# RANDOM WALK IN THE PRESENCE OF ABSORBING BARRIERS

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**1. Introduction.** The problem of random walk (along a straight line) in the presence of absorbing barriers can be stated as follows:

A particle, starting at the origin, moves in such a way that its displacements in consecutive time intervals, each of duration $\Delta t$, can be represented by independent random variables

$$X_1, X_2, X_3, \cdots$$

Moreover, if at some time the total (cumulative) displacement becomes $>p$ ($p \geq 0$) or $< -q$ ($q \geq 0$) the particle gets absorbed. The problem is to determine the probability that "the length of life" of the particle is greater than a given number $t$. This problem also admits an interpretation in terms of a game of chance in which the player quits when he loses more than $q$ or wins more than $p$. An interesting paper on this type of problem by A. Wald[1] appeared recently in the *Annals*. Wald assumes that the $X$'s are identically distributed and that their mean and standard deviation are different from 0.[2] He is then mostly interested in the limiting case when both the mean and the standard deviation become small. The object of this paper is to propose a different method of attack which in some cases leads to an answer in closed form. The method we use has been employed repeatedly in statistical mechanics in the study of the so called order-disorder problem. It is due, I believe, to E. W. Montroll[3]. As far as the author knows this method was never used in connection with the classical probability theory and this seems to furnish an additional reason for publishing this paper.

**2. The simplest discrete case.** We assume that each $X$ is capable of assuming the values 1 and $-1$ each with probability $\frac{1}{2}$, and for simplicity sake we let $\Delta t = 1$. Note that, unlike in Wald's case, the mean of $X$ is 0. Denote by $N$ the random variable which represents the "length of life" of the particle and let ($m$ an integer)

$$\delta(m) = \begin{matrix} \frac{1}{2} & m = 1 \quad \text{or} \quad m = -1, \\ 0 & \text{otherwise.} \end{matrix}$$

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[1] A. Wald "On cumulative sums of random variables," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 283–296.

[2] Since this was written Professor Wald informed the author that he can easily avoid the condition that the mean should be zero.

[3] See for instance E. W. Montroll, "Statistical Mechanics of nearest neighbor systems," *Jour. of Chem. Physics*, Vol. 9 (1941), pp. 706–721.

Clearly we have (throughout this section we assume that both $p$ and $q$ are integers)

$$\text{Prob. } \{N > n\} = \text{Prob. } \{-q \le X_1 \le p, -q \le X_1 + X_2 \le p, \cdots, -q \le X_1 + \cdots + X_b \le p\} = \Sigma\delta(m_1)\delta(m_2) \cdots \delta(m_n),$$

where the summation is extended over all integers $m_1, m_2, \cdots m_n$ for which

$$-q \le m_1 \le p, \quad -q \le m_1 + m_2 \le p, \cdots, \quad -q \le m_1 + m_2 + \cdots + m_n \le p.$$

Letting

$$l_j = q + m_1 + \cdots + m_j, \qquad (j = 1, 2, \cdots, n),$$

we see that

$$\text{(1)} \qquad \text{Prob } \{N > n\} = \sum_{l_1, \cdots, l_n = 0}^{p+q} \delta(l_1 - q)\delta(l_2 - l_1) \cdots \delta(l_n - l_{n-1}).$$

Let us now consider the $(p + q + 1)$ by $(p + q + 1)$ matrix

$$\text{(2)} \qquad A = ((\delta(i - k))) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It is easily seen that the sum in (1) is equal to the sum of the elements in the $(q + 1)$-st column (or row) of the matrix $A^n$. Thus

Prob. $\{N > n\}$ = sum of the elements of the $(q + 1)$-st column of $A^n$.

Denote by $\lambda_1, \lambda_2, \cdots \lambda_{p+q+1}$ the eigenvalues of the matrix $A$ and let

$$(x_1^{(j)}, x_2^{(j)}, \cdots, x_{p+q+1}^{(j)})$$

be the normalized eigenvector of $A$ belonging to the eigenvalue $\lambda_j$. It can be shown by elementary means[4] that

$$\lambda_j = \cos \frac{\pi j}{p + q + 2}$$

[4] Matrices of type (2) have been introduced and studied in various connections. In a paper by R. P. Boas and the present author recently accepted by the *Duke Mathematical Journal* references to several authors are given. In order to find the eigenvalues and the eigenvectors of (2) it suffices to know that

$$\begin{vmatrix} 1 & a & 0 & \cdots & \cdots \\ a & 1 & a & \cdots & \cdots \\ 0 & a & 1 & a & \cdots \\ 0 & 0 & a & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \frac{\rho_1^{m+1} - \rho_2^{m+1}}{\rho_1 - \rho_2},$$

where $m$ is the order of the matrix $\rho_1$ and $\rho_2$ roots of the equation $\rho^2 - \rho + a^2 = 0$.

and

$$x_k^{(j)} = \frac{\sqrt{2}}{\sqrt{p+q+2}} \sin \frac{\pi jk}{p+q+2}.$$

Denoting by $R$ the orthogonal matrix

$$\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_{p+q+1}^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_{p+q+1}^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{(p+q+1)} & x_2^{(p+q+1)} & \cdots & x_{p+q+1}^{(p+q+1)} \end{pmatrix}$$

and by $R'$ the transposed of $R$ we have (since the eigenvalues of $A$ are simple) by a well known theorem

$$A^n = R' \begin{pmatrix} \lambda_1^n & & & 0 \\ & \lambda_2^n & & \\ & & \ddots & \\ 0 & & & \lambda_{p+q+1}^n \end{pmatrix} R.$$

It thus follows by an easy computation that the sum of the elements of the $(q+1)$-st column (row) of $A^n$ is

$$\sum_{r=1}^{p+q+1} \sum_{j=1}^{p+q+1} \lambda_j^n x_n^{(j)} x_{q+1}^{(j)} = \sum_{j=1}^{p+q+1} \lambda_j^n x_{q+1}^{(j)} \left( \sum_{r=1}^{p+q+1} x_r^{(j)} \right).$$

We have

$$\sum_{r=1}^{p+q+1} x_r^{(j)} = \frac{\sqrt{2}}{\sqrt{p+q+2}} \sum_{r=1}^{p+q+1} \sin \frac{\pi jr}{p+q+2}$$

$$= \begin{cases} 0, & j \text{ even}, \\ \dfrac{\sqrt{2}}{\sqrt{p+q+2}} \cot \dfrac{\pi j}{2(p+q+2)}, & j \text{ odd}, \end{cases}$$

and therefore[5]

Prob. $\{M > n\}$

$$= \frac{2}{p+q+2} \sum_{j=1}^{p+q+1}{}^{*} \cos^n \frac{\pi j}{p+q+2} \sin \frac{\pi j(q+1)}{p+q+2} \cot \frac{\pi j}{2(p+q+2)},$$

where the star on the summation sign indicates that only odd $j$'s are taken under account.

The method just illustrated is quite general but in more complicated cases the job of finding the eigenvalues and eigenvectors becomes formidable.

[5] Professor Feller has called the author's attention to the fact that similar problems and formulas can be found in Chapter III of W. Burnside's *Theory of Probability* (Cambridge, 1928). He also pointed out that the problem could be treated by means of Markoff chains.

Professor G. E. Uhlenbeck has pointed out that our formula implies a known result from the theory of Brownian motion.

Consider a free Brownian particle which at $t = 0$ is at $x = x_0 (x_0 > 0)$. R. Fürth[6] has shown that the probability that between $t$ and $t + dt$ the particle will be either at $x = 0$ or at $x = d$ $(0 < x_0 < d)$ for the first time, is given by the formula

$$dt \frac{4\pi D}{d^2} \sum_{m=0}^{\infty} (2m + 1) e^{(-\pi^2 Dt/d^2)(2m+1)^2} \sin \frac{(2m + 1)\pi x_0}{d},$$

where $D$ is the "coefficient of diffusion."

If we treat the one-dimensional Brownian motion as a random walk with steps $\pm \Delta x$, each move lasting $\Delta t$, the probability that a particle starting from $x_0$ will not have reached 0 or $d$ in the time interval $(0, t)$ can be calculated by means of our formula.

We must only put $q = x_0/\Delta x$, $p = (d - x_0)/\Delta x$, $n = t/\Delta t$ and assume that as both $\Delta x$ and $\Delta t$ approach 0 the ratio $(\Delta x)^2/2\Delta t$ approaches the "coefficient of diffusion" $D$.

An elementary computation shows that in this limit the Prob. $\{N > t/\Delta t\}$ approaches

$$\frac{4}{\pi} \sum_{j=1}^{\infty}{}^{*} \frac{1}{j} e^{(-\pi^2 j^2 D/d^2)t} \sin \frac{\pi j x_0}{d}$$

and that the differential of this expression (with a minus sign) gives exactly Fürth's expression.

**3. General theory in the continuous case.** We now assume that the distribution function of $X$ possesses a continuous and even density function $\rho(x)$. We have

$$\text{Prob. } \{N > n\} = \int \cdots \int_{\Omega} \rho(x_1) \cdots \rho(x_n)\, dx_1 \cdots dx_n,$$

where the region of integration $\Omega$ is defined by the inequalities

$$-q \leq x_1 \leq p, \qquad -q \leq x_1 + x_2 \leq p, \cdots, \qquad -q \leq x_1 + \cdots + x_n \leq p$$

Introducing the new variables

$$y_j = q + x_1 + \cdots + x_j, \qquad (j = 1, 2, \cdots, n),$$

we see that the Jacobian of the transformation is 1 and

$$\text{Prob. } \{N > n\}$$

$$(3) \qquad = \int_0^{p+q} \cdots \int_0^{p+q} \rho(y_1 - q)\rho(y_2 - y_1) \cdots \rho(y_n - y_{n-1})\, dy \cdots dy_n.$$

Consider the symmetric integral equation

$$(4) \qquad \int_0^{p+q} \rho(s - t) f(t)\, dt = \lambda f(s)$$

[6] *Ann. d. Phys.* 53 (1917) p. 177.

and note that if $K_n(s, t)$ denotes the $n$-th iterated kernel of this integral equation, the right side of (3) is equal to

$$\int_0^{p+q} K_n(q, t)\, dt.$$

Thus

$$\text{Prob. } \{N > n\} = \int_0^{p+q} K_n(q, t)\, dt.$$

From the general theory of integral equations we know that

$$K_n(s, t) = \sum_{j=1}^{\infty} \lambda_j^n f_j(s) f_j(t), \qquad (n \geq 2),$$

where $\lambda_1, \lambda_2, \cdots$ are eigenvalues and $f_1(t), f_2(t), \cdots$ normalized eigenfunctions of the integral equation (4).

Since $\rho$ was assumed to be continuous it follows that the eigenfunctions are continuous and

$$\text{Prob. } \{N > n\} = \sum_{j=1}^{\infty} \lambda_j^n f_j(q) \int_0^{p+q} f_j(t)\, dt.$$

This formula is very general and provides, in a sense, a complete solution of the problem in the continuous and symmetric case. Unfortunately the usefulness of this formula is limited by the difficulties encountered in solving integral equations of the type (4).

In fact, the integral equation

$$\frac{1}{\sqrt{2\pi}} \int_0^a e^{-(s-t)^2/2} f(t)\, dt = \lambda f(s),$$

to which one is led by considering the normally distributed $X$'s, appears to be very difficult to solve.

**4. A particular case.** If we assume

$$\rho(x) = \frac{1}{2} e^{-|x|}$$

we are led to the integral equation

$$(5) \qquad \int_0^{p+q} e^{-|s-t|} f(t)\, dt = 2\lambda f(s),^7$$

which is quite easy to solve.

In fact, rewriting (5) in the form

$$(6) \qquad e^{-s} \int_0^{p+q} e^{t} f(t)\, dt + e^{s} \int_0^{p+q} e^{-t} f(t)\, dt = 2\lambda f(s)$$

[7] I have recently encountered the integral equation (5) in solving an entirely different problem. A complete discussion can be found in a restricted N.D.R.C. Report 14-305.

and differentiating twice with respect to $s$ we obtain the differential equation

$$f''(s) + \left(\frac{1}{\lambda} - 1\right) f(s) = 0.$$

Substituting the general solution of this equation in (6) we find in an entirely elementary fashion that

$$\lambda_j = \frac{1}{1 + y_j^2},$$

$$f_j(t) = \frac{\sin y_j t + y_j \cos y_j t}{\sqrt{1 + \frac{1}{2}(p + q)(1 + y_j^2)}},$$

where $y_j$ is the $j$th (positive) root of the transcendental equation

$$\tan (p + q)y = -\frac{2y}{1 - y^2}. \tag{7}$$

We have

$$\int_0^{p+q} (\sin y_j t + y_j \cos y_j t)\, dt = \frac{1}{y_j} \{1 - \cos (p + q)y_j + y_j \sin (p + q)y_j\}$$

and it is easily seen that (7) implies

$$1 - \cos (p + q)y_j + y_j \sin (p + q)y_j = \begin{cases} 0 & \text{if } \cos (p + q)y_j = \dfrac{1 - y_j^2}{1 + y_j^2}, \\ 2 & \text{if } \cos (p + q)y_j = -\dfrac{1 - y_j^2}{1 + y_j^2}. \end{cases}$$

Finally,

$$\text{Prob. } \{N > n\} = 2 \sum_{j=1}^{\infty}{}' \frac{1}{(1 + y_j^2)^n} \frac{\sin y_j q + y_j \cos y_j q}{y_j \{1 + \frac{1}{2}(p + q)(1 + y_j^2)\}},$$

where the dash on the summation sign indicates that only those $j$'s are taken under account for which

$$\cos (p + q)y_j = -\frac{1 - y_j^2}{1 + y_j^2}.$$

We omit here the discussion of various limiting cases inasmuch as our main purpose was to obtain exact formulas.

There are indications that some of the limiting cases are related to singular integral equations with continuous spectra. We may return to this subject at a later date.