k$ are independent of the $x_{i,\alpha+h}$. These criteria are similar in form to those obtained in regression analysis.

The likelihood ratio criterion for testing the hypothesis *a*) turns out to be $\lambda = (1 - R^2)^{n/2}$ where $R$ is the function given above. The moments of $R$ are obtained and from these the moments of $\lambda^{2/n}$. These moments are found to agree with those of a Pearson Type I curve to $n/2$ moments. A simple transformation gives us the moments of a ratio differing from that used by von Neumann by the addition of the term $(x_n - x_1)^2$ to the numerator. A simplification of the moments is attained by this change. In fact, if we denote this altered statistic by $\eta$ we find that $(\eta - 2)^2$ is distributed according to a Pearson Type I curve to $n/2$ moments.

The mean and variance were determined for the ratio of the sum of squares of the second successive differences to the first successive differences.

The mean and variance are obtained for the likelihood criteria for testing the hypothesis *b*) for $r = 2$, and for testing the hypothesis *c*) for $r = 2$ where $x_{\alpha+l_2}$ is the subset of $x_{\alpha+l_i}$; ($i = 1, 2$).

All the above moments were obtained under the assumption that the hypothesis to be tested was true. No results have been obtained thus far in cases *b*) and *c*) for a general $r$ nor for hypothesis *d*).

The moments for the several cases above were obtained by the use of moment generating functions which, for the criteria used, took the form of the product of $n$ terms. In the case *a*) it was shown that the product could be approximately represented by the $n$th power of a single expression which was equivalent for the purpose of obtaining the first $n$ moments. A method was developed for making analogous approximations to the generating functions for cases *b*) and *c*) since

it was not found possible to obtain the moments from the products in their original form.

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