scription of his treatment, I should perhaps have made it clearer that the derived formula (6) gives only the distance D_{ij} measured from the initially chosen and fixed gene i to an arbitrary gene j. Other distances D_{jk}, $(i < j < k)$, are deduced from it by the postulate of additivity $(D_{jk} = D_{ik} - D_{ij})$. If the origin i is changed, there will be a similar formula (6), but it should not be assumed that the function p_0 is the same. In referring to certain conditions necessary 'to assure additivity,' Geiringer evidently means conditions that the function p_0 may be the same for all origins i. These conditions would be interpreted biologically as asserting uniformity of interference along the chromosome. I agree that there are further points to be cleared up in this connection.

If I might sum up the discussion, I would say that the geneticist's conception of the distance between genes is an actual property of the corresponding chromosome segment. Geiringer's definition represents the best possible general approach to this from the limited data of the l.d. alone. Haldane's definition fits the geneticist's conception, and his investigation is an attempt to get the best estimate of the distance by making approximate assumptions as to what happens between the observed genes. It is based on the unobservable crossover-distribution of a supposed infinite set of genes, but can be applied to particular models of this infinite c.d. so as to derive results which involve only a finite and observable c.d. Finally it should be mentioned that in the paper quoted, Haldane gave also an alternative method for the case $p = 2$, leading to the same formula (7'), which is really equivalent to defining the distance as the mathematical expectation of the number of chiasmata (not crossovers in G.'s sense) in the interval (i, j)."

A CRITERION OF CONVERGENCE FOR THE CLASSICAL ITERATIVE METHOD OF SOLVING LINEAR SIMULTANEOUS EQUATIONS

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The recent development of two devices[1, 2] for solving linear simultaneous equations by means of the classical iterative method[3] has stimulated the writer to investigate convergence criteria for the method. There are in the literature[4] necessary and sufficient criteria for convergence of symmetric systems, and sufficiency criteria for general systems. So far as the writer knows, however, this is the first development of a necessary and sufficient criterion for convergence in the general case. The results obtained are applicable to any arbitrary square non-singular matrix in which $a_{ii} \neq 0$.

Let the set of equations be represented by

$$AX = G, \tag{1}$$

[1] Morgan, T. D., Crawford, F. W., "Time-saving computing instruments designed for spectroscopic analysis", *The Oil and Gas Journal*, August 26 (1944), pp. 100–105.

[2] Berry, C. E., Wilcox, D. E., Rock, S. M., Washburn, H. W., "A computer for solving linear simultaneous equations", to be published.

[3] Hotelling, Harold, "Some new methods in matrix calculation", *The Annals of Mathematical Statistics*, Vol. XIV (1943), pp. 1–34.

[4] Mises, R. von and Pollaczek-Geiringer, Hilda, "Zusammenfassende Berichte. Praktische Verfahren der Gleichungsauflösung". *Zeitschrift für angewandte Math. und Mechanik*, Vol. 9 (1929), pp. 58–77, and 152–164.

in which A is the square matrix of the coefficients, X is the column matrix of the unknowns, and G is the column matrix of the constant terms. $|A|$ is the determinant of A.

We define a matrix A_1 which contains the prediagonal and diagonal terms of A, and a matrix A_2 which contains the postdiagonal terms of A. According to this definition,

$$A_1 + A_2 = A. \tag{2}$$

In the classical iterative method, arbitrary (or approximate) values of the x's are chosen, the first equation is solved for the first unknown, the second equation for the second unknown, etc., using in each equation the most recent approximations to the x's. This process may be written

$$A_1X^{(1)} + A_2X^{(0)} = G, \tag{3}$$

in which $X^{(0)}$ is the initial approximation matrix, and $X^{(1)}$ is the approximation matrix existing at the end of the first iterative cycle. The superscripts indicate the number of the approximation. The next cycle is described by

$$A_1X^{(2)} + A_2X^{(1)} = G, \tag{4}$$

and the mth by

$$A_1X^{(m)} + A_2X^{(m-1)} = G. \tag{5}$$

The method yields a solution, i.e., converges, if

$$\lim_{m\to\infty} (X^{(m)} - X) = 0.$$

Solving (5) explicitly for $X^{(m)}$,

$$X^{(m)} = A_1^{-1}G - A_1^{-1}A_2X^{(m-1)}. \tag{6}$$

Subtracting X from each side,

$$X^{(m)} - X = A_1^{-1}G - A_1^{-1}A_2X^{(m-1)} - X, \tag{7}$$

and making use of (1) and (2)

$$X^{(m)} - X = -A_1^{-1}A_2(X^{(m-1)} - X). \tag{8}$$

Since (8) applies for any value of m, we may write

$$X^{(m)} - X = (-A_1^{-1}A_2)^2(X^{(m-2)} - X), \tag{9}$$

and continuing this process,

$$X^{(m)} - X = (-A_1^{-1}A_2)^m(X^{(0)} - X). \tag{10}$$

Now, $\lim_{m\to\infty} (X^{(m)} - X) = 0$ if and only if

$$\lim_{m\to\infty} (-A_1^{-1}A_2)^m = 0. \tag{11}$$

This is a general result, applicable to any arrangement of the terms of an arbitrary square matrix A, subject only to the conditions that $|A| \neq 0$ and that no diagonal term of A is zero. In this latter exceptional case, the iterative method itself obviously cannot be applied.

The criterion (11) clearly shows that the order in which the elements of the matrix A are arranged is important. For instance, it is plain that an arrangement in which the diagonal terms are large and the off-diagonal terms, particularly the post-diagonal terms, are small will tend to favor convergence.

A somewhat relaxed condition, which is sufficient but not necessary, is obtained through the use of an inequality used by Hotelling[3], namely,

$$N(B^m) \leq [N(B)]^m, \tag{12}$$

in which $N(B)$ is the norm of the matrix B, that is, the square root of the sum of the products of its elements by their complex conjugates, or in the case of a real matrix the square root of the sum of the squares of the elements.

The condition is that, if

$$N(A_1^{-1}A_2) < 1, \tag{13}$$

then

$$\lim_{m\to\infty} (A_1^{-1}A_2)^m = 0. \tag{14}$$

Criterion (13) is readily computed, since A_1^{-1}, the reciprocal of a triangular matrix is readily computed, and the post-multiplication by A_2 involves a number of zero terms.

A more stringent condition than (13) though still not a necessary condition, is that if some finite number p can be found such that

$$N(A_1^{-1}A_2)^p < 1, \tag{15}$$

then (14) follows. Since n matrix squarings result in a value of $p = 2^n$, the size of the norm for fairly large values of p can be investigated without excessive labor.

A REMARK ON INDEPENDENCE OF LINEAR AND QUADRATIC FORMS INVOLVING INDEPENDENT GAUSSIAN VARIABLES

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The purpose of this note is to call attention to the following useful theorem, which to the best of my knowledge was never stated explicitly.

If $X_1, X_2, X_3, \cdots X_n$ are identically distributed, independent Gaussian random variables each having mean 0, then the necessary and sufficient condition that

$$\sum_{j,k=1}^{n} a_{jk}X_jX_k \quad \textit{and} \quad \sum_{j=1}^{n} \alpha_j X_j = \alpha\cdot X$$