Let

$$f_j^*(v) = f_j(v)e^{-\frac{1}{2}\Sigma v_i^2} \qquad (j = 1, 2). \tag{19}$$

Now we shall show that for any positive values $\beta_1, \cdots, \beta_k$

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_j^*(v_1, \cdots, v_k)e^{\beta_1|v_1|+\cdots+\beta_k|v_k|}\, dv_1 \cdots dv_k < \infty. \tag{20}$$

In fact, consider the 2^k sets $(a_1, \cdots, a_k)$ where $a_i = \pm 1$ $(i = 1, \cdots, k)$. Denote by $R_{a_1 \cdots a_k}$ the subset of the k-dimensional Cartesian space which consists of all points $v = (v_1, \cdots, v_k)$ for which v_i is either zero or signum $v_i =$ signum a_i $(i = 1, \cdots, k)$. Putting $\alpha_i = a_i\beta_i$, it follows from (17) and (18) that

$$\int_{R_{a_1 \cdots a_k}} f_j^*(v_1, \cdots, v_k)e^{\beta_1|v_1|+\cdots+\beta_k|v_k|}\, dv_1 \cdots dv_k < \infty. \tag{21}$$

Since (21) holds for any of the 2^k sets $R_{a_1 \cdots a_k}$, equation (20) is proved.

From (1) it follows that

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} v_1^{r_1} \cdots v_k^{r_k}[f_1^*(v_1, \cdots, v_k) - f_2^*(v_1, \cdots, v_k)]\, dv_1 \cdots dv_k = 0,$$

for all non-negative integers $r_1, \cdots, r_k$. Hence, because of (21) and Lemma A we see that

$$f_1^*(v_1, \cdots, v_k) = f_2^*(v_1, \cdots, v_k), \tag{22}$$

except perhaps on a set of measure zero. From (22) it follows that

$$f(v_1, \cdots, v_k) = f_1(v_1, \cdots, v_k) - f_2(v_1, \cdots, v_k) = 0,$$

except perhaps on a set of measure zero. Hence Proposition I is proved.

A NOTE ON SKEWNESS AND KURTOSIS

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It is the purpose of §1 of this paper to prove the following inequality:

$$\alpha_4 \geqq \alpha_3^2 + 1. \tag{1}$$

This inequality seems to have first been stated by Pearson[1]. The inequality also follows from a result appearing in the thesis of Vatnsdal. Here we give a proof based on the theory of quadratic forms which seems to be more direct and more elementary than either of the previous proofs.

[1] "Mathematical contributions to the theory of evolution, XIX; second supplement to a memoir on skew variation," *Phil. Trans. Roy. Soc. (A)*, Vol. 216 (1916), p. 432.

The inequality (1) obviously shows that $\alpha_4 \geqq 1$. It is then natural to ask for an upper bound for α_4. In §2 we shall show that there is no universal upper bound (independent of the number N of quantities in the distribution) for α_3. In fact we find the actual dependence of the maximum possible value of α_3 as a function of N. The form of this function seems to be known but not to have been rigorously proved before. It then follows from (1) that there is no universal upper bound for α_4.

1. The inequality (1). Let us consider the quadratic form

$$G(a, b, c) = \nu_0 a^2 + 2\nu_1 ab + 2\nu_2 ac + \nu_2 b^2 + 2\nu_3 bc + \nu_4 c^2$$
$$= N^{-1}\Sigma(a + xb + x^2 c)^2.$$

It follows that $G(a, b, c)$ is a positive semi-definite quadratic form. In fact, if there are at least three distinct values of x, then $G(a, b, c)$ is a positive definite form. Consequently, its discriminant

$$\begin{vmatrix} \nu_0 & \nu_1 & \nu_2 \\ \nu_1 & \nu_2 & \nu_3 \\ \nu_2 & \nu_3 & \nu_4 \end{vmatrix}$$

must be non-negative. There is no loss of generality in supposing that $\nu_1 = 0$, $\nu_2 = 1$, in which case we find that

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & \alpha_3 \\ 1 & \alpha_3 & \alpha_4 \end{vmatrix} \geqq 0.$$

Expanding the determinant, we get the inequality (1).

We remark that equality holds in (1) if and only if there are only two distinct values of x.

2. The maximum value of α_3. It is clear that this maximum will be N^{-1} times the maximum value of the function $f(x) = \Sigma x^3$ on the bounded closed set consisting of those points x for which $g(x) = \Sigma x^2 = N$ and $h(x) = \Sigma x = 0$. According to the Lagrange multiplier rule, this latter maximum is obtained as follows. Let $F(x) = f(x) - \lambda g(x) - \mu h(x)$. Then the maximizing point satisfies the relations

$$F_{x_i} = 3x_i^2 - 2\lambda x_i - \mu = 0, \qquad \Sigma x^2 = N, \qquad \Sigma x = 0.$$

The equations $\Sigma F_x = 0$, $\Sigma x F_x = 0$ shows that $\mu = 3$, $f_{\max} = 2N\lambda/3$. Solving the equation $F_{x_i} = 0$ gives

$$x_i = [\lambda + e_i(\lambda^2 + 9)^{\frac{1}{2}}]/3, \tag{2}$$

where $e_i = \pm 1$. For these values of x_i we shall have $h(x) = 0$, $g(x) = N$ if and only if

$$\lambda = -(\lambda^2 + 9)^{\frac{1}{2}} N^{-1} \Sigma e.$$

Therefore λ has the sign opposite to that of Σe, and

$$\lambda^2[N^2 - (\Sigma e)^2] = 9(\Sigma e)^2.$$

It follows that $\Sigma e \neq \pm N$, and that

$$\lambda = -3\Sigma e/[N^2 - (\Sigma e)^2]^{\frac{1}{2}}, \tag{3}$$

$$f_{\max} = -2N\Sigma e/[N^2 - (\Sigma e)^2]^{\frac{1}{2}}.$$

We have still not obtained the maximum, however, since the minimum will also satisfy all of the relations deduced above. We distinguish the maximum from the other critical values by examining the function

$$\theta(\Sigma) = -2N\Sigma/(N^2 - \Sigma^2)^{\frac{1}{2}}.$$

Since $\Sigma e \neq \pm N$, $e_i = \pm 1$, it is clear that $N - 2 \geqq \Sigma e \geqq 2 - N$. We therefore consider $\theta(\Sigma)$ on the interval $(2 - N, N - 2)$. We find that

$$d\theta/d\Sigma = -2N^3/(N^2 - \Sigma^2)^{3/2} < 0,$$

so that θ is a decreasing function of Σ on the interval indicated. Its maximum value will therefore occur when $\Sigma = 2 - N$, and this maximum value will be

$$\theta(2 - N) = N(N - 2)/(N - 1)^{\frac{1}{2}}.$$

The value $\Sigma e = 2 - N$ occurs only when one of the e_i, say e_1, is equal to $+1$ and all the rest are equal to -1. Then we find from (3) and (2) that

$$\lambda = 3(N - 2)/2(N - 1)^{\frac{1}{2}},$$

$$x_1 = (N - 1)^{\frac{1}{2}}, \qquad x_2 = x_3 = \cdots = x_N = -(N - 1)^{-\frac{1}{2}}, \tag{4}$$

$$\alpha_3 = f(x)/N = (N - 2)/(N - 1)^{\frac{1}{2}}.$$

Since the maximum value of α_3 given by (4) approaches ∞ with N, it follows that there is no universal upper bound for α_3. More precisely, the quantity α_3 can be made as large as desired by choosing N large enough and then picking x_i as in the last paragraph. Since there is no universal upper bound for α_3, it is clear from (1) that there is no universal upper bound for α_4. It would probably be possible, although rather difficult, to derive an explicit bound for α_4 as a function of N by using the methods employed above for α_3.