THE APPROXIMATE DISTRIBUTIONS OF THE MEAN AND VARIANCE OF A SAMPLE OF INDEPENDENT VARIABLES

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1. Introduction. In this paper we shall study the mean and variance of a large number, n (a sample of size n) of mutually independent random variables:

$$\xi_1, \xi_2, \cdots, \xi_n, \tag{1}$$

having the same probability distribution represented by a (cumulative) distribution function $P(x)$. The rth moment, absolute moment, and semi-invariant of $P(x)$ are denoted by α_r, β_r, and γ_r respectively. It is assumed that for a certain integer $k \geq 3$, $\beta_k < \infty$ and that $\alpha_2 > 0$. Hence there is no loss of generality in assuming that

$$\alpha_1 = 0, \qquad \alpha_2 = 1. \tag{2}$$

The characteristic function corresponding to $P(x)$ is denoted by $p(t)$.

We put

$$\bar{\xi} = \frac{1}{n}\sum_{r=1}^{n} \xi_r, \qquad \eta = \frac{1}{n}\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2 \tag{3}$$

$$F(x) = Pr\{\sqrt{n}\,\bar{\xi} \leq x\}, \qquad G(x) = Pr\left\{\frac{\sqrt{n}(\eta - 1)}{\sqrt{\alpha_4 - 1}} \leq x\right\}. \tag{4}$$

The definition of $G(x)$ implies that $\alpha_4 < \infty$ and $\alpha_4 - 1 > 0$. The case $\alpha_4 - 1 = 0$ provides an easy degenerated case which will be treated separately (section 4).

Cramér's theorem of asymptotic expansion[1] reads as follows:

THEOREM 1. *If $P(x)$ is non-singular and if $\beta_k < \infty$ for some integer $k \geq 3$, then*

$$F(x) = \Phi(x) + \Psi(x) + R(x) \tag{5}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy. \tag{6}$$

$\Psi(x)$ is a certain linear combination of successive derivatives $\Phi^{(3)}(x), \cdots, \Phi^{(3(k-3))}(x)$ with each coefficient of the form $n^{-\frac{1}{2}\nu}$ times a quantity depending only on $k, \alpha_3, \cdots, \alpha_{k-1}$ $(1 \leq \nu \leq k-3)$ and

$$|R(x)| \leq Q/n^{\frac{1}{2}(k-2)} \tag{7}$$

where Q is a constant depending only on k and $P(x)$.

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[1] H. CRAMÉR: *Random Variables and Probability Distributions* (1937), Ch. 7. This book will be referred to as (C).

In particular, putting $k = 3$ we get that $| F(x) - \Phi(x) | \leq Qn^{-\frac{1}{2}}$ provided $P(x)$ is non-singular and $\beta_3 < \infty$. If the condition of non-singularity of $P(x)$ be removed, then Liapounoff's theorem[2] furnishes the weaker result: $| F(x) - \Phi(x) | \leq A\beta_3 n^{-\frac{1}{2}} \log n$ where A is a numerical constant.

Very recently Berry[3] succeeded in removing the factor $\log n$ from Liapounoff's theorem under no other condition than that $\beta_3 < \infty$. We state here Berry's theorem:

THEOREM 2. *If* $\beta_3 < \infty$, *then*

$$|F(x) - \Phi(x)| \leq \frac{A\beta_3}{\sqrt{n}} \tag{8}$$

where A *is a numerical constant.*

An essential step in the proof of these results is the selection of a weighting function $w(x)$ and the appraisal of the integral

$$\int_{-\infty}^{\infty} w(u)\{F(u + x) - \Phi(u + x) - \Psi(u + x)\} \, du \tag{9}$$

($\Psi \equiv 0$ when $k = 3$). In his book[1] Cramér proves Theorem 1 by taking $w(u) = \frac{1}{\Gamma(\omega)} (-u)^{\omega-1}$ when $u < 0$ and $w(u) = 0$ when

$$u \geq 0 \quad (0 < \omega < 1) \tag{10}$$

and proves Liapounoff's theorem by taking

$$w(u) = \frac{1}{\sqrt{2\pi}\epsilon} e^{-u^2/2\epsilon^2}. \tag{11}$$

On the other hand, Berry uses the following weighting function in his proof of Theorem 2:

$$w(u) = \frac{1 - \cos Tu}{u^2}. \tag{12}$$

The unfortunate selection of the function (11) accounts for the presence of the factor $\log n$ in Liapounoff's theorem.

Now Cramér's proof of Theorem 1, based on the integral (9) with $w(u)$ defined in (10), makes use of a result on that integral due to M. Riesz. A more elementary proof than this can be devised. In fact, one has only to use, with Berry, the function (12) and to adopt his elementary appraisal[4] of the integral

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[2] (C), Ch. 7.

[3] A. C. BERRY: "The accuracy of the Gaussian approximation to the sum of independent variates." *Trans. Amer. Math. Soc.*, Vol. 49 (1941), pp. 122–136. This paper will be referred to as (B).

[4] Berry proves the inequality (in our notation):

$$\left| \int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x + a) - \Phi(x + a)\} \, dx \right| \leq \int_0^T \frac{(T - t) \, |f(t) - e^{-\frac{1}{2}t^2}|}{t} \, dt$$

(9) in order to obtain the proof of Theorem 1. One of our purposes is therefore to give an elementary proof of Theorem 1, without reference to the above-mentioned result due to M. Riesz. Section 2 is devoted to this work.

We ought to add that Cramér's theorem and Berry's theorem correspond to Theorems 1 and 2 for the case in which the random variables (1) do not follow the same distribution. The proof given in Section 2 is adaptable to these more general theorems when subjected to appropriate modifications; the assumption of a common distribution function for (1) is only made for the sake of convenience.

So much for the known results for the approximate distribution of $\bar{\xi}$. By a purely formal operational method Cornish and Fisher[5] obtain terms of successive approximation to the distribution function of any random variable X with the help of its semi-invariants. It is hardly necessary to emphasize the importance of turning Cornish and Fisher's formal result (asymptotic expansion without appraisal of the remainder) into a mathematical theorem of asymptotic expansion which gives the order of magnitude of the remainder. In this paper we achieve this for the simplest function of (1) next to $\bar{\xi}$, viz. the η in (3). We do not seek to remove the assumption of a common distribution for (1), as there will be no practical significance (e.g. in statistics) of η if the variables (1) do not have the same probability distribution. Section 3 is devoted to the proof of the following theorems:

THEOREM 3. *If* $\alpha_6 < \infty$ *and* $\alpha_4 - 1 - \alpha_3^2 \neq 0$ *(it cannot be negative), then*

$$|G(x) - \Phi(x)| \leq \frac{A}{\sqrt{n}}\left(\frac{\alpha_6}{\alpha_4 - 1 - \alpha_3^2}\right)^{3/2} \tag{13}$$

where A *is a numerical constant.*

THEOREM 4. *Let* $P(x)$ *be non-singular and let* $\alpha_{2k} < \infty$ *for some integer* $k > 3$. *Then*

$$G(x) = \Phi(x) + \chi(x) + R_1(x), \tag{14}$$

where $\Phi(x)$ *is the function* (6), $\chi(x)$ *is a linear combination of the derivatives* $\Phi'(x), \cdots, \Phi^{(3(k-3))}(x)$ *with each coefficient of the form* $n^{-\frac{1}{2}\nu}$ *times a quantity depending only on* k *and* $\alpha_3, \alpha_4, \cdots, \alpha_{2k-2}$, *and*

(B), p. 128. The "appraisal" mentioned here refers to (50) which is contained in B, p. 128. But Berry's appraisal of the integral in the right-hand side of the above inequality is in default. He writes

$$\frac{\epsilon}{6}\int_0^{c/\epsilon}\left(\frac{1.1}{\epsilon} - t\right)t^2 e^{-\frac{1}{2}t^2}\,dt = \frac{1.1}{6}\sqrt{\frac{\pi}{2}} - \frac{\epsilon}{3} - \frac{1}{6}\int_{c/\epsilon}^{\infty}\left\{(1.1 - c)t^2 + c - \frac{2c}{t^2}\right\}e^{-t^2/2}\,dt$$

(B, p. 132, line 3) whilst the last integral ought to be

$$\int_{c/\epsilon}^{\infty}\{(1.1 - c)t^2 + c - 2\epsilon t\}e^{-t^2/2}\,dt.$$

[5] E. A. Cornish and R. A. Fisher: "Moments and cumulants in the specification of distributions." (Revue de l'Institut International de Statistique (1937), pp. 1–14.)

(15) $$|R_1(x)| \leq \frac{Q_k}{n^{\frac{1}{2}(k-2)}} \quad \text{if } k = 4, 5 \text{ or } 6$$

(16) $$|R_1(x)| \leq \frac{Q'_k}{n^{k(k-1)/(2k+3)}} \quad \text{if } k \geq 7$$

where Q_k and Q'_k are constants depending only on k and $P(x)$.

It may be noticed that Theorem 3 is a "Berryian" theorem about $G(x)$, its characteristic feature being the absence of any condition on the distribution function except the two on its moments, and that Theorem 4 is a "Cramerian" theorem about $G(x)$, the characteristic feature being the assumption of non-singularity of $P(x)$ besides that $\alpha_{2k} < \infty$.

In proving these theorems we have devised a method which is applicable to getting similar results about functions other than η, such as functions commonly used in applied statistics: the higher moments about the means, the moment ratios (e.g. K. Pearson's b_1 and b_2), the covariance, the coefficient of correlation, and "Student's" t-statistic. Works on such functions are being done by my university colleagues, and the results will be published shortly.

If ξ is any of the random variables (1), then

$$0 \leq \epsilon\{a(\xi^2 - 1) + b\xi\} = a^2(\alpha_4 - 1) + 2ab\alpha_3 + b^2$$

for all real (a, b). Hence $\alpha_4 - 1 - \alpha_3^2 \geq 0$, and $\alpha_4 - 1 - \alpha_3^2 = 0$ means that there is unit probability that ξ assumes exactly two values. This easily degenerated case is first eliminated in Theorem 3 by the assumption $\alpha_4 - 1 - \alpha_3^2 \neq 0$ and then considered in section 4. In Theorem 4 the condition $\alpha_4 - 1 - \alpha_3^2 \neq 0$ is implied since ξ cannot be a random variable of the nature just described owing to the non-singularity of $P(x)$.

2. Lemmas. Throughout this paper A, B, C, etc. will denote positive numerical constants; A_k, B_k (A_{km}, B_{km}), etc., will denote positive constants depending only on some integer k (integers k and m), and Q_k (Q_{km}) will denote a positive constant depending only on k (k and m) and the distribution function $P(x)$. ϑ, Θ, Θ_k, (Θ_{km}), Λ_k (Λ_{km}) will denote respectively quantities such that $|\vartheta| \leq 1$, $|\Theta| \leq A$, $|\Theta_k| \leq A_k$ ($|\Theta_{km}| \leq A_{km}$), $|\Lambda_k| \leq Q_k$ ($|\Lambda_{km}| \leq Q_{km}$). These symbols do not necessarily stand for the same quantity at each occurrence. Thus $2\vartheta = \Theta$, $k\Theta_k = \Theta_k$ etc. In particular any positive functions of $k, \alpha_3, \cdots, \alpha_k$ is a Q_k.

1.1. Cramér obtains the asymptotic expansion of the characteristic function of the distribution of $\sqrt{n}\bar{\xi}$, viz. $\epsilon(e^{it\sqrt{n}\bar{\xi}})$, when (1) do not have the same distribution, valid for $|t| \leq Q_k n^{1/6}$. Since we assume a common distribution for (1), so that the characteristic function is $\left\{p\left(\frac{t}{\sqrt{n}}\right)\right\}^n$, we are able to derive an asymptotic expansion valid for $|t| \leq Q_k\sqrt{n}$. The extension to $\left\{p\left(\frac{t_1}{\sqrt{n}},\right.\right.$

$\cdots, \frac{t_m}{\sqrt{n}}\bigg)\bigg\}^n$ presents no difficulty. This is done in the following three lemmas, of which Lemma 3 contains the final result.

LEMMA 1.

$$\log p(t) = \sum_{r=2}^{k-1} \frac{\gamma_r (it)^r}{r!} + \Theta_k \beta_k |t|^k, \quad \text{for } |t| \le \beta_k^{1/k}. \tag{17}$$

PROOF: Since $p(t) = 1 + \sum_{r=1}^{k-1} \frac{\alpha_r (it)^r}{r!} + \frac{\vartheta \beta_k |t|^k}{k!} = 1 + q(t)$ say, we have, for $\beta_k^{1/k} |t| \le 1$,

$$q(t) \le \sum_{r=2}^{k} \frac{\beta_r |t|^r}{r!} \le \sum_{r=2}^{k} \frac{(\beta_k^{1/k} |t|)^r}{r!} < \sum_{r=2}^{\infty} \frac{1}{r!} = e - 2 < \frac{3}{4}.$$

Hence

$$\log p(t) = \sum_{1 \le j \le [\frac{1}{2}(k-1)]} (-1)^{j+1} \frac{\{q(t)\}^j}{j} + \Theta |q(t)|^{[\frac{1}{2}(k+1)]}. \tag{18}$$

For $1 \le j \le [\frac{1}{2}(k-1)]$ let us expand each $(-1)^{j+1} j^{-1} \{q(t)\}^j$ to get a polynomial $q_j(t)$ of degree $k - 1$ and a remainder $r_j(t)$. In doing this we regard $q(t)$ formally as a polynomial of degree k in t. For this polynomial we have the majorating relation

$$q(t) \ll e^{\beta_k^{1/k} |t|},$$

whence

$$\frac{(-1)^j}{j} \{q(t)\}^j \ll e^{j \beta_k^{1/k} |t|},$$

which gives

$$|r_j(t)| \le \sum_{r=k}^{\infty} \frac{j^r \beta_k^{r/k} |t|^r}{r!} \le j^k \beta_k |t|^k e^{j \beta_k^{1/k} |t|} \le j^k e^j \beta_k |t|^k \le A_k \beta_k |t|^k. \tag{19}$$

Similarly,

$$|q(t)|^{[\frac{1}{2}(k+1)]} \le A_k \beta_k |t|^k. \tag{20}$$

From (18), (19), (20) we obtain

$$\log p(t) = \sum_{1 \le j \le [\frac{1}{2}(k-1)]} q_j(t) + \Theta_k \beta_k |t|^k. \tag{21}$$

Since the sum in (21) must equal the sum in (17), the Lemma is proved.

LEMMA 2. *Let* $(\zeta_1, \zeta_2, \cdots, \zeta_m)$ *be a random point with* $\epsilon(\zeta_i) = 0$ *and* $\epsilon(|\zeta_i|^k) = \beta_{ki} < \infty$ *for some integer* $k \ge 3$ $(i = 1, \cdots, m)$. *Let* $p(t_1, \cdots, t_m)$ *be the characteristic function. Then for* $|t_i| \le m^{-2+1/k} \beta_{ki}^{-1/k} \sqrt{n}$ $(i = 1, \cdots, m)$ *we have*

$$n \log p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right) = \sum_{r=2}^{k-1} \frac{i^r U_r}{r!\, n^{\frac{1}{2}(r-2)}} + \frac{\Theta_k V_k}{n^{\frac{1}{2}(k-2)}} \tag{22}$$

where U_r and V_r are the rth *semi-invariant and the absolute moment respectively of* $\Sigma t_i\zeta_i$

PROOF: If $|t_i| \leq m^{-2+1/k}\beta_{ki}^{-1/k}\sqrt{n}$, then $V_k^{1/k} \leq m^{(k-1)/k}(\Sigma\beta_{ki}|t_i|^k)^{1/k} \leq m^{(k-1)/k}(\Sigma\beta_{ki}^{1/k}|t_i|) \leq \sqrt{n}$. Since $p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right)$ is the value at $t = \frac{1}{\sqrt{n}}$ of the characteristic function of $\Sigma t_i\zeta_i$, it follows from Lemma 1 that for $\sqrt{n} \geq V_k^{1/k}$ we have (22).

LEMMA 3. *Let $(\zeta_1, \cdots, \zeta_m)$ be a random point with $\epsilon(\zeta_i) = 0$, $\epsilon(\zeta_i^2) = 1$ and $\epsilon(|\zeta_i|^k) = \beta_{ki} < \infty$ for some integer $k \geq 3$. Let $\rho_{ij} = \epsilon(\zeta_i\zeta_j)(\rho_{ii} = 1; i, j = 1, \cdots, m)$ and the matrix $\|\rho_{ij}\|$ be positive definite. Let*

$$\Delta = \text{det.}\,|\rho_{ij}|, \qquad \varphi(t_1, \cdots, t_m) = e^{-\frac{1}{2}\sum_{i,j=1}^{m}\rho_{ij}t_it_j}. \tag{23}$$

Let $p(t_1, \cdots, t_m)$ be the characteristic function. Then there exists a B_{km} such that for $|t_i| \leq \frac{B_{km}\Delta\sqrt{n}}{\beta_{ki}^{3/k}}$ $(i = 1, \cdots, m)$ we have

$$\begin{aligned}\left\{p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right)\right\}^n &= \varphi(t_1, \cdots, t_m)\{1 + \psi(it_1, \cdots, it_m)\} \\ &\quad + \frac{\Theta_{km}}{n^{\frac{1}{2}(k-2)}}\left\{\sum_{i=1}^{m}\beta_{ki}^{3(k-2)/k}(|t_i|^k \right. \\ &\quad \left. + |t_i|^{k+1} + \cdots + |t_i|^{3(k-2)})\right\}e^{-\Delta/4m^{m-1}\sum_{i=1}^{m}t_i^2}\end{aligned} \tag{24}$$

where $\psi(it_1, \cdots, it_m)$ is a polynomial each of whose terms has the form

$$\frac{1}{n^{\nu/2}}a_{\nu_1\cdots\nu_m}(it_1)^{\nu_1}\cdots(it_m)^{\nu_m},$$

with $1 \leq \nu \leq k - 3$, $3 \leq \nu_1 + \cdots + \nu_m \leq 3(k - 3)$, and $a_{\nu_1\cdots\nu_m}$ depending only on k and the moments $\epsilon(\zeta_1^{\mu_1}\cdots\zeta_m^{\mu_m})$, $3 \leq \mu_1 + \cdots + \mu_m \leq k - 1$. If $k = 3$, then $\psi = 0$.

PROOF. If $|t_i| \leq m^{-2+(1/k)}\beta_{ki}^{-3/k}\Delta\sqrt{n}$, then $|t_i| \leq m^{-2+(1/k)}\beta_{ki}^{-1/k}\sqrt{n}$ since $\Delta \leq 1$ and $\beta_{ki} \geq 1$. It follows from Lemma 2 and the fact $U_2 = \Sigma\rho_{ij}t_it_j$ that

$$\begin{aligned}\left\{p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right)\right\}^n &= \varphi(t_1, \cdots, t_m)e^s \\ &= \varphi(t_1, \cdots, t_m)\left\{1 + \sum_{j=1}^{k-3}\frac{s^j}{j!} + \frac{\vartheta|s|^{k-2}e^{|s|}}{(k-2)!}\right\}\end{aligned} \tag{25}$$

where

$$s = \frac{i^3}{\sqrt{n}}\sum_{r=0}^{k-3}\frac{i^r U_{r+3}}{(r+3)!\,n^{r/2}} + \frac{\Theta_k V_k}{n^{\frac{1}{2}(k-2)}}. \tag{26}$$

Regarding s formally as a polynomial in $n^{-\frac{1}{2}}$ let us expand each $(j!)^{-1}s^j$ $(1 \leq j \leq k-3)$ to get a polynomial s_j of degree $k-3$ in $n^{-\frac{1}{2}}$ and a remainder r_j. For the formal polynomial s we have the majorating relation

$$s \ll \frac{A_k}{\sqrt{n}} \sum_{r=0}^{k-3} \frac{V_{r+3}}{r!n^{r/2}} \ll \frac{A_k}{\sqrt{n}} \sum_{r=0}^{k-3} \frac{V_k^{(r+3)/k}}{r!\,n^{r/2}} \ll \frac{A_k V_k^{3/k}}{\sqrt{n}} e^{V_k^{1/k} n^{-\frac{1}{2}}}, \tag{27}$$

whence

$$\frac{1}{j!} s^j \ll A_k \frac{V_k^{3j/k}}{n^{j/2}} e^{jV_k^{1/k} n^{-\frac{1}{2}}},$$

which gives

$$|r_j| \leq \frac{A_k V_k^{3j/k}}{n^{j/2}} \sum_{\nu=k-2-j}^{\infty} \frac{j^\nu V_k^{\nu/k}}{\nu!\,n^{\nu/2}} \leq \frac{A_k V_k^{(k-2+2j)/k}}{n^{\frac{1}{2}(k-2)}} e^{j(V_k^{1/k}/\sqrt{n})}.$$

Since $V_k^{1/k} n^{-\frac{1}{2}} \leq 1$ as shown in the proof of Lemma 2, we have

$$|r_j| \leq \frac{A_k V_k^{(k-2+2j)/k}}{n^{\frac{1}{2}(k-2)}} \leq \frac{A_{km}(\sum_i \beta_{ki} |t_i|^k)^{(k-2+2j)/k}}{n^{\frac{1}{2}(k-2)}}$$

$$\leq \frac{A_{km}(\sum_i \beta_{ki}^{1/k} |t_i|)^{k-2+2j}}{n^{\frac{1}{2}(k-2)}} \leq \frac{A_{km} \sum_i \beta_{ki}^{(k-2+2j)/k} |t_i|^{k-2+2j}}{n^{\frac{1}{2}(k-2)}}.$$

Since $\beta_{ki} \geq 1$ we have $\beta_{ki}^{(k-2+2j)/k} \leq \beta_{ki}^{3(k-2)/k}$. Hence

$$|r_j| \leq \frac{A_{km} \sum_i \beta_{ki}^{3(k-2)/k} |t_i|^{k-2+2j}}{n^{\frac{1}{2}(k-2)}}. \tag{28}$$

Similarly

$$\frac{|s|^{k-2}}{(k-2)!} \leq \frac{A_{km} \sum_i \beta_{ki}^{3(k-2)/k} |t_i|^{3(k-2)}}{n^{\frac{1}{2}(k-2)}}. \tag{29}$$

From (25), (28), (29) we get

$$\left\{p\left(\frac{t_1}{\sqrt{n}}, \cdots, \frac{t_m}{\sqrt{n}}\right)\right\}^n = \varphi(t_1, \cdots, t_m)\left\{1 + \sum_{j=1}^{k-3} s_j + \sum_{j=1}^{k-3} r_j + \frac{\vartheta |s|^{k-2}}{(k-2)!} e^{|s|}\right\}$$

$$= \varphi(t_1, \cdots, t_m) \{1 + \psi(it_1, \cdots, it_m)\}$$

$$+ \frac{\Theta_{km}}{n^{\frac{1}{2}(k-2)}} \{\Sigma\beta_{ki}^{3(k-2)/k}(|t_i|^k + |t_i|^{k+1} + \cdots + |t_i|^{3(k-2)})\} \varphi(t_1, \cdots, t_m)e^{|s|}$$

where $\psi(it_1, \cdots, it_m)$ stands for Σs_j. The assertion about $\psi(it_1, \cdots, it_m)$ announced in the lemma can now be seen without difficulty. It remains to show that with suitable B_{km} in the lemma, we have

$$\varphi(t_1, \cdots, t_m)e^{|s|} \leq e^{-\Delta/4m^{m-1} \sum_{i=1}^m t_i^2}$$

i.e.

$$-\frac{1}{2}\sum_{i,j=1}^{m}\rho_{ij}t_it_j + |s| \leq -\frac{\Delta}{4m^{m-1}}\sum_{i=1}^{m}t_i^2. \tag{30}$$

From (27) we have

$$\begin{aligned}|s| &\leq \frac{A_k}{\sqrt{n}}V_k^{3/k} \leq \frac{A_{km}}{\sqrt{n}}\left(\sum_i \beta_{ki}|t_i|^k\right)^{3/k} \\ &\leq \frac{A_{km}}{\sqrt{n}}\left(\sum_i \beta_{ki}^{1/k}|t_i|\right)^3 \leq \frac{A_{km}}{\sqrt{n}}\sum_i \beta_{ki}^{3/k}|t_i|^3.\end{aligned} \tag{31}$$

If we choose $B_{km} \leq (4m^{m-1}A_{km})^{-1}$ (and $B_{km} \leq m^{-2+(1/k)}$ in order that the earlier results may not be affected), the A_{km} here coinciding with the last written A_{km} in (31), we have, for $|t_i| \leq B_{km}\beta_{ki}^{-3/k}\Delta\sqrt{n}$,

$$|s| \leq \frac{\Delta}{4m^{m-i}}\sum_{i=1}^{m}t_i^2. \tag{32}$$

On the other hand, if $\lambda_1, \lambda_2, \cdots, \lambda_m$ are the latent roots of $\|\rho_{ij}\|$ then each $\lambda_i \leq m$ since their sum is m. Letting λ_1 be the smallest one we have

$$\frac{1}{2}\sum_{i,j}\rho_{ij}t_it_j \geq \frac{1}{2}\lambda_1\sum t_i^2 = \frac{\lambda_1\lambda_2\cdots\lambda_m}{2\lambda_2\cdots\lambda_m}\sum t_i^2 \geq \frac{\Delta}{2m^{m-1}}\sum t_i^2. \tag{33}$$

(32) and (33) imply (30). Hence the lemma is proved.

Let us write down the particular cases $m = 1$ and $m = 2$ of (24):

$$\begin{aligned}\left\{p\left(\frac{t}{\sqrt{n}}\right)\right\}^n &= e^{-\frac{1}{2}t^2}(1+\psi(it)) \\ &+ \frac{\Theta_k}{n^{\frac{1}{2}(k-2)}}\beta_k^{3(k-2)/k}\{|t|^k + |t|^{k+1} + \cdots + |t|^{3(k-2)}\}e^{-t^2/4}, \quad \left(|t| \leq \frac{A_k\sqrt{n}}{\beta_k^{3/k}}\right)\end{aligned} \tag{34}$$

$$\begin{aligned}\left\{p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right)\right\}^n &= e^{-\frac{1}{2}(t_1^2+t_2^2+2\rho t_1t_2)}\{1+\psi(it_1, it_2)\} \\ &+ \frac{\Theta_k}{n^{\frac{1}{2}(k-2)}}\left\{\sum_{i=1}^{2}\beta_{ki}^{3(k-2)/k}(|t_i|^k + |t_i|^{k+1} + \cdots + |t_i|^{3(k-2)})\right\}e^{-(1-\rho^2)(t_1^2+t_2^2)/8} \\ &\left(|t_i| \leq \frac{A_k(1-\rho^2)\sqrt{n}}{\beta_{ki}^{3/k}}, \quad \rho = \epsilon(\zeta_1\zeta_2)\right).\end{aligned} \tag{35}$$

More specially let us rewrite (34) and (35) with $k = 3$:

$$\left\{p\left(\frac{t}{\sqrt{n}}\right)\right\}^n = e^{-\frac{1}{2}t^2} + \frac{\Theta}{\sqrt{n}}\beta_3|t|^3e^{-\frac{1}{2}t^2}, \qquad \left(|t| \leq \frac{A\sqrt{n}}{\beta_3}\right); \tag{36}$$

$$\begin{aligned}\left\{p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right)\right\}^n &= e^{-\frac{1}{2}(t_1^2+t_2^2+2\rho t_1t_2)} \\ &+ \frac{\Theta}{\sqrt{n}}(\beta_{31}|t_1|^3 + \beta_{32}|t_2|^3)e^{-(1-\rho^2)(t_1^2+t_2^2)/8}, \qquad \left(|t_i| \leq \frac{A(1-\rho^2)\sqrt{n}}{\beta_{3i}}\right).\end{aligned} \tag{37}$$

In this paper only these last four formulae are needed; they are used in the proofs of Theorems 2, 1, 3, 4 respectively. Cases of $m > 2$ of (24) will be needed for the works on other functions alluded to in the introduction.

1.2. In the following group of lemmas, which culminate in Lemma 7, one finds a generalization of the Riemann-Lebesgue theorem, viz. Lemma 6.

LEMMA 4. *Let $f(x)$ be a polynomial of degree $m > 0$, with real coefficients:*

$$f(x) = \sum_{i=0}^{m} a_i x^{m-i} \qquad (a_0 \neq 0) \tag{38}$$

Then

$$\left| \int_0^1 e^{if(x)}\, dx \right| \leq \frac{A_m}{|a_0|^{1/m}}. \tag{38}$$

PROOF: It is sufficient to prove the inequality for $\int_0^1 \cos f(x)\, dx$. Divide the interval into A_m sub-intervals in each of whose interior none of the derivatives $f^{(i)}(x)$ $(i = 1, \cdots, m)$ vanishes. It is sufficient to consider one of these sub-intervals, say (a, b). Consequently each of the polynomials $f^{(i)}(x)$ are monotonic in (a, b). Let

$$I = \int_a^b \cos f(x)\, dx. \tag{39}$$

Suppose first that $f'(x)$ is positive and increasing for $a < x \leq b$. Then

$$|I| \leq \epsilon + \left| \int_{a+\epsilon}^{b} \frac{f'(x) \cos f(x)\, dx}{f'(x)} \right|$$

$$= \epsilon + \frac{1}{f'(a+\epsilon)} \left| \int_{a+\epsilon}^{b_1} f'(x) \cos f(x)\, dx \right|, \quad (a + \epsilon \leq b_1 \leq b),$$

by the second mean-value theorem. Hence

$$|I| \leq \epsilon + \frac{2}{f'(a + \epsilon)}. \tag{40}$$

Now $0 < f'(a + \frac{1}{2}\epsilon) = f'(a + \epsilon) - \epsilon f''(a + \theta\epsilon)/2$, $\frac{1}{2} \leq \theta \leq 1$. Hence $f'(a + \epsilon) > \frac{1}{2}\epsilon f''(a + \theta\epsilon)$. Since $f''(x)$ is monotonic, we have either $f'(a + \epsilon) > \frac{1}{2}\epsilon f''(a + \epsilon)$ or $f'(a + \epsilon) > \frac{1}{2}\epsilon f''(a + \frac{1}{2}\epsilon)$. In other words, there exists a constant C_2, independent of a or ϵ, such that $\frac{1}{2} \leq C_2 \leq 1$ and $f'(a + \epsilon) > \frac{1}{2}\epsilon f(a + C_2\epsilon)$.

If $f'''(x) \geq 0$, we have, as before $f''(a + C_2\epsilon) > \frac{1}{2}C_2\epsilon f'''(a + C_3\epsilon)$, where C_3 is independent of a or ϵ and $\frac{1}{4} \leq C_3 \leq 1$. If $f'''(x) < 0$, then, since $0 < f''(a + 2C_2\epsilon) = f''(a + C_2\epsilon) + C_2\epsilon f'''(a + \theta_1 C_2\epsilon)$, $\frac{1}{2} \leq \theta_1 \leq 1$, we have $f''(a + C_2\epsilon) > -C_2\epsilon f'''(a + 2\theta_1 C_2\epsilon)$. As $f'''(x)$ is monotonic, either $f''(a + C_2\epsilon) > -C_2\epsilon f'''(a + C_2\epsilon)$ or $f''(a + C_2\epsilon) > -C_2\epsilon f'''(a + 2C_2\epsilon)$. In all cases we obtain $f''(a + C_2\epsilon) > B_3\epsilon\, |f'''(a + C_3\epsilon)|$, where B_3 and C_3 are independent of a or ϵ, and $\frac{1}{4} \leq C_3 \leq 2$. Hence $f'(a + \epsilon) > \frac{1}{2}B_3\epsilon^2\, |f'''(a + C_3\epsilon)|$. Arguing with $\pm f'''(a + C_3\epsilon)$ as we did with $f''(a + C_2\epsilon)$, and so on until we come to $f^{(m)}$, we obtain $f'(a + \epsilon) > B_m\epsilon^{m-1} \,|\, f^{(m)}(a + C_m\epsilon) \,| = B_m\epsilon^{m-1} \,|\, a_0 \,|$. Substituting in (40) and putting $\epsilon = |\, a_0 \,|^{-1/m}$ we obtain $|\, I \,| \le A_m \,|\, a_0 \,|^{-1/m}$. The proof presupposes that $C_m\epsilon < b - a$. If the reverse inequality is true, then $|\, I \,| \le b - a < C_m \,|\, a_0 \,|^{-1/m}$. Hence the lemma is true for $f'(x)$ positive and increasing in (a, b).

If $f'(x)$ is positive and decreasing in (a, b), then $I = \int_0^{b-a} \cos(-f(b - y))\, dy$, $-f(b - y)$ being a polynomial with the leading coefficient $\pm a_0$ and the first derivative $f'(b - y)$, which is positive and increasing. This case reduces therefore to the preceding one. Finally, if $f'(x)$ is negative, we have only to notice that $I = \int_a^b \cos(-f(x))\, dx$. Hence the lemma is proved.

LEMMA 5. *Let $f(x)$ be the polynomial* (38a), *and let $a_r \ne 0$ for some r, $0 \le r < m$. Then*

$$\left| \int_0^1 e^{if(x)}\, dx \right| \le \frac{A_m}{|\, a_r \,|^{A_m}}. \tag{41}$$

PROOF: We may assume that $|\, a_r \,| \ge 1$, (41) being trivial if $|\, a_r \,| < 1$. If $r = 0$ this reduces to Lemma 4. Suppose that the lemma is true for $a_0, a_1, \cdots, a_{r-1}$. Let $f_1(x) = a_0x^m + \cdots + a_{r-1}x^{m-r+1}$, $f_2(x) = f(x) - f_1(x)$ and divide $(0, 1)$ into A_m sub-intervals in each of which $f_1(x)$ is monotonic. It is sufficient to consider one of these sub-intervals, say, (a, b). We have

$$I = \int_a^b \cos\{f_1(x) + f_2(x)\}\, dx$$

$$= \int_a^b \cos f_1(x) \cos f_2(x)\, dx - \int_a^b \sin f_1(x) \sin f_2(x)\, dx.$$

We have only to consider the integral of cosines, say J. Divide (a, b) into sub-intervals in each of whose interior $\cos f_1(x)$ is monotonic and does not vanish. The number of such intervals does not exceed $(\frac{1}{2}\pi)^{-1} \,|\, f_1(b) - f_1(a) \,| \le (\frac{1}{2}\pi)^{-1}(|\, f_1(b) \,| + |\, f_1(a) \,|) < 2(|\, a_0 \,| + \cdots + |\, a_{r-1} \,|)$. Then, by the second mean-value theorem,

$$|\, J \,| \le 2(|\, a_0 \,| + \cdots + |\, a_{r-1} \,|) \left| \int_a^{b_1} \cos f_2(x)\, dx \right| \qquad (a \le b_1 \le b).$$

Hence, applying Lemma 4 to $f_2(x)$, we get

$$|\, I \,| \le \frac{A_m(|\, a_0 \,| + \cdots + |\, a_{r-1} \,|)}{|\, a_r \,|^{1/(m-r)}} \le \frac{A_m(|\, a_0 \,| + \cdots + |\, a_{r-1} \,|)}{|\, a_r \,|^{1/m}}. \tag{42}$$

On the hypothesis of induction we have $|\, I \,| \le A_m \,|\, a_i \,|^{-B_m}$ $(i = 0, \cdots, r - 1)$. If $|\, a_i \,| \ge |\, a_r \,|^{1/2m}$ for some $i < r$, then $|\, I \,| \le A_m \,|\, a_r \,|^{-B_m/2m}$; if $|\, a_i \,| < |\, a_r \,|^{1/2m}$, then by (42), $|\, I \,| \le A_m \,|\, a_r \,|^{-1/2m}$. The proof is therefore complete.

LEMMA 6. *Let $f(x)$ be the polynomial* (38a) *and $g(x)$ be summable over* $(-\infty, \infty)$. *Then for every r we have*

$$(43)\qquad \lim_{|a_r|\to\infty} \int_{-\infty}^{\infty} e^{if(x)} g(x)\, dx = 0, \quad \textit{uniformly in } a_i (i \neq r).$$

PROOF: By Lemma 5 We have

$$\lim_{|a_r|\to\infty} \int_0^1 e^{if(x)}\, dx = 0, \quad \text{uniformly in } a_i(i \neq r).$$

Hence

$$(44)\qquad \lim_{|a_r|\to\infty} \int_a^b e^{if(x)}\, dx = 0, \quad \text{uniformly in } a_i(i \neq r)$$

for if $a \neq 0$ and $b \neq 0$, then (a, b) is the sum or the difference of two intervals of the form $(0, c)$ or $(c, 0)$, and for the latter intervals the transformation $x = \pm cy$ reduces the interval of integration to $(0, 1)$.

Let G be any open set of finite measure. Then G is the sum of a sequence $\{I_\nu\}$ of non-overlapping intervals. Since $\Sigma mI_\nu = mG < \infty$, we have

$$\sum_{\nu \geq n} mI_\nu < \epsilon, \qquad n \geq N.$$

Hence

$$\left| \int_G e^{if(x)}\, dx \right| < \epsilon + \sum_{\nu=1}^{N} \left| \int_{I_\nu} e^{if(x)}\, dx \right|$$

which, together with (44), implies

$$(45)\qquad \lim_{|a_r|\to\infty} \int_G e^{if(x)}\, dx = 0 \quad \text{uniformly in } a_i(i \neq r).$$

Let S be any set of finite measure. Then there is an open set G such that $G \supset S$ and $m(G - S) < \epsilon$. Hence

$$\left| \int_S e^{if(x)}\, dx \right| < \epsilon + \left| \int_G e^{if(x)}\, dx \right|.$$

Hence, by (45),

$$(46)\qquad \lim_{|a_r|\to\infty} \int_S e^{if(x)}\, dx = 0 \quad \text{uniformly in } a_i(i \neq r).$$

Now let $h(x)$ be any positive "simple" summable function, i.e. $h(x) = a_\nu > 0$ for $x \in S$ $(\nu = 1, 2, \cdots, n)$ and $h(x) = 0$ otherwise. Since $h(x)$ is summable, each S_ν must be of finite measure. Hence

$$\left| \int_{-\infty}^{\infty} e^{if(x)} h(x)\, dx \right| \leq \sum_{\nu=1}^{n} a_\nu \left| \int_{S_\nu} e^{if(x)}\, dx \right|$$

which, together with (46), implies

$$\lim_{|a_r|\to\infty} \int_{-\infty}^{\infty} e^{if(x)} h(x)\, dx = 0 \quad \text{uniformly in } a_i(i \neq r).$$

Finally, let $g(x)$ be any summable function ≥ 0. Then by a well-known theorem[6] we have $g(x) = \lim h_n(x)$, where $\{h_n(x)\}$ is an ascending sequence of positive summable simple functions. Hence

$$\left|\int_{-\infty}^{\infty} e^{if(x)} g(x)\, dx\right| \leq \left|\int_{-\infty}^{\infty} e^{if(x)} h_n(x)\, dx\right| + \int_{-\infty}^{\infty} (g(x) - h_n(x))\, dx.$$

By monotonic convergence the last integral tends to 0 as $n \to \infty$. Hence

$$\left|\int_{-\infty}^{\infty} e^{if(x)} g(x)\, dx\right| \leq \epsilon + \left|\int_{-\infty}^{\infty} e^{if(x)} h_n(x)\, dx\right|,$$

which implies (43). If $g(x)$ is any summable function, we have only to consider the customary expression of $g(x)$ as the difference of two non-negative functions. This completes the proof.

LEMMA 7. *Let $P(x)$ be a non-singular distribution function of a random variable X, and let*

$$p(t_1, t_2, \cdots, t_m) = \int_{-\infty}^{\infty} e^{i \sum_{r=1}^{m} t_r x^r}\, dP. \tag{47}$$

Then for every r and every positive constant c we have

$$\underset{|t_r| \geq c}{\text{l.u.b.}} |p(t_1, \cdots, t_m)| < 1. \tag{48}$$

PROOF: We have $P(x) = a_1 P_1(x) + a_2 P_2(x)$, where $P_1(x)$ is absolutely continuous, P_2 is singular, $a_1 > 0$, $a_1 + a_2 = 1$. Hence

$$|p(t_1, t_2, \cdots, t_m)| \leq a_1 \left|\int_{-\infty}^{\infty} e^{i \sum_{r=1}^{m} t_r x^r} P_1'(x)\, dx\right| + a_2.$$

By Lemma 6 we may find $C > 0$ such that

$$|p(t_1, t_2, \cdots, t_m)| \leq \tfrac{1}{2} a_1 + a_2 < 1, \quad \text{if any} \quad |t_i| > C.$$

Suppose that

$$\underset{|t_r| \geq c}{\text{l.u.b.}}\, p(t_1, \cdots, t_m) = 1,$$

then $c < C$ and we must have

$$\underset{c \leq |t_r| \leq C, |t_i| \leq C (i \neq r)}{\text{l.u.b.}} |p(t_1, \cdots, t_m)| = 1. \tag{49}$$

Since $p(t_1, \cdots, t_m)$ is a continuous function, it must attain its least upper bound in any bounded closed set. It follows that there is a point $(t_1^0, \cdots, t_m^0)$ such that[7] $t_r^0 \neq 0$ ($|t_r^0| \geq c$) and $p(t_1^0, \cdots, t_m^0) = 1$. But this implies that the distribution of $\Sigma t_i^0 X^i$ is discrete, i.e. that the distribution of X itself is discrete,

[6] H. Kestelman: *Modern Theories of Integration* (1937), p. 108.

[7] Cf. (C), p. 26.

which contradicts the non-singularity of $P(x)$. Hence (49) is false and (48) is true.

1.3. In his cited work Berry[8] shows that if $F(x)$ is any distribution function and if $\Phi(x)$ is the function (6), then there is a constant a such that

$$\left|\int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x + a) - \Phi(x + a)\}\, dx\right| \geq \sqrt{\frac{2}{\pi}}\, T\delta \left\{3 \int_0^{P\delta} \frac{1 - \cos x}{x^2}\, dx - \pi\right\} \tag{50}$$

where $\delta = \sqrt{\frac{\pi}{2}}$ l.u.b. $| F(x) - \Phi(x) |$. This is easily extended to the following lemma, which needs no further proof.

LEMMA 8. *Let $F(x)$ be a distribution function and $F_1(x)$ be a function having the following properties:* (i) *$F_1(x)$ is bounded for all x,* (ii) *$F_1(x) \to 1$ as $x \to \infty$, $F_1(x) \to 0$ as $x \to -\infty$,* (iii) *$F_1(x)$ has a bounded derivative, $| F_1'(x) | \leq M$. Let*

$$\delta = \frac{1}{2M} \text{l.u.b.} | F(x) - F_1(x) |.$$

Then there exists a constant a such that

$$\left|\int_{-\infty}^{\infty} \frac{1 - \cos Tx}{x^2} \{F(x + a) - F_1(x + a)\}\, dx\right| \geq 2MT\delta \left\{3 \int_0^{T\delta} \frac{1 - \cos x}{x^2}\, dx - \pi\right\}. \tag{51}$$

1.4. In section 3 we define, for given ϵ, k, λ and z, a function

$$G(x, y) = e^{-\epsilon y^{2k}} \quad \text{if} \quad z < x \leq z + \lambda y^2, \quad G(x, y) = 0 \quad \text{otherwise.} \tag{52}$$

The introduction of $G(x, y)$ and the appraisal of its Fourier transform constitute the essence of our method of solving the problem of the asymptotic expansion of the distribution function $G(x)$. The solution of the same problem about other functions of (1) alluded to in section 3 is based on the introduction of functions playing the role of $G(x, y)$. We now prove the following lemma:

LEMMA 9. *Let $G(x, y)$ be defined by* (52) *and let*

$$g(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x - it_2y} G(x, y)\, dx\, dy. \tag{53}$$

Then

(i) $$| g(t_1, t_2) | \leq \frac{\lambda A_k}{\epsilon^{3/2k}}$$

(ii) $$| g(t_1, t_2) | \leq \frac{A}{| t_2 |^3}\left(\lambda + \frac{\lambda^2 | t_1 |}{\epsilon^{1/3}} + \frac{\lambda^3 | t_1 |^2}{\epsilon^{2/3}}\right) \quad \text{if } k = 3,$$

(iii) $$| g(t_1, t_2) | \leq \frac{A_k}{| t_2 |^2}\left(\frac{\lambda}{\epsilon^{1/2k}} + \frac{\lambda^2 | t_1 |}{\epsilon^{3/2k}}\right).$$

[8] (B), p. 128.

PROOF:

(i) $$|g(t_1, t_2)| \leq \int_{R_2} G(x, y)\, dx\, dy = \lambda \int_{-\infty}^{\infty} y^2 e^{-\epsilon y^{2k}}\, dy = \frac{A_k \lambda}{\epsilon^{3/2k}}$$

(ii) Putting $k = 3$ we have

$$g(t_1, t_2) = \frac{e^{-it_1 x}}{it_1} \int_{-\infty}^{\infty} e^{-\epsilon y^6 - it_2 y}(1 - e^{-it_1 \lambda y^2})\, dy,$$

$$|g(t_1, t_2)| \leq \frac{1}{|t_1||t_2|^3} \left| \int_{-\infty}^{\infty} u(y) v'''(y)\, dy \right|,$$

where $u(y) = e^{-\epsilon y^6}(1 - e^{-it_1 \lambda y^2})$, $v(y) = e^{-it_2 y}$. On integrating by parts we obtain

(54) $$|g(t_1, t_2)| \leq \frac{1}{|t_1||t_2|^3} \left| \int_{-\infty}^{\infty} v(y) u'''(y)\, dy \right| \leq \frac{1}{|t_1||t_2|^3} \int_{-\infty}^{\infty} |u'''(y)|\, dy.$$

Elementary calculation establishes that

$$\frac{|u'''(y)|}{|t_1|} \leq e^{-\epsilon y^6}(216\lambda\epsilon^3 |y|^{17} + 756\lambda\epsilon^2 |y|^{11} + 336\lambda\epsilon |y|^5 + 8\lambda^3 |t_1|^2 |y|^3 + 12\lambda^2 |t_1||y|).$$

Substituting in (54) and making the transformation $y = \epsilon^{-1/6} x$ we get the result.

(iii) We have

$$|g(t_1, t_2)| \leq \frac{1}{|t_1|} \left| \int_{-\infty}^{\infty} e^{-\epsilon y^{2k} - it_2 y}(1 - e^{-it_1 \lambda y^2})\, dy \right|.$$

Integrating by parts twice we obtain

$$|g(t_1, t_2)| \leq \frac{1}{|t_1||t_2|^2} \int_{-\infty}^{\infty} \left| \frac{d^2}{dy^2} \{e^{-\epsilon y^{2k}}(1 - e^{-it_1 \lambda y^2})\} \right| dy.$$

By elementary calculations we get

$$|g(t_1, t_2)| \leq \frac{1}{|t_2|^2} \int_{-\infty}^{\infty} (4k^2 \lambda \epsilon y^{4k} + 2k(k + 3)\lambda\epsilon y^{2k} + 4\lambda^2 |t_1| y^2 + 2\lambda) e^{-\epsilon y^{2k}}\, dy$$

which, on the transformation $y = \epsilon^{-1/2k} x$, gives the result.

1.5. We prove a few additional lemmas used in the proof of Theorems 3 and 4.

LEMMA[9] 10. *Let $u(x_1, \cdots, x_m) \geq 0$ be summable in the m-dimensional space and let*

(55) $$v(t_1, \cdots, t_m) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-it_1 x_1 - \cdots - it_m x_m} u(x_1, \cdots, x_m)\, dx_1 \cdots dx_m.$$

[9] Although the author believes that this lemma is almost classical, a proof is given owing to lack of reference.

If $v(t_1, \cdots, t_m)$ is summable in the m-dimensional space, then

$$u(x_1, \cdots, x_m) = \frac{1}{(2\pi)^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{it_1x_1+\cdots+it_mx_m} v(t_1, \cdots t_m)\, dt_1 \cdots dt_m. \tag{56}$$

PROOF: Except for a constant factor the function $u(x_1, \cdots, x_m)$ may be regarded as a probability density function. Hence by the well-known inversion formula of (55),

$$\begin{aligned} &\int \cdots \int_{a_i \le x_i \le b_i \ (i=1,\cdots,m)} u(x_1, \cdots, x_m)\, dx_1 \cdots dx_m \\ &\quad = \frac{1}{(2\pi)^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\prod_{j=1}^{m} \frac{e^{it_jb_j} - e^{it_ja_j}}{it_j} \right) v(t_1, \cdots, t_m)\, dt_1 \cdots dt_m . \end{aligned} \tag{57}$$

Now $u(x_1, \cdots, x_m)$ is almost everywhere the symmetric derivative of the interval function in the left-hand side of (57):

$$u(x_1, \cdots, x_m) = \lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^m} \int \cdots \int_{x_i - \epsilon \le y_i \le x_i + \epsilon \ (i=1,2,\cdots,m)} u(y_1, \cdots, y_m)\, dy_1 \cdots dy_m .$$

Hence

$$\begin{aligned} u(x_1, \cdots, x_m) &= \frac{1}{(2\pi)^m} \lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \\ &\cdot \left(\prod_{j=1}^{m} \frac{e^{it_j\epsilon} - e^{-it_j\epsilon}}{it_j} \right) e^{it_1x_1+\cdots+it_mx_m} v(t_1, \cdots, t_m)\, dt_1 \cdots dt_m . \end{aligned} \tag{58}$$

Owing to dominated convergence the order of the limit sign and the integration sign in (58) may be inverted: Hence (56) is true.

LEMMA 11. *We have*

$$\int_{-\infty}^{\infty} e^{-itu} \frac{1 - \cos Tu}{u^2}\, du = \begin{cases} \pi(T - |t|) & \text{if } |t| \le T, \\ 0 & \text{if } |t| > T. \end{cases} \tag{59}$$

PROOF: The Fourier transform of the function in the right-hand side of (59) is

$$\pi \int_{-T}^{T} e^{itu}(T - |t|)\, dt = \frac{2\pi}{u^2} (1 - \cos Tu).$$

Hence (59) follows from (56).

LEMMA 12.

$$| \epsilon(\xi_1 + \cdots + \xi_n)^k | \le A_k n^{k/2} \beta_k \tag{60}$$

PROOF. As (60) is true for $k = 1$, let us assume, for induction, that it is true for $1, 2, \cdots, k$. Then, by symmetry,

$$\epsilon(\xi_1 + \cdots + \xi_n)^{k+1} = n\epsilon\{\xi_1(\xi_1 + \cdots + \xi_n)^k\} = n \sum_{r=0}^{k} \binom{k}{r} \epsilon(\xi_1^{r+1} U^{k-r})$$

where $U = \xi_2 + \cdots + \xi_k$. Since $\epsilon(\xi_1) = 0$, we have

$$\epsilon(\xi_1 + \cdots + \xi_n)^{k+1} = n \sum_{r=1}^{k} \binom{k}{r} \epsilon(\xi_1^{r+1} U^{k-r}).$$

On the hypotheses of induction we have $| \epsilon(U^{k-r}) | \leq A_k(n - 1)^{\frac{1}{2}(k-r)} \beta_{k-r} < A_k n^{\frac{1}{2}(k-1)} \beta_{k-r}$. Hence

$$\left| \epsilon(\xi_1 + \cdots + \xi_n)^{k+1} \right| \leq k! A_k n^{\frac{1}{2}(k+1)} \Sigma \beta_{r+1} \beta_{k-r} \leq A_{k+1} n^{\frac{1}{2}(k+1)} \beta_{k+1}.$$

Therefore the induction is complete.

3. Elementary Proof of Theorem 1. 2.1 We have defined

$$F(x) = Pr\{\sqrt{n}\xi \leq x\}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \tag{61}$$

with the characteristic functions

$$f(t) = \left\{ p\left(\frac{t}{\sqrt{n}} \right) \right\}^n, \qquad \varphi(t) = e^{-\frac{1}{2}t^2}. \tag{62}$$

Following Berry[10] we use the equation

$$\int_{-\infty}^{\infty} \{F(x) - \Phi(x)\} e^{itx} dx = \frac{f(t) - \varphi(t)}{-it}. \tag{63}$$

Let $\psi(it)$ be the polynomial in (34), and let us define $\Psi(x)$ as the function obtained from $\psi(it)$ through the replacement of each power $(it)^\nu$ by $(-1)^\nu \Phi^{(\nu)}(x)$. Integration by parts shows $(-1)^{\nu-1} \int_{-\infty}^{\infty} e^{itx} \Phi^{(\nu)}(x)\, dx = (it)^{\nu-1} \varphi(t)$, whence

$$\int_{-\infty}^{\infty} \Psi(x) e^{itx} dx = \frac{\psi(it)\varphi(t)}{-it}. \tag{64}$$

From (63) and (64) we obtain

$$\int_{-\infty}^{\infty} \{F(x) - \Phi(x) - \Psi(x)\} e^{itx} dx = \frac{f(t) - \varphi(t)\{1 + \psi(it)\}}{-it}. \tag{65}$$

The function $\Psi(x)$ defined here is precisely the $\Psi(x)$ appearing in (5) under Theorem 1. Our task is to prove that

$$| F(x) - \Phi(x) - \Psi(x) | \leq \frac{Q_k}{n^{(k-2)/2}}. \tag{66}$$

Following Berry[11] we replace x by $x + a$ in (65), getting

$$\begin{aligned} &\int_{-\infty}^{\infty} \{F(x + a) - \Phi(x + a) - \Psi(x + a)\} e^{itx} dx \\ &\qquad = \frac{e^{-ita}[f(t) - \varphi(t)\{1 + \psi(it)\}]}{-it} \end{aligned} \tag{67}$$

[10] (B), p. 127, Equation (23).

[11] (B), p. 127.

multiply both sides of (67) by $T - |t|$ and integrate with respect to t in $(-T, T)$:

$$2\int_{-\infty}^{\infty} \frac{1-\cos Tx}{x^2}\{F(x+a) - \Phi(x+a) - \Psi(x+a)\}\,dx$$
$$= \int_{-T}^{T} \frac{(T-|t|)e^{-ita}[f(t) - \varphi(t)\{1+\psi(it)\}]}{-it}\,dt$$

the reversion of order of integration involved is obviously justifiable. Hence

$$\left|\int_{-\infty}^{\infty} \frac{1-\cos Tx}{x^2}\{F(x+a) - \Phi(x+a) - \Psi(x+a)\}\,dx\right| \tag{68}$$
$$\leq T\int_0^T \frac{|f(t) - \varphi(t)\{1+\psi(it)\}|}{t}\,dt.$$

2.2. When in particular $k = 3$, (68) becomes

$$\left|\int_{-\infty}^{\infty} \frac{1-\cos Tx}{x^2}\{F(x+a) - \Phi(x+a)\}\,dx\right| \leq T\int_0^T \frac{|f(t) - \varphi(t)|}{t}\,dt. \tag{69}$$

If we choose a to be the a in (50), the left-hand side of (69) is not less than

$$\sqrt{\frac{2}{\pi}}\,T\delta\left\{3\int_0^{T\delta}\frac{1-\cos x}{x^2}\,dx - \pi\right\}, \qquad \delta = \sqrt{\frac{\pi}{2}}\,\text{l.u.b.}\,|F(x) - \Phi(x)|.$$

On the other hand, taking $T = \dfrac{A\sqrt{n}}{\beta_3}$ as in (36) the right-hand side of (69) is not greater than

$$A\int_0^{\infty} t^2 e^{-\frac{1}{4}t^2}\,dt = A.$$

Hence

$$T\delta\left\{3\int_0^{T\delta}\frac{1-\cos x}{x^2}\,dx - \pi\right\} \leq A. \tag{70}$$

Now the left-hand side of (70), as a function of $T\delta$, is positive and increasing for sufficiently large $T\delta$, and becomes infinite as $T\delta \to \infty$. Hence (70) implies that $T\delta \leq A$, i.e.

$$\text{l.u.b.}\,|F(x) - \Phi(x)| \leq \frac{A}{T} = \frac{A\beta_3}{\sqrt{n}},$$

giving Theorem 2.

2.3. Coming back to the general case, we see that the function $\Phi(x) + \Psi(x)$ has a bounded derivative: $|\Phi'(x) + \Psi'(x)| \leq Q_k$, and also has all the properties of the function $F_1(x)$ in Lemma 8. On choosing a in (69) to be the a in (51) we obtain

(71) $$Q_k T\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} dx - \pi \right\} \leq T \int_0^T \frac{|f(t) - \varphi(t)\{1 + \psi(it)\}|}{t} dt,$$

where

$$\delta = Q_k \text{ l.u.b. } |F(x) - \Phi(x) - \Psi(x)|.$$

Let us take $T = (A_k \beta_k^{-3/k} \sqrt{n})^{k-2}$ with A_k in accordance with (34). Then

(72) $$T \int_0^T \frac{|f(t) - \varphi(t)\{1 + \psi(it)\}|}{t} dt = Q_k n^{\frac{1}{2}(k-2)} \int_0^{T^{1/(k-2)}} + Q_k n^{\frac{1}{2}(k-2)} \int_{Q_k\sqrt{n}}^T = J_1 + J_2 \quad \text{say.}$$

By (34) we have

(73) $$J_1 \leq Q_k \int_0^\infty (t^{k-1} + \cdots + t^{3k-7}) e^{-\frac{1}{4}t^2} dt = Q_k.$$

Also,

(74) $$J_2 \leq Q_k n^{\frac{1}{2}(k-2)} \int_{Q_k\sqrt{n}}^T \frac{|p(t/\sqrt{n})|^n}{t} dt + Q_k n^{\frac{1}{2}(k-2)} \int_{Q_k\sqrt{n}}^T \frac{\varphi(t)\,|1 + \psi(it)|}{t} dt.$$

The second term in the right-hand side of (74) is evidently $\leq Q_k$. The first term does not exceed

(75) $$Q_k n^{\frac{1}{2}(k-3)} T \underset{t \geq Q_k}{\text{l.u.b.}} |p(t)|^n.$$

At this step we make use of the non-singularity of $P(x)$ and apply Lemma 7 for $m = 1$. We have

$$\underset{t \geq Q_k}{\text{l.u.b.}} |p(t)| = e^{-Q_k}.$$

Hence (75) does not exceed $Q_k n^{\frac{1}{2}(2k-5)} e^{-Q_k n} \leq Q_k$. We have therefore

(76) $$T\delta \left\{ 3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} dx - \pi \right\} \leq Q_k, \qquad T = Q_k n^{\frac{1}{2}(k-2)}.$$

Arguing with (76) as we did with (70) we conclude that

$$\text{l.u.b.} |F(x) - \Phi(x) - \Psi(x)| \leq \frac{Q_k}{T} = \frac{Q_k}{n^{\frac{1}{2}(k-2)}}.$$

(72) is valid for $T \geq 1$. If $T < 1$, we have only to suppress the term J_2. Hence Theorem 1 is proved.

4. Proof of Theorem 3 and Theorem 4. 3.1. In connection with the random variables (1), we assume that $\beta_{2k} < \infty$ for some integer $k \geq 3$ and define

(77) $$\eta = \frac{1}{n} \sum_{r=1}^n (\xi_r - \bar{\xi})^2, \qquad G(z) = Pr\left\{ \frac{\sqrt{n}(\eta - 1)}{\sqrt{\alpha_4 - 1}} \leq z \right\}.$$

Now,

$$\eta = \frac{1}{n} \sum \xi_r^2 - \bar{\xi}^2 = 1 + \sqrt{\frac{\alpha_4 - 1}{n}}\, X - \frac{Y^2}{n}$$

where

(78) $$X = \frac{1}{\sqrt{n}} \sum \frac{(\xi_r^2 - 1)}{\sqrt{\alpha_4 - 1}}, \qquad Y = \sqrt{n}\, \bar{\xi}.$$

Hence

(79) $$G(z) = Pr\{X - \lambda Y^2 \leq z\}$$

with

(80) $$\lambda = \frac{1}{\sqrt{n(\alpha_4 - 1)}}.$$

Let W be the probability function of the distribution of the random point (X, Y) and $f(t_1, t_2)$ be the characteristic function:

(81) $$W(S) = Pr\{(X, Y) \epsilon S\} \quad \text{for every Borel set } S \text{ in } R_2,$$

(82) $$f(t_1, t_2) = \epsilon(e^{it_1 X + it_2 Y}) = \left\{p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right)\right\}^n$$

(83) $$p(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_1(x^2-1)/(\sqrt{\alpha_4 - 1}) + it_2 x}\, dP.$$

Let $G_1(z)$ be the distribution function of X. Then

(84) $$G(z) - G_1(z) = \iint\limits_{z < x \leq z + \lambda y^2} dW = K(z), \quad \text{say}.$$

Let

(85) $$K_\epsilon(z) = \iint\limits_{z < x \leq z + \lambda y^2} e^{-\epsilon y^{2k}}\, dW.$$

If we define (for fixed z) the function $G(x, y)$ by

(86) $G(x, y) = e^{-\epsilon y^{2k}}$ if $z < x \leq z + \lambda y^2$, $\quad G(x, y) = 0$ otherwise,

then

(87) $$K_\epsilon(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y)\, dW.$$

Letting

(88) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1 x - it_2 y}\, G(x, y)\, dx\, dy = g(t_1, t_2),$$

we replace x by $x - u$ in the integral and get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}G(x-u,y)\,dx\,dy = e^{-it_1u}g(t_1,t_2). \tag{89}$$

Multiplying both sides by $\dfrac{1-\cos Tu}{u^2}$ and integrating with respect to u we obtain, with the help of (59), Lemma 11,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}\,dx\,dy\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}G(x-u,y)\,du = \begin{cases}\pi(T-|t_1|)g(t_1,t_2) & \text{if } |t_1|\le T,\\ 0 & \text{if } |t_1| > T;\end{cases} \tag{90}$$

the reversion of order of integration in the left-hand side is obviously justifiable. By Lemma 9 the right-hand side of (90) is summable in the whole plane of (t_1, t_2). Hence, by Lemma 10,

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}G(x-u,y)\,du = \frac{1}{4\pi}\iint_{|t_1|\le T}(T-|t_1|)g(t_1,t_2)e^{it_1x+it_2y}\,dt_1\,dt_2. \tag{91}$$

If we integrate both sides with respect to the probability function W, we obtain, on reversing the order of integration,

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\,du\iint_{R_2}G(x-u,y)\,dW = \frac{1}{4\pi}\iint_{|t_1|\le T}(T-|t_1|)g(t_1,t_2)f(t_1,t_2)\,dt_1\,dt_2. \tag{92}$$

By (86) and (87),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x-u,y)\,dW = K_\epsilon(u+z). \tag{93}$$

Hence

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}K_\epsilon(u+z)\,du = \frac{1}{4\pi}\iint_{|t_1|\le T}(T-|t_1|)g(t_1,t_2)f(t_1,t_2)\,dt_1\,dt_2. \tag{94}$$

We now take the functions

$$\varphi(t_1,t_2) = e^{-\frac{1}{2}(t_1^2+t_2^2+2\rho t_1t_2)} \tag{95}$$

and $\psi(it_1, it_2)$ as in (35), where

$$\rho = \int_{-\infty}^{\infty} \frac{(x^2 - 1)x}{\sqrt{\alpha_4 - 1}} \, dP = \frac{\alpha_3}{\sqrt{\alpha_4 - 1}}. \tag{96}$$

Since the condition $\alpha_4 - 1 - \alpha_3^2 \neq 0$ is assumed in Theorem 3 and implied in Theorem 4, we have $|\rho| < 1$. Let

$$w(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} e^{-(1/2(1-\rho^2))(x^2+y^2-2\rho xy)} \tag{97}$$

and let $\gamma(x, y)$ be the function obtained from $\psi(it_1, it_2)$ through the replacement of each power $(it_1)^{\nu_1} (it_2)^{\nu_2}$ by $(-1)^{\nu_1+\nu_2} W_{\nu_1\nu_2}(x, y) = (-1)^{\nu_1+\nu_2} \dfrac{\partial^{\nu_1+\nu_2} w(x, y)}{\partial x^{\nu_1} \partial y^{\nu_2}}$. Since

$$w(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-it_1x-it_2y} \varphi(t_1, t_2) \, dt_1 dt_2, \tag{98}$$

we have

$$w_{\nu_1\nu_2}(x, y) = \frac{(-1)^{\nu_1+\nu_2}}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (it_1)^{\nu_1} (it_2)^{\nu_2} e^{-it_1x-it_2y} \varphi(t_1, t_2) \, dt_1 dt_2, \tag{99}$$

whence, by Fourier inversion,

$$(it_1)^{\nu_1} (it_2)^{\nu_2} \varphi(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1x+it_2y} w_{\nu_1\nu_2}(x, y) \, dxdy. \tag{100}$$

From the definition of $\gamma(x, y)$ it follows therefore

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1x+it_2y} \{w(x, y) + \gamma(x, y)\} \, dx \, dy = \varphi(t_1, t_2)\{1 + \psi(it_1, it_2)\}. \tag{101}$$

A comparison of (101) with $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1x+it_2y} \, dW = f(t_1, t_2)$ shows that (94) will remain true if $K_\epsilon(u)$ be replaced by

$$\iint\limits_{u < x \leq u+\lambda y^2} e^{-\epsilon y^{2k}} (w(x, y) + \gamma(x, y)) \, dx \, dy = L_\epsilon(u), \text{ say,} \tag{102}$$

and $f(t_1, t_2)$ be replaced by $\varphi(t_1, t_2)\{1 + \psi(it_1, it_2)\}$. Hence

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{K_\epsilon(u + z) - L_\epsilon(u + z)\} \, du \\ &= \frac{1}{4\pi} \iint\limits_{|t_1| \leq T} (T - |t_1|) g(t_1, t_2) \{f(t_1, t_2) \\ &\qquad - \varphi(t_1, t_2)[1 + \psi(it_1, it_2)]\} \, dt_1 dt_2. \end{aligned} \tag{103}$$

Let also

$$(104)\qquad H(z) = \iint\limits_{x-\lambda y^2 \le z} \{w(x, y) + \gamma(x, y)\}\, dx\, dy,$$

$$H_1(z) = \iint\limits_{x \le z} \{w(x, y) + \gamma(x, y)\}\, dx\, dy,$$

$$(105)\qquad L(z) = H(z) - H_1(z) = \iint\limits_{z < x \le z+\lambda y^2} \{w(x, y) + \gamma(x, y)\}\, dx\, dy.$$

3.2. We now consider the particular case $k = 3$ and prove Theorem 3. For $k = 3$ we have $\psi \equiv \gamma \equiv 0$ and so

$$(106)\qquad \begin{aligned} H(z) &= \iint\limits_{x-\lambda y^2 \le z} w(x, y)\, dx\, dy, \\ H_1(z) &= \iint\limits_{x \le z} w(x, y)\, dx\, dy = \Phi(z), \end{aligned}$$

$$(107)\qquad \begin{aligned} L(z) &= H(z) - H_1(z), \\ L_\epsilon(z) &= \iint\limits_{z < x \le z+\lambda y^2} e^{-\epsilon y^6} w(x, y)\, dx\, dy, \end{aligned}$$

$$(108)\qquad \begin{aligned} &\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{K_\epsilon(u + x) - L_\epsilon(u + x)\}\, du \\ &= \frac{1}{4\pi} \iint\limits_{|t_1| \le T} (T - |t_1|) g(t_1, t_2) \{f(t_1, t_2) - \varphi(t_1, t_2)\} dt_1\, dt_2 . \end{aligned}$$

Now

$$\begin{aligned} K_\epsilon(u) - L_\epsilon(u) = \{G(u) - \Phi(u)\} - \{H(u) - \Phi(u)\} - \{G_1(u) - \Phi(u)\} \\ - \{K(u) - K_\epsilon(u)\} + \{L(u) - L_\epsilon(u)\}, \end{aligned}$$

$$\begin{aligned} 0 \le H(u) - \Phi(u) &= \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy \int_u^{u+\lambda y^2} e^{-(1/2(1-\rho^2))(x-\rho y)^2}\, dx \\ &\le \frac{\lambda}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} y^2 e^{-\frac{1}{2}y^2}\, dy = \frac{\lambda}{\sqrt{2\pi(1 - \rho^2)}}, \end{aligned}$$

$$|G_1(u) - \Phi(u)| \le \frac{A}{\sqrt{n}} \int_{-\infty}^{\infty} \left| \frac{x^2 - 1}{\sqrt{\alpha_4 - 1}} \right|^3 dP \le \frac{A\alpha_6}{(\alpha_4 - 1)^{3/2}\sqrt{n}} \text{ by Theorem 2,}$$

$$0 \le K(u) - K_\epsilon(u) \le \epsilon\mathcal{E}(Y^6) \le A\alpha_6\epsilon \text{ by Lemma 12,}$$

$$0 \le L(u) - L_\epsilon(u) \le A\epsilon.$$

Hence

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1-\cos Tu}{u^2}\{G(u+\lambda)-\Phi(u+\lambda)\}\,du \\
&\quad = \Theta T\left\{\alpha_6\epsilon + \frac{\alpha_6}{(\alpha_4-1)^{3/2}\sqrt{n}} + \frac{1}{\sqrt{n}\sqrt{(\alpha_4-1)(1-\rho^2)}}\right. \\
&\qquad + \Theta T \iint_{|t_1|\le T} |g(t_1,t_2)|\cdot|f(t_1,t_2)-\varphi(t_1,t_2)|\,dt_1dt_2 .
\end{aligned} \tag{109}
$$

It is easy to verify that

$$
\frac{\alpha_6}{(\alpha_4-1)^{3/2}} + \frac{1}{\sqrt{(\alpha_4-1)(1-\rho^2)}} \le \left(\frac{\alpha_6}{\alpha_4-1-\alpha_3^2}\right)^{3/2}.
$$

For the left-hand side of (109) we refer to (50) and take x to be the number a therein. Hence

$$
\begin{aligned}
&T\delta\left\{3\int_0^{T\alpha}\frac{1-\cos u}{u^2}\,du-\pi\right\} \le AT\left\{\alpha_6\epsilon+\frac{1}{\sqrt{n}}\left(\frac{\alpha_6}{\alpha_4-1-\alpha_3^2}\right)^{3/2}\right\} \\
&\qquad + AT \iint_{|t_1|\le T,|t_2|\le T} |g(t_1,t_2)|\cdot|f(t_1,t_2)-\varphi(t_1,t_2)|\,dt_1dt_2 \\
&\qquad + AT \iint_{|t_1|\le T,|t_2|> T} |g(t_1,t_2)|\,dt_1dt_2 .
\end{aligned} \tag{110}
$$

By Lemma 9 (ii) we have

$$
\begin{aligned}
&T \iint_{|t_1|\le T,|t_2|>T} |g(t_1,t_2)|\,dt_1\,dt_2 \\
&\quad \le AT \iint_{|t_1|\le T,|t_2|>T} \frac{1}{|t_2|^3}\left(\lambda+\frac{\lambda^2|t_1|}{\epsilon^{\frac{1}{2}}}+\frac{\lambda^3|t_1|^2}{\epsilon^{\frac{3}{2}}}\right)dt_1\,dt_2. \\
&\quad \le A\left(\lambda+\frac{\lambda^2T}{\epsilon^{\frac{1}{2}}}+\frac{\lambda^3T^2}{\epsilon^{\frac{3}{2}}}\right).
\end{aligned} \tag{111}
$$

Hence

$$
\begin{aligned}
&T\delta\left\{3\int_0^{T\delta}\frac{1-\cos u}{u^2}\,du-\pi\right\} \\
&\quad \le A\left\{\alpha_6T\epsilon+\left(\frac{\alpha_6}{\alpha_4-1-\alpha_3^2}\right)^{3/2}\frac{T}{\sqrt{n}}+\lambda+\frac{\lambda^2T}{\epsilon^{\frac{1}{2}}}+\frac{\lambda^3T^2}{\epsilon^{\frac{3}{2}}}\right\} \\
&\qquad + AT\iint_{|t_1|\le T,|t_2|\le T} |g(t_1,t_2)|\,|f-\varphi|\,dt_1\,dt_2 .
\end{aligned} \tag{112}
$$

By Lemma 9 (i) with $k = 3$ we have

$$(113)\quad T \iint_{|t_1|\le T,|t_2|\le T} |g|\cdot|f-\varphi|\,dt_1\,dt_2 \le \frac{AT\lambda}{\epsilon^{\frac{1}{2}}} \iint_{|t_1|\le T,|t_2|\le T} |f-\varphi|\,dt_1\,dt_2 .$$

By (37) under Lemma 3,

$$(114)\quad |f-\varphi| \le \frac{A}{\sqrt{n}}(\beta_{31}|t_1|^3+\beta_{32}|t_2|^3)e^{-\frac{1}{4}(1-\rho^2)(t_1^2+t_2^2)} \quad \text{for } |t_i| \le \frac{A(1-\rho^2)\sqrt{n}}{\beta_{3i}}$$

with

$$(115)\quad \begin{aligned} \beta_{31} &= \int_{-\infty}^{\infty}\left|\frac{x^2-1}{\sqrt{\alpha_4-1}}\right|^3 dP \le \frac{4}{(\alpha_4-1)^{\frac{3}{2}}}\int_{-\infty}^{\infty}(x^6+1)\,dP \\ &\le \frac{8\alpha_6}{(\alpha_4-1)^{\frac{3}{2}}}, \qquad \beta_{32} = \int_{-\infty}^{\infty}|x|^3\,dP = \beta_3 . \end{aligned}$$

We now take

$$(116)\quad T = \frac{A}{8}\left(\frac{\alpha_4-1-\alpha_3^2}{\alpha_6}\right)^{\frac{3}{2}}\sqrt{n},$$

the A coinciding with that in (114). Then

$$(117)\quad \begin{aligned} \frac{A(1-\rho^2)\sqrt{n}}{\beta_{31}} &\ge \frac{A(1-\rho^2)(\alpha_4-1)^{\frac{3}{2}}\sqrt{n}}{8\alpha_6} \\ &= \frac{A(\alpha_4-1-\alpha_3^2)\sqrt{\alpha_4-1}\,\sqrt{n}}{8\alpha_6} \ge \frac{A(\alpha_4-1-\alpha_3^2)^{\frac{3}{2}}\sqrt{n}}{8\alpha_6^{3/2}} = T \end{aligned}$$

$$(118)\quad \begin{aligned} \frac{A(1-\rho^2)\sqrt{n}}{\beta_{32}} &= \frac{A(\alpha_4-1-\alpha_3^2)\sqrt{n}}{(\alpha_4-1)\beta_3} \\ &\ge \frac{A(\alpha_4-1-\alpha_3^2)^{\frac{3}{2}}\sqrt{n}}{\alpha_4^{3/2}\beta_3} \ge \frac{A(\alpha_4-1-\alpha_3^2)^{\frac{3}{2}}\sqrt{n}}{\alpha_6^{3/2}} > T. \end{aligned}$$

Hence (114) is true for $|t_1| \le T$ and $|t_2| \le T$. Using this fact on (113) we obtain

$$(119)\quad \begin{aligned} T &\iint_{|t_1|\le T,|t_2|\le T} |g||f-\varphi|\,dt_1\,dt_2 \\ &\le \frac{AT\lambda}{\epsilon^{\frac{1}{2}}}\frac{1}{\sqrt{n}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{\alpha_6}{(\alpha_4-1)^{\frac{3}{2}}}|t_1|^3+\beta_3|t_2|^3\right\}e^{-\frac{1}{4}(1-\rho^2)(t_1^2+t_2^2)}\,dt_1\,dt_2 \\ &\le \frac{AT\lambda}{\epsilon^{\frac{1}{2}}\sqrt{n}}\left(\frac{\alpha_6}{(\alpha_4-1)^{3/2}}+\beta_3\right)\frac{1}{(1-\rho^2)^{5/2}} \\ &= \frac{AT\lambda}{\sqrt{n\epsilon}}(\alpha_6(\alpha_4-1)+\beta_3(\alpha_4-1)^{5/2})\frac{1}{(\alpha_4-1-\alpha_3^2)^{5/2}} \\ &= \frac{AT}{n\sqrt{\epsilon}}(\alpha_6\sqrt{\alpha_4-1}+\beta_3(\alpha_4-1)^2)\frac{1}{(\alpha_4-1-\alpha_3^2)^{5/2}} \\ &\le \frac{AT\alpha_6^{11/6}}{n\sqrt{\epsilon}(\alpha_4-1-\alpha_3^2)^{5/2}} . \end{aligned}$$

Substituting in (112), setting $\epsilon = (\alpha_6 T)^{-1}$ and using (116) we obtain after some easy reduction

$$\begin{aligned}(120)\quad & T\delta\left\{3\int_0^{T\delta}\frac{1-\cos u}{u^2}\,du - \pi\right\} \\ & \leq A\left[1+\frac{1}{\sqrt{n(\alpha_4-1)}}+\left(\frac{\alpha_6}{n(\alpha_4-1-\alpha_3^2)}\right)^{\frac{1}{2}}+\left(\frac{\alpha_6}{n(\alpha_4-1-\alpha_3^2)}\right)^{\frac{1}{4}}\right].\end{aligned}$$

If $n \geq (\alpha_4 - 1 - \alpha_3^2)^{-1}\alpha_6$, then the right-hand side of (120) is $\leq A$, and so, arguing with (120), as we did with (70), we obtain

$$(121)\qquad \text{l.u.b.}\,|G(u)-\Phi(u)| \leq \frac{A}{T} = \frac{A}{\sqrt{n}}\left(\frac{\alpha_6}{\alpha_4-1-\alpha_3^2}\right)^{\frac{1}{2}}.$$

For $n < (\alpha_4 - 1 - \alpha_3^2)^{-1}\alpha_6$, however, the right-hand side of (121) $\geq A(\alpha_4 - 1 - \alpha_3^2)^{-1}\alpha_6 \geq A$ and (121) becomes a triviality. Hence Theorem 3 is proved.

3.3. To prove Theorem 4, we start again with the identity (103). We have

$$\begin{aligned}(122)\quad K_\epsilon(u) - L_\epsilon(u) &= \{G(u)-H(u)\} - \{G_1(u)-H_1(u)\} \\ &\qquad - \{K(u)-K_\epsilon(u)\} + \{L(u)-L_\epsilon(u)\},\end{aligned}$$

$$(123)\qquad 0 \leq K(u) - K_\epsilon(u) \leq \epsilon\epsilon(Y^{2k}) \leq Q_k\epsilon \quad \text{by Lemma 12,}$$

$$(124)\quad 0 \leq L(u) - L_\epsilon(u) \leq \epsilon\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^{2k}(w(x,y)+|\gamma(x,y)|)\,dx\,dy \leq Q_k\epsilon.$$

Let us show that

$$(125)\qquad |G_1(u) - H_1(u)| \leq Q_k/n^{\frac{1}{2}(k-1)}.$$

The function $X = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\xi_i^2-1}{\sqrt{\alpha_4-1}}\right)$ has the same structure as $\sqrt{n}\,\xi$ (with $(\alpha_4-1)^{-\frac{1}{2}}(\xi_i^2-1)$ playing the role of ξ_i); hence, by Theorem 1, there exists an asymptotic expansion of the distribution function $G_1(u)$. We shall see that the terms of this asymptotic expansion are precisely $H_1(u)$, whence (125) follows from Theorem 1.

It is obvious that for the polynomial $\psi(it_1, it_2)$ in (35) $\psi(it, 0)$ coincides with the polynomial $\psi(it)$ in (34). Hence the terms of the asymptotic expansion of $G_1(u)$ are the inversion of $e^{-\frac{1}{2}t^2}\{1+\psi(it,0)\}$ viz.

$$(126)\qquad \Phi(u) + \frac{1}{2\pi}\int_{-\infty}^{u}dx\int_{-\infty}^{\infty}e^{-itx-\frac{1}{2}t^2}\psi(it,0)\,dt.$$

On the other hand, by (104),

$$(127)\qquad H_1(u) = \Phi(u) + \int_{-\infty}^{u}dx\int_{-\infty}^{\infty}\gamma(x,y)\,dy,$$

and by (101) with $t_2 = 0$,

$$(128)\qquad \int_{-\infty}^{\infty}e^{itx}\,dx\int_{-\infty}^{\infty}\gamma(x,y)\,dy = e^{-\frac{1}{2}t^2}\psi(it,0).$$

Inversion of (118) gives

$$\int_{-\infty}^{\infty} \gamma(x, y)\, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx-\frac{1}{2}t^2} \psi(it, 0)\, dt \tag{129}$$

which establishes the equality of $H_1(u)$ and (126).

Using (122), (123), (124), (125) on (103) we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{G(u + z) - H(u + z)\}\, du &= \Lambda_k T \left(\epsilon + \frac{1}{n^{\frac{1}{2}(k-2)}}\right) \\ &+ \Theta T \int\int |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)[1 + \psi(it_1, it_2)]|\, dt_1\, dt_2 . \end{aligned} \tag{130}$$

If we expand

$$H(u) = \int\int_{-\lambda_n y^2 \leq u} \{w(x, y) + \gamma(x, y)\}\, dx\, dy \tag{131}$$

in powers of $n^{-\frac{1}{2}}$ up to and including the term $n^{-\frac{1}{2}(k-3)}$, the remainder is obviously $\Lambda_k n^{-\frac{1}{2}(k-2)}$ Hence

$$H(u) = \Phi(u) + \chi(u) + \Lambda_k / n^{\frac{1}{2}(k-2)}, \tag{132}$$

where $\Phi(u) + \chi(u)$ is the group of terms of the Taylor expansion of (131) in powers of $n^{-\frac{1}{2}}$ up to and including the term $n^{-\frac{1}{2}(k-3)}$. From (130) and (132) we get

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2} \{G(u + z) - \Phi(u + z) - \chi(u + z)\}\, du \right| \\ &\qquad \leq Q_k T \left(\epsilon + \frac{1}{n^{\frac{1}{2}(k-2)}}\right) + AI, \end{aligned} \tag{133}$$

where

$$I = T \int\int_{|t_1| \leq T} |g(t_1, t_2)| \cdot |f(t_1, t_2) - \varphi(t_1, t_2)\{1 + \psi(it_1, it_2)\}|\, dt_1\, dt_2 . \tag{134}$$

We are going to prove that the function $\chi(u)$ here defined satisfies all the requirements of the function $\chi(u)$ in Theorem 4. The structure of $\chi(u)$ announced in Theorem 4 is easily verifiable. It remains to prove the inequalities (15) and (16) satisfied by

$$| G(u) - \Phi(u) - \chi(u) | .$$

It is obvious that the function $\Phi(u) + \chi(u)$ has all the properties of the function $F_1(u)$ in Lemma 8, having a bounded derivative $| \Phi'(u) + \chi'(u) | \leq Q_k$. Hence, on taking z in (133) to be the number a in (51), the left-hand side of (133) does not exceed

$$Q_k T\delta \left(3 \int_0^{T\delta} \frac{1 - \cos u}{u^2}\, du - \pi\right), \qquad \delta = Q_k \text{l.u.b.} | G(u) - \Phi(u) - \chi(u) |.$$

Hence

$$T\delta\left(3\int_0^{T\delta}\frac{1-\cos u}{u^2}\,du-\pi\right)\le Q_kT\left(\epsilon+\frac{1}{n^{\frac{1}{2}(k-2)}}\right)+Q_kI. \tag{135}$$

In order to appraise I we recall (35) under Lemma 3 (replacing therein each β_{ki} by the larger number $\beta_{k1}\beta_{k2}$, and merging the latter into Q_k)

$$\begin{aligned}|f(t_1,t_2)-\varphi(t_1,t_2)\{1+\psi(it_1,it_2)\}|\le\frac{Q_k}{n^{\frac{1}{2}(k-2)}}\{\Sigma(|t_i|^k+\cdots\\+|t_i|^{3(k-2)})\}e^{-(1-\rho^2)(t_1^2+t_2^2)/8}\end{aligned} \tag{136}$$

for

$$|t_i|\le Q_k\sqrt{n}. \tag{137}$$

Put $T=(Q_k\sqrt{n})^l$, with Q_k here coinciding with that in (137) and then (136) is valid for $|t_1|\le T^{1/l}$ and $|t_2|\le T^{1/l}$. Write

$$I=T\iint_{|t_1|\le T^{1/l},|t_2|\le T^{1/l}}+T\iint_{|t_1|\le T,|t_2|>T^{1/l}}+T\iint_{\substack{T^{1/l}<|t_1|\le T\\|t_2|\le T^{1/l}}}=I_1+I_2+I_3.$$

By Lemma 9 (i),

$$I_1\le\frac{Q_kT}{n^{\frac{1}{2}}\epsilon^{3/2k}}\iint|f-\varphi(1+\psi)|\,dt_1dt_2, \tag{138}$$

whence, by (136)

$$\begin{aligned}I_1\le\frac{Q_kT}{n^{\frac{1}{2}(k-1)}\epsilon^{3/2k}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{i=1}^{2}(|t_i|^k+\cdots+|t_i|^{3(k-2)})\right)\\\cdot e^{-(1-\rho^2)(t_1^2+t_2^2)/8}\,dt_1dt_2\le\frac{Q_kT}{n^{\frac{1}{2}(k-1)}\epsilon^{3/2k}}.\end{aligned} \tag{139}$$

By Lemma 9 (iii) we have

$$\begin{aligned}I_2\le Q_kT\iint_{|t_1|\le T,|t_2|>T^{1/l}}\frac{1}{|t_2|^2}\left(\frac{1}{\sqrt{n}\epsilon^{1/2k}}\right.\\\left.+\frac{|t_1|}{n\epsilon^{3/2k}}\right)\{|f(t_1,t_2)|+\varphi(t_1,t_2)|1+\psi(it_1,it_2)|\}\,dt_1dt_2.\end{aligned}$$

Obviously,

$$\operatorname*{l.u.b.}_{t_2>T^{1/k-2}}\varphi(t_1,t_2)|1+\psi(it_1,it_2)|=e^{-nQ_k}. \tag{140}$$

On the assumption of non-singularity of $P(x)$ we have, by Lemma 7,

$$\begin{aligned}\operatorname*{l.u.b.}_{|t_2|>T^{1/k-2}}|f(t_1,t_2)|=\operatorname*{l.u.b.}_{|t_2|>Q_k\sqrt{n}}\left|p\left(\frac{t_1}{\sqrt{n}},\frac{t_2}{\sqrt{n}}\right)\right|^n\\=\operatorname*{l.u.b.}_{|t_2|\ge Q_k}\left|p\left(\frac{t_1}{\sqrt{n}},t_2\right)\right|^n=e^{-nQ_k}.\end{aligned} \tag{141}$$

Hence

$$I_2 \leq Q_k T e^{-nQ_k} \iint_{|t_1|\leq T, |t_2|>T^{1/l}} \frac{1}{|t_2|^2}\left(\frac{1}{\sqrt{n}\epsilon^{1/2k}} + \frac{|t_1|}{n\epsilon^{3/2k}}\right) dt_1\, dt_2 \tag{142}$$
$$= Q_k \left(\frac{n^{l-1}}{\epsilon^{1/2k}} + \frac{n^{(3/2)(l-1)}}{\epsilon^{3/2k}}\right) e^{-nQ_k}.$$

For I_3 we have $|t_1| > T^{1/l} = Q_k\sqrt{n}$, and so Lemma 7 is applicable to I_3 in the same manner as to I_2. Using Lemma 9 (i) on the factor $|g(t_1, t_2)|$ we get

$$I_3 \leq \frac{Q_k n^l e^{-nQ_k}}{\epsilon^{3/2k}}. \tag{143}$$

Combining (135), (138), (139), (142), (143) we obtain

$$T\delta\left(3\int_0^{T\delta} \frac{1-\cos u}{u^2}\, du - \pi\right) \leq Q_k\left(n^{l/2}\epsilon + \frac{n^{l/2}}{n^{\frac{1}{2}(k-2)}} + \frac{n^{l/2}}{n^{\frac{1}{2}(k-1)}\epsilon^{3/2k}}\right) \tag{144}$$
$$+ Q_k\left(\frac{n^{l-1}}{\epsilon^{1/2k}} + \frac{n^{3/2(l-1)}}{\epsilon^{3/2k}} + \frac{n^l}{\epsilon^{3/2k}}\right) e^{-nQ_k}.$$

Putting $\epsilon = \dfrac{1}{n^{k(k-1)/(2k+3)}}$ we get, as the last term in (144) is $\leq Q_k$,

$$T\delta\left(3\int_0^{T\delta} \frac{1-\cos u}{u^2}\, du - \pi\right) \leq Q_k + Q_k n^{l/2}\left(\frac{1}{n^{k(k-1)/(2k+3)}} + \frac{1}{n^{\frac{1}{2}(k-2)}}\right).$$

If $4 \leq k \leq 6$, we take $l = k - 2$ and get

$$T\delta\left(3\int_0^{T\delta} \frac{1-\cos u}{u^2}\, du - \pi\right) \leq Q_k + Q_k\left(\frac{1}{n^{(6-k)/(2(2k+3))}} + 1\right) \leq Q_k.$$

Hence, by the argument following (70),

$$\text{l.u.b.}\,|G(u) - \Phi(u) - \chi(u)| \leq \frac{Q_k}{T} = \frac{Q_k}{n^{\frac{1}{2}(k-2)}},$$

giving (15). If $k \geq 7$, we take $l = \dfrac{2k(k-1)}{2k+3}$ and get

$$T\delta\left(3\int_0^{T\delta} \frac{1-\cos u}{u^2}\, du - \pi\right) \leq Q_k + Q_k\left(1 + \frac{1}{n^{(k-6)/(2(k+3))}}\right) \leq Q_k.$$

Hence

$$\text{l.u.b.}\,|G(u) - \Phi(u) - \chi(u)| \leq \frac{Q_k}{T} = \frac{Q_k}{n^{k(k-1)/2(k+3)}},$$

giving (16). Therefore Theorem 4 is proved.

5. When $\alpha_4 - 1 - \alpha_3^2 = 0$. If $\alpha_4 - 1 - \alpha_3^2 = 0$, then there is unit probability that ξ_i assumes exactly two values:

$$Pr\{\xi_i = a\} = p, \qquad Pr\{\xi_i = b\} = q, \qquad p + q = 1.$$

Let $\zeta_i = 1$ with probability p and $\zeta_i = 0$ with probability q. Then $\xi_i = b + (a - b)\zeta_i$, $\eta = (a - b)^2 \frac{1}{n} \Sigma(\zeta_i - \bar{\zeta})^2$. Hence it is sufficient to consider the variable $\frac{1}{n} \sum (\zeta_i - \bar{\zeta})^2 = \eta$. Letting $\Sigma\zeta_i = r = np + \sqrt{npq}\, X$we have $\eta_1 = r - \frac{r^2}{n} = npq + (q - p)\sqrt{npq}\, X - pqX^2$. We now consider two distinct cases:

Case (*i*). $p \neq q$. Here

$$F(z) = Pr\left\{\frac{\eta_1 - n/\partial q}{|p - q|\sqrt{npq}} \leq z\right\}$$

$$= Pr\{(X + c\sqrt{n})^2 \geq c^2 n - 2|c|\sqrt{n}\, z\}, \quad c = \frac{p - q}{2\sqrt{pq}}.$$

Thus $F(z) = 1$ if $z \geq \frac{1}{2} | c | \sqrt{n}$. If $z < \frac{1}{2} | c | \sqrt{n}$, then

$$F(z) = Pr\{X \leq -cn - (c^2 n - 2 | c | \sqrt{n} z)^{\frac{1}{2}}\}$$

$$+ Pr\{X \geq -c\sqrt{n} + (c^2 n - 2 | c | \sqrt{n} z)^{\frac{1}{2}}\} = F_1(z) + F_2(z).$$

To the random variable X Theorem 2 can be applied. Suppose that $c < 0$; then, by Tchebycheff's inequality,

$$F_2(z) \leq Pr\{X \geq -cn\} \leq \frac{1}{c^2 n} \leq \frac{1}{(p - q)^2 n}.$$

By Theorem 2,

$$F_1(z) = Pr\{X \leq -cn - (c^2 n - 2|c|\sqrt{n}\, z)^{\frac{1}{2}}\}$$

$$= \Phi(z) + \frac{\Theta z^2}{\sqrt{n}\,|p - q|} + \frac{\Theta(p^2 + q^2)}{\sqrt{npq}}.$$

Hence

$$(145) \qquad |F(z) - \Phi(z)| \leq A \left\{\frac{p^2 + q^2}{\sqrt{npq}} + \frac{z^2}{\sqrt{n}\,|p - q|} + \frac{1}{n(p - q)^2}\right\}.$$

The same inequality holds also for $c > 0$.

Case (*ii*). $p = q = 1/2$. Here $\eta_1 = \frac{1}{4}(n - X^2)$; hence

$$(146) \quad Pr\left\{\eta_1 \geq \frac{n - z}{4}\right\} = Pr\{X^2 \leq z\} = \frac{1}{\sqrt{2\pi}} \int_0^z x^{-\frac{1}{2}} e^{-x/2}\, dx + \frac{\Theta}{\sqrt{n}}.$$

There is no asymptotic expansion for the distribution function of η_1. (See (C), p. 83.)