or returning to the original notation and retaining terms in $1/N$,

$$r \sim r_\infty\left(1 + \frac{1}{2N}\right). \tag{3}$$

If x_p is defined by $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_p} e^{-t^2/2}\,dt = p$ we know from [3] that

$$\frac{\chi_\beta^2}{n} \sim 1 + \frac{\sqrt{2}\,x_{1-\beta}}{\sqrt{n}} + \frac{2}{3}\frac{x_{1-\beta}^2 - 1}{n}. \tag{4}$$

Proceeding formally and retaining terms in $1/N$ we obtain

$$\left(\frac{n}{\chi_\beta^2}\right)^{\frac{1}{2}} = \left(1 - \frac{x_{1-\beta}}{\sqrt{2N}} + \frac{4 + 5x_{1-\beta}^2}{12N}\right)$$

and multiplying by the expression for r given by equation (3) we find the desired expansion for λ.

$$\lambda \sim r_\infty\left(1 - \frac{x_{1-\beta}}{\sqrt{2N}} + \frac{5x_{1-\beta}^2 + 10}{12N}\right). \tag{5}$$

Recall that both r_∞ and $x_{1-\beta}$ are readily obtainable from tables of the normal curve; in fact, r_∞ is defined by

$$\frac{1}{\sqrt{2\pi}}\int_{-r_\infty}^{r_\infty} e^{-t^2/2}\,dt = \gamma \text{ and } x_{1-\beta} \text{ is defined by } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_{1-\beta}} e^{-t^2/2}\,dt = 1 - \beta.$$

A comparative table of approximate and exact values of λ is given in Table 1. From the table we see that for $N \geqq 800$ the error is less than 1 in the 4th significant figure, and for $N \geqq 160$ the error is less than 1 in the 3rd significant figure within the limits of β and γ considered. The approximation will be less exact for higher values of β and γ.

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THE PROBABILITY DISTRIBUTION OF THE MEASURE OF A RANDOM LINEAR SET

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1. Introduction. Consider a random sample $0_n(x_1, \cdots, x_n)$ of n values of a one-dimensional random variable x having cumulative distribution function $F(x)$. Let there be associated with each x an interval of length D centered at x

(D a positive constant). Let $\bar{S}(0_n)$ denote the random set which is the point-set sum of the n intervals associated with 0_n; $\bar{S}(0_n)$ is a set of one or more intervals. Let S denote the measure of $\bar{S}(0_n)$ (S is the sum of lengths of the intervals composing $\bar{S}(0_n)$). Given F, n and D, what is the probability function of S? This note contains a solution of the problem for $F(x) = x$, $(0 \leq x \leq 1)$; the case of $F(x) = \int_0^x He^{-Ht}\,dt$, $(0 \leq x < \infty; H > 0)$, is also treated.

2. Sampling from a uniform distribution. Let $y = S - D$. The range of y is $0 \leq y \leq m$, where m denotes the minimum of 1 and $(n - 1)D$. Let $x_1, \cdots, x_n$ be the sample values arranged in increasing order of magnitude. Make the transformation

$$\begin{aligned} y_0 &= x_1 \\ y_i &= x_{i+1} - x_i, \qquad (i = 1, \cdots, n-1). \end{aligned} \tag{2.1}$$

y can be expressed as $\sum_{i=1}^{n-1} m(y_i, D)$, where $m(y_i, D)$ denotes the minimum of y_i and D. The probability function of $(y_0, y_1, \cdots, y_{n-1})$ is $n! \prod_{u=0}^{n-1} dy_u$, $\left(y_u \geq 0; \sum_{u=0}^{n-1} y_u \leq 1\right)$. If $m = (n-1)D$, then $y = (n-1)D$ if and only if $y_i \geq D$, $(i = 1, \cdots, n-1)$; for a fixed y_0 it can be shown by use of the Dirichlet integral that the volume of the $(n-1)$ dimensional region in which any point $(y_0, y_1, \cdots, y_{n-1})$ satisfies this condition is $\frac{(1 - y_0 - (n-1)D)^{n-1}}{(n-1)!}$. It follows that:

$$\begin{aligned} \Pr\{y = (n-1)D\} &= n \int_{y_0=0}^{1-(n-1)D} [1 - y_0 - (n-1)D]^{n-1}\,dy_0 \\ &= [1 - (n-1)D]^n, \qquad ((n-1)D \leq 1). \end{aligned} \tag{2.2}$$

The probability that $Y < y < Y + \Delta Y$ (where $Y < m$ and ΔY denotes an arbitrarily small positive increment in Y) can be evaluated by determining volumes of certain regions contained in the tetrahedron defined by $y_u \geq 0$, $\sum_{u=0}^{n-1} y_u \leq 1$. Consider the following conditions:

(a) $qD \leq Y < (q+1)D$ $\quad$ ($q = 0, 1, \cdots, M$; M denotes the minimum of $(n-2)$ and the greatest integer less than $\frac{1}{D}$),

(b) $\quad y_u \geq D$ $\quad$ $(u = 1, \cdots, j; j \leq q)$,

(c) $\sum_{u=0}^{j} y_u \leq 1 - y_0 - y + jD$,

(d) $\quad y_v < D$ $\quad$ $(v = j+1, \cdots, n-1)$.

The probability that $Y < y < Y + \Delta Y$ and that (b), (c) and (d) are satisfied is:

$$n! \int_{y=Y}^{Y+\Delta Y} B_j(y) \int_{y_0=0}^{1-y} A_j(y, y_0)\, dy_0 \frac{dy}{\sqrt{n-j-1}}, \tag{2.3}$$

where $A_j(y, y_0)$ denotes the j dimensional volume of the region in which any point $(y_1, \cdots, y_j)$ satisfies (b) and (c), and $B_j(y)$ denotes the $(n - j - 2)$ dimensional volume of intersection of the hyperplane $\sum_{v=j+1}^{n-1} y_v = y - jD$ with an $(n - j - 1)$ dimensional cube $(0 \leq y_v \leq D)$. It is clear that if any other of the $\binom{n-1}{j}$ combinations of j y's out of the set of $(n - 1)$ y's had been specified in (b) and the $(n - j - 1)$ complementary y's had been specified in (d), the corresponding $A_j(y, y_0)$ and $B_j(y)$ would be equal to those given in (2.3); hence

$$\Pr\{Y < y < Y + \Delta Y\} = n! \sum_{j=0}^{q} \binom{n-1}{j} \int_{y=Y}^{Y+\Delta Y} B_j(y) \cdot \int_{y_0=0}^{1-Y} A_j(y, y_0)\, dy_0 \frac{dy}{\sqrt{n-j-1}}, \tag{2.4}$$

$$qD \leq Y < (q+1)D,\ Y < m, \qquad (q = 0, 1, \cdots, M).$$

$A_j(y, y_0) = \dfrac{(1 - y_0 - y)^j}{j!}$, and (see [1] and [2])

$$B_j(y) = \frac{\sqrt{n-j-1}}{(n-j-2)!} \sum_{r=0}^{q-j} (-1)^r \binom{n-j-1}{r} [y - D(j+r)]^{n-j-2}. \tag{2.5}$$

From (2.4) and (2.5) it follows that the probability function of y, say $f_n(y)$, is:

$$f_n(y) = n \sum_{j=0}^{q} \sum_{r=0}^{q-j} (-1)^r \binom{n-1}{j}\binom{n-1}{j+1} \cdot \binom{n-j-1}{r} (1-y)^{j+1} [y - D(j+r)]^{n-j-2}, \tag{2.6}$$

$$qD \leq y < (q+1)D, \quad (q = 0, \cdots, M), \qquad y < m.$$

$f_n(y)$ is not defined at $(n - 1)D$ if $(n - 1)D < 1$ (see (2.2)); if $m = 1$, the range of definition of $f_n(y)$ as given in (2.6) is $y \leq 1$.

The cumulative distribution function of y is continuous with the exception, in the case of $(n - 1)D < 1$, of a saltus of amount $[1 - (n - 1)D]^n$ at $y = (n - 1)D$ (see (2.2)). The probability function $f_n(y)$ is continuous over the range $0 \leq y < m$ with the exception, in the case of $n \geq 3$ and $(n - 2)D < 1$, of a simple discontinuity at $y = (n - 2)D$.

For $n = 2$ and $D < 1$,

$$f_2(y) = 2(1 - y), \qquad (0 \leq y < D),$$

and $\Pr\{y = D\} = (1 - D)^2$.

For $n = 3$ and $2D < 1$,

$$f_3(y) = 6(1 - y)y, \qquad (0 \leq y < D),$$

$$f_3(y) = 6(1 - y)y - 12(1 - y)(y - D) + 6(1 - y)^2, \quad (D \leq y < 2D),$$

and $\Pr\{y = 2D\} = (1 - 2D)^3$.

The expected value, say $E(y)$, of y is:

$$(2.7) \qquad \begin{aligned} E(y) &= \frac{(n-1)}{(n+1)}[1 - (1 - D)^{n+1}] & (D \leq 1); \\ &= \frac{(n-1)}{(n+1)} & (D > 1). \end{aligned}$$

The expected value of S is $D + E(y)$. $E(y)$ can be derived by use of (2.6) or by use of a theorem of Robbins [3].

3. Probability that random linear set covers range of variate. Given that $F(x) = x$, $(0 \leq x \leq 1)$, and $nD > 1$, what is the probability, say ${}_nP_D$, that $\bar{S}(0_n)$ contains the interval $(0 \leq x \leq 1)$? If $D < 1$, the interval is covered if and only if (i), (ii) and (iii) below are all satisfied:

(i) $$y_u \leq D, \qquad (u = 1, \cdots, n - 1),$$

(ii) $$\sum_{u=1}^{n-1} y_u \geq \left(1 - y_0 - \frac{D}{2}\right),$$

(iii) $$y_0 \leq \frac{D}{2}.$$

${}_nP_D$ can be expressed as follows:

$$(3.1) \qquad {}_nP_D = n! \int_{y_0=0}^{D/2} \int_{z=1-y_0-D/2}^{1-y_0} C_{n-1}(z) \frac{dz}{\sqrt{n-1}}\, dy_0,$$

where $C_{n-1}(z)$ (see [2]) denotes the $(n - 2)$ dimensional volume of the intersection of the hyperplane $\sum_{u=1}^{n-1} y_u = z$ with an $(n - 1)$ cube $0 \leq y_u \leq D$. It follows from (2.5) and (3.1) that

$$(3.2) \qquad \begin{aligned} {}_nP_D = \sum_{u=0}^{[1/D]} (-1)^u \binom{n-1}{u}(1 - uD)^n & \\ - 2 \sum_{u=0}^{[(1/D)-\frac{1}{2}]} (-1)^u \binom{n-1}{u}\left(1 - uD - \frac{D}{2}\right)^n & \\ + \sum_{u=0}^{[(1/D)-1]} (-1)^u \binom{n-1}{u}(1 - uD - D)^n, & \end{aligned}$$

where $D < 1$ and $[x]$ denotes the greatest integer less than x. If $1 \leq D < 2$, ${}_nP_D = 1 - 2\left(1 - \frac{D}{2}\right)^n$.

4. Sampling from $F(x) = \int_0^x He^{-Ht}\,dt$, $(0 \leq x < \infty; H > 0)$. If $F(x) = \int_0^x He^{-Ht}\,dt$, the probability function of S can be determined but is very cumbersome in the form in which it is known to the writer. The characteristic function, say $g(\theta)$, of the probability function of S will be given instead. By use of (2.1) it can be shown that:

$$g(\theta) = e^{iD\theta} \prod_{\lambda=1}^{n-1} \left\{ \frac{i\theta e^{D(i\theta - \lambda H)} - \lambda H}{i\theta - \lambda H} \right\}, \tag{4.1}$$

where $i = \sqrt{-1}$.

The expected value, $E(S)$, and variance, σ_S^2, of S are:

$$\begin{aligned} E(S) &= D + \frac{1}{H} \sum_{\lambda=1}^{n-1} \frac{(1 - e^{-DH\lambda})}{\lambda}, \\ \sigma_S^2 &= \frac{1}{H^2} \sum_{\lambda=1}^{n-1} \frac{(1 - e^{-2DH\lambda})}{\lambda^2} - \frac{2D}{H} \sum_{\lambda=1}^{n-1} \frac{e^{-DH\lambda}}{\lambda}. \end{aligned} \tag{4.2}$$

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INFORMATION GIVEN BY ODD MOMENTS

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The widespread use of the third moment about the mean as a measure of skewness and the belief engendered by this use that a distribution is symmetric if its third moment is zero prompt the question of how much information about a distribution can be deduced from a knowledge of its odd moments. An answer to this question is: *Let* $F(x)$, *a cumulative distribution function;* $\{\mu_{2n-1}\}$, $(n = 1, 2, \cdots)$, *a sequence of real numbers; and* $\epsilon > 0$ *be arbitrary. There exists a c.d.f.,* $F^*(x)$, *having as odd moments the terms of the given sequence and such that*

$$|F(x) - F^*(x)| \leq \epsilon, \ \underline{\text{all}}\ x. \tag{1}$$

If the mean of $F(x)$ is equal to μ_1 and the variance of $F(x)$ is not zero, it can be shown that $F^*(x)$ may be chosen so that in addition the variance of $F^*(x)$ is equal to that of $F(x)$.

An immediate consequence of our statement is that a distribution need not be