4. Sampling from $F(x) = \int_0^x He^{-Ht}\,dt$, $(0 \le x < \infty; H > 0)$. If $F(x) = \int_0^x He^{-Ht}\,dt$, the probability function of S can be determined but is very cumbersome in the form in which it is known to the writer. The characteristic function, say $g(\theta)$, of the probability function of S will be given instead. By use of (2.1) it can be shown that:

$$g(\theta) = e^{iD\theta} \prod_{\lambda=1}^{n-1} \left\{ \frac{i\theta e^{D(i\theta - \lambda H)} - \lambda H}{i\theta - \lambda H} \right\}, \tag{4.1}$$

where $i = \sqrt{-1}$.

The expected value, $E(S)$, and variance, σ_S^2, of S are:

$$E(S) = D + \frac{1}{H} \sum_{\lambda=1}^{n-1} \frac{(1 - e^{-DH\lambda})}{\lambda},$$

$$\sigma_S^2 = \frac{1}{H^2} \sum_{\lambda=1}^{n-1} \frac{(1 - e^{-2DH\lambda})}{\lambda^2} - \frac{2D}{H} \sum_{\lambda=1}^{n-1} \frac{e^{-DH\lambda}}{\lambda}. \tag{4.2}$$

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INFORMATION GIVEN BY ODD MOMENTS

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The widespread use of the third moment about the mean as a measure of skewness and the belief engendered by this use that a distribution is symmetric if its third moment is zero prompt the question of how much information about a distribution can be deduced from a knowledge of its odd moments. An answer to this question is: *Let $F(x)$, a cumulative distribution function; $\{\mu_{2n-1}\}$, $(n = 1, 2, \cdots)$, a sequence of real numbers; and $\epsilon > 0$ be arbitrary. There exists a c.d.f., $F^*(x)$, having as odd moments the terms of the given sequence and such that*

$$|F(x) - F^*(x)| \le \epsilon, \ \underline{all}\ x. \tag{1}$$

If the mean of $F(x)$ is equal to μ_1 and the variance of $F(x)$ is not zero, it can be shown that $F^*(x)$ may be chosen so that in addition the variance of $F^*(x)$ is equal to that of $F(x)$.

An immediate consequence of our statement is that a distribution need not be symmetric even though all its odd moments vanish. Such an asymmetric distribution, due to Stieltjes, is given by:

$$(2)\quad dF(x) = 1/48\, e^{-|x|^{\frac{1}{4}}} \cdot (1 - k \sin |x|^{\frac{1}{4}})\, dx, \; -\infty < x < \infty, \; k = -1 \text{ if } x < 0,$$
$$k = 1 \text{ if } x \geq 0.$$

The proof of our statement will follow easily from the following:

LEMMA. *Let $\{m_{2n-1}\}$, a sequence of real numbers be given. There exists a c.d.f. having as odd moments the given numbers.*

We construct a sequence $\{H_n\}$ of increasing step-functions in such a manner that for every n, the first n moments of H_n are the first of the given numbers, and such that this sequence converges to a monotone function having all the desired moments. A slight modification of this function will give the desired c.d.f.

Let H_0 be identically zero. We form H_1 by adding to H_0 a jump or mass of $\frac{1}{2}$ at $x = 2m_1$. In general, H_k is formed from H_{k-1} by adding to it k masses chosen so that their first $(k - 1)$ odd moments are zero and so that the kth odd moment of H_k is m_{2k-1}. This we do by adding the masses $|x_j|$, $(j = 1, 2, \cdots, k)$, at the points $e_j jp$ where the x_j's are the solutions of:

$$\begin{aligned}
px_1 + \quad 2px_2 + \cdots + \quad kpx_k &= 0\\
p^3x_1 + (2p)^3x_2 + \cdots + (kp)^3x_k &= 0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\\
p^{2k-3}x_1 + (2p)^{2k-3}x_2 + \cdots + (kp)^{2k-3}x_k &= 0\\
p^{2k-1}x_1 + (2p)^{2k-1}x_2 + \cdots + (kp)^{2k-1}x_k &= m_{2k-1} - m(H_{k-1}),
\end{aligned}$$

$m(H_{k-1})$ is the kth odd moment of H_{k-1}, e_j is the sign of x_j and p is a parameter. Since the determinant of this system is a Vandermonde determinant, there exists a unique set of solutions for every non-zero value of the parameter. The masses thus chosen clearly have the specified moments. Eliminating p from the left sides of the equations by division, it is apparent that the x_j's are all linear functions of $p^{-(2k-1)}$. Thus we may choose p so large that the sum of the masses added at this step does not exceed $1/2^k$. The absolute odd moments of orders less than $(2k - 1)$ of these k masses are also linear functions of negative powers of p. We may thus insure by further increasing our choice of p that the $(2k - 1 - 2r)$th absolute moment of H_k does not exceed the corresponding moment of H_{k-1} by more than $1/2^r$. For definiteness, we choose p as the smallest number satisfying these requirements.

The first of these restrictions on p insures that for each value of x, the sequence $H_n(x)$ is increasing and bounded from above by one. The sequence of functions thus converges to a monotone function $H^*(x)$ with the property that $H^*(-\infty) = 0$, $H^*(\infty) \leq 1$. The other restrictions on p insure that the sequences of absolute odd moments of all orders are uniformly bounded, a bound for the absolute moments of order $2k - 1$ being one greater than the absolute moment of this order of H_k. This in turn insures that the odd moments of $H^*(x)$ exist and that they have the desired values. By adding a jump of $1 - H^*(\infty)$ at the origin we obtain $H(x)$, a c.d.f. with the given odd moments.

The main statement of this note is an immediate consequence of the lemma. Let the kth odd moment of $F(x)$ be M_{2k-1}, which we assume to be finite, and let the sequence $\{m_{2k-1}\}$ be defined by the relationships:

$$\mu_{2k-1} = (1 - \epsilon)M_{2k-1} + \epsilon m_{2k-1}, \qquad (k = 1,2, \cdots).$$

Let $H(x)$ have the m's as odd moments. The c.d.f. $F^*(x)$ defined by

$$F^*(x) = (1 - \epsilon)F(x) + \epsilon H(x)$$

clearly has the properties stated above, and our statement is proved. If the moments of $F(x)$ are not all finite, the proof will need only minor modifications.

If one asks in addition that F^* have a finite range, F^* will, in general, not exist. If, for example, the range of F is finite and its odd moments are zero, then F must be symmetric about the origin, for F^* defined by $dF^*(x) = dF(-x)$ would have the same moments as F. But a c.d.f. with finite range is determined by its moments; hence $F(x) = F^*(x)$.

SOME ORDER STATISTIC DISTRIBUTIONS FOR SAMPLES OF SIZE FOUR

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1. Summary. Let x_1, x_2, x_3, x_4 represent the values of a sample of size four drawn from a normal population. There is no loss of generality in assuming that the distribution function of this population has zero mean and unit variance. Denote it by $N(0, 1)$. Let $x_{(i)}$ be the ith largest of x_1, x_2, x_3, x_4. The purpose of this note is to determine the joint distribution of

$x_{(4)} + x_{(3)} - x_{(2)} - x_{(1)}$, $x_{(4)} - x_{(3)} + x_{(2)} - x_{(1)}$, and $x_{(4)} - x_{(3)} - x_{(2)} + x_{(1)}$, and derive from this joint distribution the joint distributions of these statistics taken in pairs, also the distribution of each statistic itself.

2. Analysis. Consider the joint distribution of

$$r_1 = \tfrac{1}{2}(x_4 + x_3 - x_2 - x_1)$$

$$r_2 = \tfrac{1}{2}(x_4 - x_3 + x_2 - x_1)$$

$$r_3 = \tfrac{1}{2}(x_4 - x_3 - x_2 + x_1).$$