If $\sigma(t)$ is such that $dt/d\sigma$ can be found as an explicit function of σ, then (16) can be written with advantage as

$$E(R) = \int_0^1 \{1 - \sigma^n - (1 - \sigma)^n\} \frac{dt}{d\sigma} \, d\sigma. \tag{17}$$

For example, suppose the random variable Y has the probability density function

$$\varphi(y) = \frac{e^y}{(1 + e^y)^2}, \tag{18}$$

and hence the c. d. f.

$$\sigma(y) = \frac{e^y}{1 + e^y}. \tag{19}$$

Then

$$t = \log \frac{\sigma}{1 - \sigma}, \qquad \frac{dt}{d\sigma} = \frac{1}{\sigma(1 - \sigma)}. \tag{20}$$

Hence from (17), the expected value of the range in a sample of n is

$$E(R) = \int_0^1 \frac{1 - \sigma^n - (1 - \sigma)^n}{\sigma(1 - \sigma)} \, d\sigma. \tag{21}$$

The indicated division in the integrand may be carried out, and the result, a polynomial in σ of degree $\leq (n - 2)$, when integrated between 0 and 1, gives an explicit formula for $E(R)$. Thus for samples of $n = 2, 3, 4$ we find the values of $E(R)$ to be respectively 2, 3, 11/3. Incidentally, it is always true that the expected value of the range for $n = 3$ is three-halves that for $n = 2$. This follows from (16) and the algebraic identity

$$\{1 - \sigma^3 - (1 - \sigma)^3\} = \tfrac{3}{2}\{1 - \sigma^2 - (1 - \sigma)^2\}. \tag{22}$$

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[1] H. E. Robbins, "On the measure of a random set," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 70–74.

ON RELATIVE ERRORS IN SYSTEMS OF LINEAR EQUATIONS

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Some time ago in these *Annals*[1], L. B. Tuckerman discussed the effect of relative coefficient errors on relative solution errors for a non-singular linear algebraic system; his discussion was confined to errors so small that their squares and higher powers can be neglected. Dr. Tuckerman's paper was principally concerned

[1] L. B. Tuckerman, "On the mathematically significant figures in the solution of simultaneous linear equations," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 307–316.

with the important problem of limiting errors incurred while solving the system, and has suggested the desirability of a non-infinitesimal treatment of relative errors. Such a treatment follows; the method is a variant of that used in a paper on absolute errors[2]. The computations provide (1) a criterion for allowable relative errors in the coefficients to assure non-vanishing of the determinant; (2) a bound (subject to this criterion) for the relative error in each solution; (3) a specification of accuracy in the coefficients to assure a pre-assigned accuracy in the solution.

We consider a system of linear equations

$$(1) \qquad \sum_{i=1}^{n} a_{ij} x_j = c_i \qquad i = 1, 2, \cdots, n,$$

where none of the following vanish: the $n(n + 1)$ coefficients a_{ij}, c_i; the determinant Δ of the system; and the n solution-components x_i. Under these conditions it is possible to speak of "relative errors" in the a's, c's, and x's. Let ϵ_{ij}, σ_i be the relative errors in a_{ij} and c_i respectively, so that a_{ij} is perturbed to $a_{ij}(1 + \epsilon_{ij})$; c_i to $c_i(1 + \sigma_i)$. We inquire as to limitations on ϵ_{ij} and σ_i which will permit solution for $x_j(1 + \rho_j)$ of the system

$$(2) \qquad \sum_{j=1}^{n} a_{ij}(1 + \epsilon_{ij}) x_j (1 + \rho_j) = c_i(1 + \sigma_i) \qquad i = 1, 2, \cdots, n,$$

where ρ_j is the relative error induced in x_j; and we seek to limit $| \rho_j |$ in terms of the ϵ's and σ's. We shall assume that for all i, j

$$(3) \qquad | \epsilon_{ij} |, \; | \sigma_i | < \delta,$$

where δ will be suitably restricted later.

Combining (1) and (2) we get

$$(4) \qquad \sum_{j=1}^{n} a_{ij} x_j \rho_j = c_i \sigma_i - \sum_{j=1}^{n} a_{ij} \epsilon_{ij} x_j - \sum_{j=1}^{n} a_{ij} \epsilon_{ij} x_j \rho_j \qquad i = 1, 2, \cdots, n,$$

Since by hypothesis the determinant Δ of (1) is not zero, matrix $A = (a_{ij})$ has the inverse $A^{-1} = (b_{ij}) = (A_{ji}/\Delta)$, where A_{ij} is the cofactor of a_{ij} in Δ. Multiplying (4) by b_{ki} and summing on i we get

$$(5) \qquad x_k \rho_k = \sum_{i=1}^{n} b_{ki} c_i \sigma_i - \sum_{i=1}^{n} b_{ki} \sum_{j=1}^{n} a_{ij} \epsilon_{ij} x_j - \sum_{i=1}^{n} b_{ki} \sum_{j=1}^{n} a_{ij} \epsilon_{ij} x_j \rho_j \qquad k = 1, 2, \cdots, n,$$

and by (3)

$$(6) \qquad \rho \leqq \delta(M_k + N_k | \rho_k |),$$

[2] A. T. Lonseth, "Systems of linear equations with coefficients subject to error," *Annals of Math. Stat.*, Vol. 13 (1942), pp. 332–337.

where ρ is the greatest $|\rho_k|$, and

$$M_k = \frac{1}{|x_k|}\sum_{i=1}^{n}|b_{ki}|\left(|c_i| + \sum_{j=1}^{n}|a_{ij}x_j|\right), \tag{7}$$

$$N_k = \frac{1}{|x_k|}\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ki}a_{ij}x_j|, \tag{8}$$

so that

$$M_k = N_k + \frac{1}{|x_k|}\sum_{i=1}^{n}|b_{ki}c_i|.$$

Denote by M, N the maximum values of M_k, N_k respectively over $k = 1, 2, \cdots, n$. From (6),

$$\rho \leqq \delta(M + N\rho),$$

whence, if

$$\delta < 1/N, \tag{9}$$

it follows that

$$\rho \leqq \delta M/(1 - \delta N), \tag{10}$$

which of course bounds each individual $|\rho_k|$, though rather crudely. To bound $|\rho_k|$ more genuinely in terms of δ, M_k, N_k, M and N it remains only to use this inequality with (6):

$$|\rho_k| \leqq \delta(M_k + \delta MN_k/(1 - \delta N)), \qquad k = 1, 2, \cdots, n; \tag{11}$$

with M_k, N_k as given in (7) and (8).

If (9) holds, then, it follows that $|\rho_k|$ is bounded by (11)—if ρ_k exists. This essential point can be established by solving (5) for ρ_k by iteration[2]; (9) is a sufficient condition for convergence of the resulting series, and hence for non-singularity of the perturbed matrix $(a_{ij} + \epsilon_{ij}a_{ij})$.

In order to be sure that $|\rho_k| \leqq \eta$, a pre-assigned number, for all k, it suffices by (10) to choose δ so that

$$\delta M/(1 - \delta N) \leqq \eta,$$

whence

$$\delta \leqq \eta/(M + N\eta).$$

A less simple inequality whose satisfaction by δ will guarantee that $|\rho_k| \leqq \eta_k$ follows from (11), namely

$$\delta \leqq (A - B)/2C,$$

where $A = \{(M_k - N\eta_k)^2 + 4MN_k\eta_k\}^{\frac{1}{2}}$, $B = M_k + N\eta_k$, $C = |MN_k - NM_k|$.