SOME COMBINATORIAL FORMULAS ON MATHEMATICAL EXPECTATION

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The main problem considered here may be stated as follows:

Let $f_1(x), \cdots, f_n(x)$ be n polynomials. It is the purpose of this paper to establish formulas concerning the mathematical expectation (probable value) of the product

$$f_1(x_1) \cdots f_n(x_n),$$

where $x_1, \cdots, x_n$ are positive random variables and the sum of these is supposed known.

Before establishing the formulas let us introduce some notations for convenience.

1. Notation. (A) In this paper the notation $(m; k; x_1, \cdots, x_n)$ or $(m; k; x)$ is used to denote that a set of numbers $(x_1, \cdots, x_n)$ is over all different compositions of m into n parts with each $x \geqq k$, i.e. over all different integer solutions of the equation $x_1 + \cdots + x_n = m$ with each $x \geqq k$.

(B) Let m, δ be two positive real numbers. The notation $E(m, \delta, [f_1] \cdots [f_n])$ denotes the mathematical expectation of the product $f_1(x_1) \cdots f_n(x_n)$ in which the sum $m = x_1 + \cdots + x_n$ is known and for every $x_\nu(\nu = 1, \cdots, n)$ the value of x_ν/δ is a positive integer. The notation $E(m, \delta, [f_1] \cdots [f_n])$ thus implies that the value of m is a multiple of δ. We call the δ a "varying unit", i.e. the least possible difference between two different quantities x_i and x_j $i \neq j$. The notation $E(m\delta, [f]^n)$ is merely a special case that denotes the mathematical expectation of the product $f_1(x_1) \cdots f_n(x_n)$ under the known conditions

$$f_1 = \cdots f_n = f, \qquad x_1 + \cdots + x_n = m, \qquad \frac{x_\nu}{\delta} = \left[\frac{x_\nu}{\delta}\right] \geqq 1, \qquad (\nu = 1, \cdots, n),$$

where [ ] represents "integral part of".

(C) In order to simplify our formulas we always denote $f(x)$ by $f^{(x)}$, $f_{\nu_1} + \cdots + f_{\nu_s}$ by $f_{\nu_1 \cdots \nu_s}$ and $1.p_1 + \cdots + k.p_k$ by $\sigma(p)$ or σ. It is a convention that $\binom{m}{n} = 0$ for $m < n$.

2. Lemmas. Lemma 1. *Let $m, r_1, \cdots, r_n$ be non-negative integers. Then*

$$(1) \qquad \sum_{(m;0;x)} \prod_{\nu=1}^{n} \binom{x_\nu}{r_\nu} = \binom{m + n - 1}{r_1 + \cdots + r_n + n - 1}.$$

PROOF: The lemma follows immediately by considering the coefficient of the term $x^{m-(r_1+\cdots+r_n)}$ on both sides of

$$\left(\frac{1}{1-x}\right)^{r_1+1}\cdots\left(\frac{1}{1-x}\right)^{r_n+1}=\left(\frac{1}{1-x}\right)^{r_1+\cdots+r_n+n}$$

LEMMA 2. *Let $a, b, c, \cdots$ be any constants, and $k_1, k_2, k_3, \cdots$ any positive integers. Then*

$$\sum_{(m;1;x)}\prod_{\nu=1}^{n}\left[a\binom{x_\nu}{k_1}+b\binom{x_\nu}{k_2}+c\binom{x_\nu}{k_3}+\cdots\right] \tag{2}$$
$$= n!\sum_{(n;0;\alpha,\beta,\gamma,\cdots)}\binom{m+n-1}{\alpha k_1+\beta k_2+\gamma k_3+\cdots+n-1}\frac{a^\alpha}{\alpha!}\frac{b^\beta}{\beta!}\frac{c^\gamma}{\gamma!}\cdots.$$

PROOF: Expanding the left-hand side of (2) we see that the coefficient of the term $a^\alpha b^\beta c^\gamma\cdots$ is equal to

$$\frac{n!}{\alpha!\beta!\gamma!}\sum_{(m;0;x)}\binom{x_1}{k_1}\cdots\binom{x_\alpha}{k_1}\binom{x_{\alpha+1}}{k_2}\cdots\binom{x_{\alpha+\beta}}{k_2}\binom{x_{\alpha+\beta+1}}{k_3}\cdots\binom{x_{\alpha+\beta+\gamma}}{k_3}\cdots.$$

By Lemma 1 it becomes

$$\frac{n!}{\alpha!\beta!\gamma!}\binom{m+n-1}{\alpha k_1+\beta k_2+\gamma k_3+\cdots+n-1}.$$

Hence the lemma.

LEMMA 3. *Let m, $n(\leqq m)$ be two positive integers. Then, for any given polynomial $f(x)$ of the kth degree, we have*

$$\sum_{(m;1;x)}f(x_1)\cdots f(x_n)=n!\sum_{(n;0;p)}\binom{m+n-1}{\sigma+n-1}\prod_{\nu=0}^{k}\frac{[(f-1)^{(\nu)}]^{p_\nu}}{p_\nu!}, \tag{3}$$

where $f^{(x)}=f(x)$, $\sigma=\sigma(p)=1.p_1+\cdots+kp_k$.

PROOF: Since $f(x)$ is a polynomial of the kth degree, there exist $(k+1)$ values $\beta_k, \cdots, \beta_0$ such that

$$\sum_{i=0}^{k}\beta_i\binom{x}{i}=f(x).$$

By putting $x = 0, 1, \cdots, k$, it is orderly determined that

$$\beta_\nu=f^{(\nu)}-\binom{\nu}{1}f^{(\nu-1)}+\cdots+(-1)^\nu\binom{\nu}{\nu}f^{(0)}=(f-1)^{(\nu)},\qquad(\nu=0,1,\cdots,k).$$

The lemma is thus obtained by (2).

For convenience we denote the summation $\sum_{(m;1;x)}(m;1;x)\,f_1(x_1)\cdots f_n(x_n)$ by $S(m,[f_1]\cdots[f_n])$. Thus the formula (3) can be written as

$$S(m,[f]^n)=n!\sum_{(n;0;p)}\binom{m+n-1}{\sigma+n-1}\prod_{\nu=0}^{k}\frac{[(f-1)^{(\nu)}]^{p_\nu}}{p_\nu!}.$$

LEMMA 4. *Let* $f_1(x), \cdots, f_n(x)$ *be* n *given polynomials. Then*

$$(4) \qquad S(m, [f_1] \cdots [f_n]) = \frac{1}{n!} \sum_{\substack{(\nu_1 \cdots \nu_n) \\ 1 \le k \le n}} (-1)^{n-k} S(m, [f_{\nu_1} + \cdots + f_{\nu_k}]^n),$$

where $(\nu_1 \cdots \nu_k)$ *runs over all different combinations out of* $(1 \cdots n)$, $k = 1, \cdots, n$.

PROOF: The proof depends essentially on the formal logic theorem. Considering a typical term

$$\frac{n!}{q_1! \cdots q_t!} S(m, [f_{\nu_1}]^{q_1} \cdots [f_{\nu_t}]^{q_t}), \qquad 1 \le t \le n, \qquad q_1 + \cdots + q_t = n,$$

we see that it is contained in the last $(n - t + 1)$ summations of the righthand side of (4), i.e. in the summations $(\nu_1 \cdots \nu_k)$ as $k = t, t + 1, \cdots, n$. The number of occurrences of the term in the right-hand side of (4) is therefore

$$\sum_{\nu=0}^{n-t} (-1)^\nu \binom{n-t}{\nu} = \begin{matrix} 0 & \text{if} & t > n \\ 1 & \text{if} & t = n. \end{matrix}$$

The term vanishes generally except when $q_1 = \cdots = q_t = 1$. Hence the right-hand side gives

$$S(m, [f_1] \cdots [f_n]).$$

3. Theorems with formulas. In the following statements of theorems and corollaries, the notation $(x_1 \cdots x_n)$ is always to denote a set of undetermined quantities, though the kind of the quantities of the set is stated.

THEOREM 1. *Let* $(x_1 \cdots x_n)$ *be a set of natural numbers under a known condition* $x_1 + \cdots + x_n = m$. *Then, for any given polynomial* $f(x)$ *of the kth degree, we have*

$$(5) \qquad E(m, 1, [f]^n) = \frac{n!}{\binom{m-1}{n-1}} \sum_{(n;0;p)} \binom{m+n-1}{\sigma+n-1} \prod_{\nu=0}^{k} \frac{[(f-1)^\nu]^{p_\nu}}{p_\nu!}.$$

PROOF: Let $m' = m + nr$. By lemma 1 we then have

$$\sum_{(m;0;x)} \binom{x_1}{0} \cdots \binom{x_n}{0} = \sum_{(m';r;x)} 1 = \binom{m' - nr + n - 1}{n-1}.$$

This is the number of compositions of m' into n parts with each part $\geqq r$. In particular, for $r = 1$ we see that the number of compositions of m into n parts is $\binom{m-1}{n-1}$. Thus by the definition of mathematical expectation, the required value is equal to

$$\frac{S(m, [f]^n)}{S(m, [1]^n)}, \qquad \text{i.e.} \qquad \binom{m-1}{n-1}^{-1} S(m, [f]^n).$$

The theorem is therefore proved by Lemma 3.

COROLLARY 1. *Let $(x_1 \cdots x_n)$ be a set of positive quantities, of which the varying unit is δ, and the sum is m. Then, for any given polynomial $f(x)$ of the kth degree, we have*

$$(6) \qquad E(m, \delta, [f]^n) = \frac{n!}{\binom{\frac{m}{\delta} - 1}{n - 1}} \sum_{(n;0;p)} \binom{\frac{m}{\delta} + n - 1}{\sigma + n - 1} \prod_{\nu=0}^{k} \frac{[(q - 1)^{(\nu)}]^{p_\nu}}{p_\nu!},$$

where

$$g(x) = f(\delta x), \qquad \sigma = 1p_1 + \cdots + kp_k .$$

PROOF: It is deduced by the relation $E(m, \delta, [f(x)]^n) = E(m/\delta, 1, [f(\delta x)^n)$.

COROLLARY 2. *Let $(x_1 \cdots x_n)$ be a set of non-negative real numbers under a known condition $x_1 + \cdots + x_n = m$. Then, for any given polynomial $f(x) = a_0 + \cdots + a_k x^k$, we have*

$$(7) \qquad E(m, 0, [f]^n) = \frac{(n!)^2}{n} \sum_{(n;0;q)} \frac{m^\sigma}{(\sigma + n - 1)!} \frac{(0!a_0)^{q_0}}{q_0!} \cdots \frac{(k!a_k)^{q_k}}{q_k!},$$

where

$$a_k \neq 0, \qquad \sigma = \sigma(q) = q_1 + \cdots + kq_k .$$

PROOF: The proof of the corollary depends essentially on the concept that two different real numbers may differ by an arbitrarily small number h.

Let h be an arbitrary positive number and let $f(xh) = h^k g(x, h)$, where the number k is the degree of $f(x)$. Then, since

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (n - \nu)^p = \begin{cases} 0 & \text{if } p > n \\ n! & \text{if } p = n \\ \binom{n + 1}{2} n! & \text{if } p = n + 1, \end{cases}$$

we may write

$$\sum_{s=0}^{\nu} (-1)^s \binom{\nu}{s} g(\nu - s, h) = h^{\nu - k}[\nu! a_\nu + h \cdot R_\nu(h)],$$

$$\text{where} \quad \lim_{h \to 0} R_\nu(h) = \binom{\nu + 1}{2} \nu! \, a_{\nu+1} .$$

Now we pass to the limit $h \to 0$, in which it is assumed that h runs through a sequence of rational numbers of the form $1/N$. Thus by Corollary 2 we have

$$\lim_{h \to 0} E(m, h, [f]^n) = n!(n - 1)! \sum_{(n;0;p)} \frac{m^\sigma}{(\sigma + n - 1)!} \prod_{\nu=0}^{k} \frac{(\nu! a_\nu)^{p_\nu}}{p_\nu!} .$$

Hence the corollary.

It may be noted that this corollary can also be independently deduced by the proportion of the two integrals:

$$\int \cdots \int_R f(x_1) \cdots f(x_n) \, dx_1 \cdots dx_{n-1} : \qquad \int \cdots \int_R dx_1 \cdots dx_{n-1},$$

where the integrals are all taken over the region $R: x_1 + \cdots + x_n = m, x_1 \geq 0, \cdots x_n \geq 0$.

COROLLARY 3. *Let $(x_1 \cdots x_n)$ be a set of positive real numbers under a known condition $a < x_1 + \cdots + x_n < b$, where a, b are non-negative numbers. Then, for any given polynomial $f(x) = a_\nu + \cdots + a_k x^k$ $(a_k \neq 0)$, the mathematical expectation of the product $f(x_1) \cdots f(x_n)$, which we denote by $E((ab), 0, [f]^n)$, is given by the formula*

$$E(a, b), 0, [f]^n) = \frac{n!(n-1)!}{b-a} \times \sum_{(n;0;q)} \frac{b^{1+\sigma(q)} - a^{1+\sigma(q)}}{(1+\sigma(q))\cdot(n-1+\sigma(q))!} \frac{a_0^{q_0}}{q_0!} \cdots \frac{(k!a_k)^{q_k}}{q_k!}. \tag{8}$$

PROOF: Since the required mathematical expectation is the mean

$$\frac{1}{b-a}\int_a^b E(u, 0, [f]^n)\, du,$$

Corollary 3 follows from Corollary 2.

On the other hand we see that

$$\lim_{h\to 0} E(a, a+h), 0, [f]^n) = E(a, 0, [f]^n).$$

Hence Corollary 2 can also be deduced from Corollary 3.

THEOREM 2. (First generalization of Theorem 1). *Let $f_1(x), \cdots f_n(x)$ be n given polynomials, of which the highest degree is k. Then we have*

$$E(m, 1, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \cdots \nu_s) \\ 1 \leq s \leq n}} \sum_{(n;0;p)} (-1)^{n-s} \times \frac{\binom{m+n-1}{\sigma+n-1}}{\binom{m-1}{n-1}} \prod_{\mu=0}^{k} \frac{[(f_{\nu_1\cdots\nu_s} - 1)^{(\mu)}]^{p_\mu}}{p_\mu!}, \tag{9}$$

where

PROOF: In the proof of theorem 1 we have seen that

$$E(m, 1, [f]^n) = \binom{m-1}{n-1}^{-1} S(m, [f]^n).$$

Thus, by similar reasoning and lemma 4, we have

$$E(m, 1, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \cdots \nu_s) \\ 1 \leq s \leq n}} \frac{(-1)^{n-s}}{n!\binom{m-1}{n-1}} S(m, [f_{\nu_1\cdots\nu_s}]^n).$$

The theorem is proved by lemma 3.

COROLLARY 1. *Let δ be a varying unit. Then*

$$E(m, \delta, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \cdots \nu_s) \\ 1 \le s \le n}} \sum_{(n;0;p)} (-1)^{n-s}$$

$$\times \frac{\begin{pmatrix} \frac{m}{\delta} + n - 1 \\ \sigma + n - 1 \end{pmatrix}}{\begin{pmatrix} \frac{m}{\delta} - 1 \\ n - 1 \end{pmatrix}} \prod_{\mu=0}^{k} \frac{[(g_{\nu_1 \cdots \nu_s} - 1)^{(\mu)}]^{p_\mu}}{p_\mu !}, \tag{10}$$

where

$$g_\nu(x) = f_\nu(\delta x), \qquad g_{\nu_1 \cdots \nu_s} = g_{\nu_1} + \cdots + g_{\nu_s}.$$

PROOF: By the relation $E(m, \delta, [f_1(x)] \cdots [f_n(x)]) = E(m/\delta, 1, [f_1(\delta x)] \cdots [f_n(\delta x)])$ we obtain the corollary.

COROLLARY 2. *For any positive real number m, we have*

$$E(m, 0, [x^{p_1}] \cdots [x^{p_n}]) = \frac{p_1! \cdots p_n!(n-1)!}{(p_1 + \cdots + p_n + n - 1)!} m^{p_1 + \cdots + p_n}. \tag{11}$$

PROOF: Since $E(m, \delta, [f_1] \cdots [f_n]) = \sum (-1)^{n-s}/n! \, E(m, \delta[f_{\nu_1} \cdots {}_{\nu_s}]^n)$, we have, by letting $\delta \to 0$,

$$E(m, 0, [f_1] \cdots [f_n]) = \sum \frac{(-1)^{n-s}}{n!} E(m, 0, [f_{\nu_1 \cdots \nu_s}]^n).$$

The corollary is therefore deduced by (7).

THEOREM 3. (Second generalization of Theorem 1). *Let $(x_1 \cdots x_n)$ be a set of integers under known conditions $x_1 + \cdots + x_n = m$, $a \le x_i \le b$, where m, a, b are given integers. Then, for any given polynomial $f(x)$, the mathematical expectation of the product $f(x_1) \cdots f(x_n)$, denoted by $\underset{(ab)}{E}(m, 1, [f]^n$, is given by the formula*

$$\underset{(a,b)}{E}(m, 1, [f]^n) = \frac{\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} S(m', [g]^\nu [h]^{n-\nu})}{\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \binom{m'-1}{n-1}}, \tag{12}$$

where

$g(x) = f(b + x)$, $h(x) = f(a + x - 1)$ *and* $m' = m - (a-1)n + (a - b - 1)\nu$.

PROOF: Define $S(m, [f]^n) = 0$ for $m < n$, and $S(m, [f]^0) = \begin{matrix} 0 \text{ for } m > 0 \\ 1 \text{ for } m = 0 \end{matrix}$. We shall now prove that

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} S(m', [g]^\nu [h]^{n-\nu}) = \sum_{\substack{(x_1 \cdots x_n) \\ a \le x \le b}} f(x_1) \cdots f(x_n),$$

where on the right-hand side of the expression the set $(x_1, \cdots x_n)$ under the summation runs over all different compositions of m into n parts and

$$a \leq x_\nu \leq b, \qquad \nu = 1, \cdots, n.$$

For convenience we denote the left-hand side of the expression by $\mathfrak{S}$, that is,

$$\begin{aligned} \mathfrak{S} &= \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} S(m', [g]^\nu [h]^{n-\nu}) \\ &= \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \sum_{\bar{m}=\nu}^{m'} S(\bar{m}, [f(x+b)]^\nu) S(m' - \bar{m}, [f(x+a-1)]^{n-\nu}). \end{aligned}$$

Let $f(\bar{x}_1) \cdots f(\bar{x}_n)$ be a product term contained in $\mathfrak{S}$, i.e., $\bar{x}_1 + \cdots + \bar{x}_n \doteq m$; $\bar{x}_1 \geqq a, \cdots, \bar{x}_n \geqq a$. We assume that $\bar{x}_{\nu_1} \geq b + 1, \cdots, \bar{x}_{\nu_t} \geq b + 1$, where $\nu_1 \neq \nu_j$ if $i \neq j$. Then it is seen that the number of occurrences of the product term in $\mathfrak{S}$ is given by

$$\sum_{s=0}^{t} (-1)^s \binom{t}{s} = \begin{matrix} \textit{if} \quad t \geq 1 \\ \textit{if} \quad t = 0. \end{matrix}$$

Thus the product term $f(\bar{x}_1) \cdots f(\bar{x}_n)$ of $\mathfrak{S}$ vanishes except when

$$a \leq x_\nu \leq b, \qquad \nu = 1, \cdots, n.$$

Hence we have

$$\mathfrak{S} = \sum_{a \leq x \leq b} f(x_1) \cdots f(x_n).$$

Next, we shall find the number of different compositions of m into n parts with each $a \leq x_\nu \leq b$, i.e., the number of product terms of $\mathfrak{S}$. By the above result we see that the number is given by

$$\sum_{\nu=0}^{n} \sum_{m=\nu}^{m'} (-1)^\nu \binom{n}{\nu} \sum_{(\bar{m};1;x)} 1 \sum_{(m'-\bar{m};1;x)} 1 = \sum_{\nu=0}^{m} (-1)^\nu \binom{n}{\nu} \binom{m'-1}{n-1}.$$

Hence the theorem.

This theorem shows that the mathematical expectation $\underset{(ab)}{E}(m, 1, [f]^n)$ can be expressed by $S(\bar{m}[g]^\nu)$ and is therefore expressible in terms of linear combinations of the coefficients of the polynomial $f(x)$.

COROLLARY 1. *Let δ be a varying unit for which $\frac{m}{\delta}, \frac{a}{\delta}, \frac{b}{\delta}$ are all integers. Then*

$$\underset{(ab)}{E}(m, \delta, [f(x)]^n) = \underset{((a/\delta),(b/\delta))}{E}\left(\frac{m}{\delta}, 1, [f(\delta x)]^n\right).$$

COROLLARY 2. *Let $f_1(x), \cdots f_n(x)$ be n given polynomials. Then*

$$\underset{(ab)}{E}(m, 1, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \cdots \nu_s) \\ 1 \leq s \leq n}} \frac{(-1)^{n-s}}{n!} \underset{(a,b)}{E}(m, 1, [f_{\nu_1 \cdots \nu_s}]^n).$$

COROLLARY 3. *The number of integral solutions of the equation* $x_1 + \cdots + x_n = m$ *with* $a_1 \le x_1 \le b_1, \cdots, a_n \le x_n \le b_n$ *is equal to*

$$\sum_{\nu_1=0,\cdots,\nu_n=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_n} \cdot \binom{m+n-(a_1+\cdots+a_n)+(a_1-b_1-1)\nu_1+\cdots+(a_n-b_n-1)\nu_n-1}{n-1}.$$

PROOF: We have shown that the number of integral solutions of the equation $x_1 + \cdots + x_n = m$ with $a \le x_\nu \le b$ is given by

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu}\binom{m-(a-1)n+(a-b-1)\nu-1}{n-1}.$$

Hence the number of integral solutions of the equation $x_{11} + \cdots + x_{1n_1} + \cdots + x_{s1} + \cdots + x_{sn_s} = m$ with $a_\nu \le x_{\nu\mu} \le b_\nu$, $(\nu = 1 \cdots s, \mu = 1, \cdots n_\nu)$, is given by

$$\sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_s=0}^{n_s} (-1)^{\nu_1+\cdots+\nu_s} \prod_{i=1}^{s}\binom{n_i}{\nu_i} \cdot \left[\sum_{(m;1;m_i)} \prod_{i=1}^{s} \binom{m_i-(a_i-1)n_i+(a_i-b_i-1)\nu_i-1}{n_i-1}\right]$$

$$= \sum_{\nu_1=0,\cdots,\nu_s=0}^{n_1,\cdots,n_s} (-1)^{\nu_1+\cdots+\nu_s}\binom{n_1}{\nu_1}\cdots\binom{n_s}{\nu_s} \cdot \binom{m-(a_1-1)n_1-\cdots-(a_s-1)n_s+(a_1-b_1-1)\nu_1+\cdots+(a_s-b_s-1)\nu_s-1}{n_1+\cdots+n_s-1}.$$

The corollary follows at once by putting $n_1 = \cdots = n_s = 1, s = n$

This corollary can be restated in a more interesting manner as follows:

Let there be n store rooms, and let $b_1, \cdots, b_n$ be the numbers of stocks contained in 1st, 2nd, $\cdots$, n-th storerooms respectively. Then m stocks containing at least a_i stocks of the i-th storeroom $(i = 1, \cdots, n)$ can be chosen from these n storerooms in

$$\sum_{\nu_1=0,\cdots,\nu_n=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_n} \binom{m+n+(a_1-b_1-1)\nu_1+\cdots+(a_n-b_n-1)\nu_n-a_1-\cdots-a_n-1}{n-1}$$

different ways.

So far we have established several combinatorial formulas concerning the mathematical expectation of the product $f_1(x_1) \cdots f_n(x_n)$ under certain conditions. In the next section, we shall explain how to apply these formulas.

4. Applications. (a) *A criterion.* In order to make the above formulas applicable to practical problems we state a criterion as follows: The mathematical expectation of a function $F(x_1, \cdots, x_n)$ can be estimated by the above combinatorial formulas if and only if the sum of these undetermined quantities $x_1, \cdots, x_n$ is known and there exist n polynomials $f_1(x), \cdots, f_n(x)$ such that $F \propto f_1, \cdots, F \propto f_n$, where the quantities $x_1, \cdots, x_n$ may or may not be continuous. When the quantities are discontinuous, the varying unit is certainly given.

(b) *Some approximations.* For $f(x) = \beta_0 + \cdots + \beta_k x^k (\beta_k \neq 0)$ we may write

$$(f - 1)^{(\nu)} = \sum_{s=0}^{k} \nu! \beta_s S_{\nu,s},$$

where $S_{\nu,s}$ is a Stirling number of the second kind, as used by Jordan, and defined by

$$\nu! S_{\nu,s} = \sum_{x=0}^{\nu} (-1)^{\nu-x} \binom{\nu}{x} x^s.$$

Thus, the formulas (5) and (9) can be written as follows:

$$E(m, 1, [f]^n) = \sum_{(n;0;p)} \frac{(m + n - 1)!\,(m - n)!\,n!\,(n - 1)!}{(m - \sigma)!\,(\sigma + n - 1)!\,(m - 1)!} \cdot \prod_{\nu=0}^{k} \frac{(\beta_\nu \bar{S}_{\nu,\nu} + \cdots + \beta_k \bar{S}_{\nu,k})^{p_\nu}}{p_\nu!} \tag{5'}$$

$$E(m, 1, [f_1] \cdots [f_n]) = \sum_{\substack{(\nu_1 \cdots \nu_s) \\ 1 \le s \le n}} \sum_{(n;0;p)} (-1)^{n-s} \cdot \frac{(m + n - 1)!\,(m - n)!\,n!\,(n - 1)!}{(m - \sigma)!\,(\sigma + n - 1)!\,(m - 1)!} \prod_{\nu=0}^{k} \frac{(B_\nu \bar{S}_{\nu,\nu} + \cdots + B_k \bar{S}_{\nu k})^{p_\nu}}{p_\nu! \cdot}, \tag{9'}$$

where

$$\bar{S}_{\nu s} = \nu! S_{\nu s}, \qquad f_i = \beta_{i0} + \cdots + \beta_{ik} x^k, \qquad B_i = \beta_{1i} + \cdots + \beta_{ni}.$$

Now we state some convenient formulas concerning the number $\bar{S}_{\nu s}$.

If m is sufficiently large and t is smaller than m, the following recurrence relation is useful:

$$\begin{aligned} S_{m,m+t-1} &= \lambda_0 \binom{m + t - 1}{t} + \lambda_1 \binom{m + t - 1}{t + 1} + \cdots + \lambda_{t-2} \binom{m + t - 1}{2t - 2} \\ S_{m,m+t} &= \binom{m + t}{t + 1} + [(t + 1)\lambda_0 + 2\lambda_1] \binom{m + t}{t + 2} + \cdots + [(2t - 1)\lambda_{t-2} + t\lambda_{t-1}] \binom{m + t}{2t}, \end{aligned} \tag{13}$$

where $\lambda_\nu \equiv 1$, $\lambda_{t-1} \equiv 0$ and $\lambda_1, \cdots, \lambda_{t-2}$ are all independent of m.

Starting from the first equality and using the recurrence relation $S_{m,n+1} = mS_{m,n} + S_{m-1,n}$ successively we have

$$\begin{aligned} S_{m,m+t} &= \sum_{\nu=1}^{m} (m - \nu + 1) S_{m-\nu+1,m+t-\nu} \\ &= \sum_{j=0}^{t-2} \lambda_j \left[\sum_{\nu=1}^{m} \binom{m+t-\nu}{t+j+1} (t+j+1) + \sum_{\nu=1}^{m} \binom{m+t-\nu}{t+j} (j+1) \right] \\ &= \sum_{j=0}^{t-2} \lambda_j \left[\binom{m+t}{t+j+2} (t+j+1) + \binom{m+t}{t+j+1} (j+1) \right] \\ &= \sum_{j=0}^{t-1} [(t+j)\lambda_{j-1} + (1+j)\lambda_j] \binom{m+t}{t+j+1}, \end{aligned}$$

where $\lambda_{-1} = \lambda_{t-1} = 0$. The recurrence relation is thus deduced.

Writing

$$S_{m,m+t} = \binom{m+t}{t+1} + \lambda_1 \binom{m+t}{t+2} + \cdots + \lambda_{t-1} \binom{m+t}{2t},$$

and using the recurrence relation as obtained above, the coefficients $\lambda_1, \cdots, \lambda_{t-1}$ may be exhibited as follows:

t	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
1								
2	3							
3	10	5						
4	25	105	105					
5	56	490	1260	945				
6	119	1918	9450	17325	10395			
7	246	6825	56980	190575	270270	135135		
8	501	22935	302995	1636635	4099095	4729725	2027025	
9	1012	74316	1487200	12122110	47507460	94594500	91891800	34459425

Now let

$$\bar{S}_{n,n+t} = \left[\binom{n+t}{t+1} + \lambda_1(t) \binom{n+t}{t+2} + \cdots + \lambda_{t-1}(t) \binom{n+t}{2t} \right] n!.$$

The recurrence relation obtained above gives

$$\begin{aligned} \lambda_{t-1}(t) &= (2t-1)\lambda_{t-2}(t-1) \\ \lambda_{t-2}(t) &= 2(t-1)\lambda_{t-3}(t-1) + (t-1)\lambda_{t-2}(t-1). \end{aligned}$$

Thus we obtain

$$\lambda_{t-1}(t) = \frac{(2t)!}{t!\,2^t}.$$

$$\lambda_{t-2}(t) = (t-1)! \sum_{\nu=1}^{t-1} 2^{t-2\nu-1} \nu \cdot \binom{2\nu}{\nu}.$$

Let

$$\theta(t) = \sum_{x=1}^{t-1} \frac{x}{2^{2x}} \binom{2x}{x}.$$

Since the orders of $\binom{n+t}{t+1}, \cdots, \binom{n+t}{2t-1}$ are all less than $2t$ as $n \to \infty$, and since

$$\begin{aligned}
\binom{n+t}{2t} \lambda_{t-1}(t) &= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \prod_{x=o}^{2t-1}\left(1 + \frac{t-x}{n}\right) \\
&= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \left(1 + \frac{t}{n}\right) \prod_{x=o}^{t-1}\left(1 - \frac{x^2}{n^2}\right) \\
&= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t \left(1 + \frac{t}{n}\right)(1 - O(n^{-2})) \\
&= \left(1 + \frac{t - O(n^{-1})}{n}\right)\left(\frac{n^2}{2}\right)^t \frac{1}{t!}, \\
\binom{n+t}{2t-1} \lambda_{t-2}(t) &= \frac{2t}{n-t+1}\binom{n+t}{2t}(t-1)!\, 2^{t-1}\theta(t) \\
&= \frac{4^t\theta(t)}{n-t+1}\binom{2t}{t}^{-1}\binom{n+t}{2t}\lambda_{t-1}(t) \\
&= \frac{4^t\theta(t)}{n-t+1}\binom{2t}{t}^{-1}\left(1 + \frac{t - O(n^{-1})}{n}\right)\left(\frac{n^2}{2}\right)^t \frac{1}{t!} \\
&= \frac{4^t\theta(t)\binom{2t}{t}^{-1} + O(n^{-1})}{n}\left(\frac{n^2}{2}\right)^t \frac{1}{t!}.
\end{aligned}$$

We may write (by Stirling's formula)

$$\bar{S}_{n,n+t} = \left(\frac{n}{e}\right)^n \left(\frac{n^2}{2}\right)^t \frac{\sqrt{2\pi n}}{t!}\left(1 + \frac{t + 4^t\binom{2t}{t}^{-1}\theta(t) + \epsilon_n}{n}\right),$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Now it is easily proved that the inequality

$$\sqrt{\frac{x}{\pi}} > \frac{x}{2^{2x}}\binom{2x}{x} > \sqrt{\frac{x-1}{\pi}}$$

holds for every positive integer x. We have, therefore,

$$\theta(t) < \sum_{x=1}^{t-1}\sqrt{\frac{x}{\pi}} < \int_1^t \sqrt{\frac{x}{\pi}}\, dx = \frac{2}{3\sqrt{\pi}}(t^{\frac{3}{2}} - 1);$$

$$\theta(t) > \sum_{x=0}^{t-2}\sqrt{\frac{x}{\pi}} > \int_0^{t-2} \sqrt{\frac{x}{\pi}}\, dx = \frac{2}{3\sqrt{\pi}}(t-2)^{\frac{3}{2}};$$

and

$$\sqrt{t\pi} < 4^t \binom{2t}{t}^{-1} < \frac{t}{\sqrt{t-1}} \sqrt{\pi}.$$

Using these inequalities we have

$$l_t = \frac{2}{3} \sqrt{t}\, (t+2)^{\frac{3}{2}} < 4^t \binom{2t}{t}^{-1} \theta(t) < \frac{2}{3} \sqrt{\frac{t^2}{t-1}}\, (t^{\frac{3}{2}} - 1) = u_t,$$

where it may be noted that

$$\lim_{t \to \infty} \frac{u_t}{l_t} = 1.$$

Hence we have in conclusion

$$\sum_{x=0}^{n} (-1)^{n-x} \binom{n}{x} x^{n+t} = \left(\frac{n}{e}\right)^n \left(\frac{n^2}{2}\right)^t \frac{\sqrt{2\pi n}}{t!} \left(1 + \frac{k(t) + t + \epsilon_n}{n}\right), \tag{14}$$

where

$$\frac{2}{3} (t-2)^{\frac{3}{2}} < \frac{k(t)}{\sqrt{t}} < \frac{2}{3} \sqrt{\frac{t}{t-1}}\, (t^{\frac{3}{2}} - 1).$$

Evidently the formula (14) implies (15) and (16):

$$\begin{aligned} &\sum_{x=0}^{n} (-1)^{n-x} \binom{n}{x} x^{n+t} \\ &\quad \sim \left(\frac{n}{e}\right)^n \left(\frac{n^2}{2}\right)^t \frac{\sqrt{2\pi n}}{t!} \left(1 + \frac{2t^2}{3n}\right), \qquad t = 0(n^{1-\epsilon}), \qquad \epsilon > 0. \end{aligned} \tag{15}$$

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}. \qquad \text{(Stirling's formula)}. \tag{16}$$

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