NOTE ON A LEMMA

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In a previous paper on the power function of the analysis of variance test[1], the author stated the following lemma (designated there as Lemma 2):

Lemma 2. *Let $v_1, \cdots, v_k$ be k normally and independently distributed variates with a common variance σ^2. Denote the mean value of v_i by α_i ($i = 1, \cdots, k$) and let $f(v_1, \cdots, v_k, \sigma)$ be a function the variables $v_1, \cdots, v_k$ and σ which does not involve the mean values $\alpha_1, \cdots, \alpha_k$. Then, if the expected value of $f(v_1, \cdots, v_k, \sigma)$ is equal to zero, $f(v_1, \cdots, v_k, \sigma)$ is identically equal to zero, except perhaps on a set of measure zero.*

In the paper mentioned above it was intended to state this lemma for bounded functions $f(v_1, \cdots, v_k)$ and the lemma was used there only in a case where $f(v_1, \cdots, v_k)$ is bounded. Through an oversight this restriction on $f(v_1, \cdots, v_k)$ was not stated explicitly.[2] The published proof of Lemma 2 is adequate if $f(v_1, \cdots, v_k)$ is assumed to be bounded. From the fact that the moments of a multivariate normal distribution determine uniquely the distribution it is concluded there that if for any set $(r_1, \cdots, r_k)$ of non-negative integers

$$(1) \qquad \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} v_1^{r_1} \cdots v_k^{r_k} f(v_1, \cdots, v_k) e^{-\frac{1}{2}\Sigma(v_i-\alpha_i)^2}\, dv_1 \cdots dv_k = 0$$

identically in the parameters $\alpha_1, \cdots, \alpha_k$ then $f(v_1, \cdots, v_k)$ must be equal to zero except perhaps on a set of measure zero. This conclusion is obvious if $f(v_1, \cdots, v_k)$ is bounded. In fact, from (1) and the boundedness of $f(v_1, \cdots, v_k)$ it follows that there exists a finite value A such that

$$\varphi(v_1, \cdots, v_k) = \frac{1}{(2\pi)^{k/2}} \left[1 - \frac{1}{A} f(v_1, \cdots, v_k)\right] e^{-\frac{1}{2}\Sigma(v_i-\alpha_i)^2}$$

is a probability density function with moments equal to those of the normal distribution

$$\psi(v_1, \cdots, v_k) = \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2}\Sigma(v_i-\alpha_i)^2}.$$

Hence $f(v_1, \cdots, v_k)$ must be equal to zero except perhaps on a set of measure zero. However, this conclusion is not so immediate if no restriction is imposed on $f(v_1, \cdots, v_k)$ except that

$$(2) \qquad \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |f(v_1, \cdots, v_k)|\, e^{-\frac{1}{2}\Sigma(v_i-\alpha_i)^2}\, dv_1 \cdots dv_k < \infty$$

for all values of the parameters $\alpha_1, \cdots, \alpha_k$. It is the purpose of this note to prove this. In other words, we shall prove the following proposition:

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[1] A. Wald, "On the power function of the analysis of variance test," *Annals of Math. Stat.*, Vol. 13 (1942), pp. 434.

[2] I wish to thank Prof. J. Neyman for calling my attention to this omission.

PROPOSITION I. *If* (2) *holds for all values of the parameters* $\alpha_1, \cdots, \alpha_k$ *and if for any set* $(r_1, \cdots, r_k)$ *of non-negative integers equation* (1) *holds identically in* $\alpha_1, \cdots, \alpha_k$, *then* $f(v_1, \cdots, v_k)$ *must be equal to zero except perhaps on a set of measure zero.*

On the basis of Proposition I and the arguments given on p. 438 of the paper mentioned before, it can be seen that restriction (2) on the function $f(v_1, \cdots, v_k)$ is sufficient for the validity of Lemma 2.

To prove Proposition I, we shall first show that the following lemma holds.

LEMMA A. *If* $h(v_1, \cdots, v_k)$ *is a probability density function and if*

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h(v_1, \cdots, v_k) e^{\delta \sum_i |v_i|} dv_1 \cdots dv_k < \infty \tag{3}$$

for some $\delta > 0$, *then the problem of moments is determined for the moments of the distribution* $h(v_1, \cdots, v_k)$.

This lemma was proved by G. H. Hardy for $k = 1$.[3] I shall prove it for $k > 1$. Since

$$\sum_{n=0}^{\infty} \frac{\delta^{2n}(\sum_i |v_i|)^{2n}}{(2n)!} < e^{\delta \Sigma |v_i|} \tag{4}$$

we obtain from (3)

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h(v_1, \cdots, v_k) \left[\sum_{n=0}^{\infty} \frac{\delta^{2n}(\sum |v_i|)^{2n}}{(2n)!}\right] dv_1 \cdots dv_k < \infty. \tag{5}$$

Hence

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h(v_1, \cdots, v_k) \left[\sum_{n=0}^{\infty} \frac{\delta^{2n}\left(\sum_{i=1}^{k} v_i^{2n}\right)}{(2n)!}\right] dv_1 \cdots dv_k < \infty. \tag{6}$$

Denote the $2n$th moment of v_i by $\mu_{2n}^{(i)}$. Because of (3) the moments $\mu_{2n}^{(i)}$ are finite. Furthermore, denote $\sum_{i=1}^{k} \mu_{2n}^{(i)}$ by λ_{2n}. Then we obtain from (6)

$$\sum_{n=0}^{\infty} \frac{\delta^{2n}\lambda_{2n}}{(2n)!} < \infty. \tag{7}$$

From (7) it follows that

$$\limsup_{n=\infty} \frac{\delta^{2n}\lambda_{2n}}{(2n)!} < 1. \tag{8}$$

Hence

$$\limsup_{n=\infty} \left(\frac{\delta^{2n}\lambda_{2n}}{(2n)!}\right)^{1/2n} \leq 1. \tag{9}$$

[3] See for instance, SHOHAT and TAMARKIN, "The problem of moments," *Math. Surveys No.* 1, *Amer. Math. Soc.*, New York, 1943, p. 20.

Since according to Stirling's formula

$$\lim_{n=\infty} (2n)!/(2n)^{2n}e^{-2n}\sqrt{4\pi n} = 1$$

we obtain from (9)

$$\limsup_{n=\infty} \frac{\delta\lambda_{2n}^{1/2n}}{2ne^{-1}} \leq 1. \tag{10}$$

Taking reciprocals we obtain

$$\liminf_{n=\infty} \frac{2n\lambda_{2n}^{-1/2n}}{e\delta} \geq 1 \tag{11}$$

or

$$\liminf_{n=\infty} n\lambda_{2n}^{-1/2n} \geq \frac{e\delta}{2} > 0. \tag{12}$$

But (12) implies the existence of a positive value ρ so that

$$\lambda_{2n}^{-1/2n} \geq \frac{\rho}{n} \qquad (n = 1, 2, \cdots, \text{ad inf.}) \tag{13}$$

From (13) it follows that

$$\sum_{n=1}^{\infty} \lambda_{2n}^{-1/2n} = \infty. \tag{14}$$

But (14) is Carleman's sufficient condition for the determinateness of the problem of moments. Hence Lemma A is proved.

On the basis of Lemma A we can prove Proposition I as follows: From (2) we obtain

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |f(v_1, \cdots, v_k)| e^{-\frac{1}{2}\Sigma v_i^2 + \Sigma \alpha_i v_i} dv_1 \cdots dv_k < \infty \tag{15}$$

for all values $\alpha_1, \cdots, \alpha_k$. Let $f_1(v) = f(v)$ for all points $v = (v_1, \cdots, v_k)$ for which $f(v) \geq 0$, and $f_1(v) = 0$ for all points v for which $f(v) < 0$. Similarly, let $f_2(v) = -f(v)$ for all points v for which $f(v) \leq 0$, and $f_2(v) = 0$ for all points v for which $f(v) > 0$. Then $f_1(v)$ and $f_2(v)$ are non-negative functions and

$$f(v) = f_1(v) - f_2(v). \tag{16}$$

From (15) it follows that

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_1(v) e^{-\frac{1}{2}\Sigma v_i^2 + \Sigma \alpha_i v_i} dv_1 \cdots dv_k < \infty \tag{17}$$

and

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_2(v) e^{-\frac{1}{2}\Sigma v_i^2 + \Sigma \alpha_i v_i} dv_1 \cdots dv_k < \infty. \tag{18}$$

Let

$$f_j^*(v) = f_j(v)e^{-\frac{1}{2}\Sigma v_i^2} \qquad (j = 1, 2). \tag{19}$$

Now we shall show that for any positive values $\beta_1, \cdots, \beta_k$

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_j^*(v_1, \cdots, v_k)e^{\beta_1|v_1|+\cdots+\beta_k|v_k|}\, dv_1 \cdots dv_k < \infty. \tag{20}$$

In fact, consider the 2^k sets $(a_1, \cdots, a_k)$ where $a_i = \pm 1$ $(i = 1, \cdots, k)$. Denote by $R_{a_1 \cdots a_k}$ the subset of the k-dimensional Cartesian space which consists of all points $v = (v_1, \cdots, v_k)$ for which v_i is either zero or signum $v_i =$ signum a_i $(i = 1, \cdots, k)$. Putting $\alpha_i = a_i\beta_i$, it follows from (17) and (18) that

$$\int_{R_{a_1 \cdots a_k}} f_j^*(v_1, \cdots, v_k)e^{\beta_1|v_1|+\cdots+\beta_k|v_k|}\, dv_1 \cdots dv_k < \infty. \tag{21}$$

Since (21) holds for any of the 2^k sets $R_{a_1 \cdots a_k}$, equation (20) is proved.

From (1) it follows that

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} v_1^{r_1} \cdots v_k^{r_k}[f_1^*(v_1, \cdots, v_k) - f_2^*(v_1, \cdots, v_k)]\, dv_1 \cdots dv_k = 0,$$

for all non-negative integers $r_1, \cdots, r_k$. Hence, because of (21) and Lemma A we see that

$$f_1^*(v_1, \cdots, v_k) = f_2^*(v_1, \cdots, v_k), \tag{22}$$

except perhaps on a set of measure zero. From (22) it follows that

$$f(v_1, \cdots, v_k) = f_1(v_1, \cdots, v_k) - f_2(v_1, \cdots, v_k) = 0,$$

except perhaps on a set of measure zero. Hence Proposition I is proved.

A NOTE ON SKEWNESS AND KURTOSIS

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It is the purpose of §1 of this paper to prove the following inequality:

$$\alpha_4 \geqq \alpha_3^2 + 1. \tag{1}$$

This inequality seems to have first been stated by Pearson[1]. The inequality also follows from a result appearing in the thesis of Vatnsdal. Here we give a proof based on the theory of quadratic forms which seems to be more direct and more elementary than either of the previous proofs.

[1] "Mathematical contributions to the theory of evolution, XIX; second supplement to a memoir on skew variation," *Phil. Trans. Roy. Soc. (A)*, Vol. 216 (1916), p. 432.