SAMPLE CRITERIA FOR TESTING EQUALITY OF MEANS, EQUALITY OF VARIANCES, AND EQUALITY OF COVARIANCES IN A NORMAL MULTIVARIATE DISTRIBUTION

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Summary. In this paper statistical test criteria are developed for testing equality of means, equality of variances and equality of covariances in a normal multivariate population of k variables on the basis of a sample. More specifically, three statistical hypotheses are considered: (i) H_{mvc}, the hypothesis that the means are equal, the variances are equal, and the covariances are equal, (ii) H_{vc}, the hypothesis that variances are equal and covariances are equal, irrespective of the values of the means, and (iii) H_m, the hypothesis of equal means, assuming variances are equal and covariances are equal.

Test criteria L_{mvc}, L_{vc}, and L_m are developed by the Neyman-Pearson method of likelihood ratios for testing H_{mvc}, H_{vc} and H_m respectively. The exact moments of each of the three test criteria when the three corresponding hypotheses are true are determined for any number k of variables and for any size, n, of the sample for which the distributions exist. The exact distributions of L_{mvc} and L_{vc} are determined for $k = 2$ and $k = 3$, and the exact distribution of L_m is found for any k; these are all beta (Pearson Type I) distributions. Tables of 5% and 1% points of L_{mvc}, L_{vc} and L_m, based on Thompson's tables of percentage points of the Incomplete Beta Function, are given for certain values of k and n (Tables I and II). Also tables of values of approximate 5% and 1% points of $-n \ln L_{mvc}$, $-n \ln L_{vc}$ and $-n(k-1) \ln L_m$ for large values of n are given (Table III), based on the fact that these three quantities are approximately distributed according to chi-square laws for large values of n with $\frac{1}{2}k(k+3)-3$, $\frac{1}{2}k(k+1) - 2$, and $k - 1$ degrees of freedom respectively. A table (Table IV) is given which shows how accurate the resulting approximate 5% and 1% points of L_{mvc}, L_c and L_m are.

The paper is written in two parts. In Part I the problem of testing the three hypotheses is discussed and the mathematical results are presented together with an illustrative example. Part II is given for the reader who wishes to study the mathematical derivation of the results.

I. The Problem and a Statement of Results

1.1. Introduction. Situations occasionally arise, in which it may be desired to test the hypothesis that the means are equal, the variances are equal and the covariances are equal in a multivariate population in which the variables are correlated, the test to be made on the basis of a sample from such a population. In the case of a normal multivariate distribution this means testing the hypothesis that the distribution is symmetric with respect to the variables.

As an example[1] suppose three "parallel forms" of a test are constructed and all are given to a group of n college entrance students. On the basis of the scores of the n students on the three tests, how could one test the hypothesis that the three tests are really parallel forms, as far as means, variances and covariances are concerned? In other words, how could one test the hypothesis that the scores can be regarded as being from a sample of individuals from a college entrance population of individuals in which the distribution function of the three variables is such that the means of the three variables are all equal, the variances are equal and the covariances are equal? Actually, as far as practical considerations are concerned in testing work, it is frequently sufficient to consider only normally distributed populations. So therefore one may raise the question as to how to test the hypothesis that the three-variable sample can be considered as having come from a normal three-variable population which is symmetrical in the three variables, i.e. a normal population in which the means are equal, the variances are equal, and the covariances are equal. Or more generally, one may raise the analogous question for the case of k variables.

Similarly, one could mention biological examples which have been treated by intra-class correlation methods and raise the question as to whether the underlying multivariate distribution can be judged to be symmetric in the variables on the basis of information supplied by the sample.

To attempt to deal with this problem by comparing means, or variances or covariances two at a time or performing what might appear to be extensions of existing tests for two or more *independent* samples of one variable leads to complications because of correlation among the variables in the original population. What is needed is some kind of a comprehensive test which will take into account all means, variances and covariances at one time. If it turns out that the hypothesis of equal means, equal variances and equal covariances is not supported by the sample, then one can raise the question as to whether the sample supports the hypothesis that the variances are equal and covariances are equal irrespective of means. If the answer is yes here, one can ask the further question as to whether the sample supports the hypothesis of equal means. Such tests will be developed in this paper for samples from a normal multivariate population. More specifically three tests are developed. (i) Test L_{mvc} for testing the hypothesis H_{mvc} that all means are equal, all variances are equal and all covariances are equal, (ii) test L_{vc} for the hypothesis H_{vc} that all variances are equal and all covariances are equal, irrespective of the values of the means, and (iii) test

[1] The problem treated in this paper arose from discussions with Professor Harold O. Gulliksen, of the Psychology Department of Princeton University, in connection with the problem of testing whether two or more forms of an examination can be considered as "parallel forms". The author would like to take this opportunity to acknowledge various helpful discussions he has also had with his colleague Professor John W. Tukey in connection with this paper.

L_m for the hypothesis H_m that the means are equal, assuming that H_{vc} is true, i.e. that the variances are equal and the covariances equal.

There are rather obvious extensions of the hypotheses H_{mvc}, H_{vc} and H_m and their corresponding test criteria. For example, one could divide the variables in the multivariate population into two sets, and consider the hypothesis $H_{mvc}^{(2)}$ (say), analogous to H_{mvc}, that the means are equal, the variances are equal and the covariances are equal within each of the two sets and that the covariances of variables between the two sets are all equal. Similarly, $H_{vc}^{(2)}$ and $H_m^{(2)}$ could be defined so as to be analogous to H_{vc} and H_m. However, these extensions will not be considered in this paper.

In Part I of this paper we shall discuss the problem of testing hypotheses regarding equality of means, equality of variances, and equality of covariances in a normal multivariate population, and summarize the mathematical results which have been obtained. An illustrative example will also be given. The derivation of the test criteria and their sampling theory is presented in Part II of the paper.

1.2. The hypotheses to be tested. We assume that there is a k-variate population Π in which the variables $x_1, x_2, \cdots, x_k$ are distributed according to a normal k-variate probability density function such that the mean value of x_i is a_i $(i = 1, 2, \cdots, k)$ and the variance-covariance matrix of $x_1, x_2, \cdots, x_k$ is $\| \rho_{ij}\sigma_i\sigma_j \|$, ρ_{ij} being the correlation coefficient between x_i and $x_j (i \neq j)$, and σ_i being the standard deviation of x_i.

In specifying the hypotheses to be considered it will be convenient to define three conditions on the parameters of population Π:

Condition C_m: that the means of the x_i are all equal.

Condition C_v: that the variances of the x_i are all equal.

Condition C_c: that the covariances of the x_i and x_j $(i \neq j)$ are all equal.

The hypotheses regarding Π to be tested are as follows:

H_{mvc}: that conditions C_m, C_v, and C_c hold

H_{vc}: that conditions C_v and C_c hold

H_m: that condition C_m holds, assuming that H_{vc} is true.

A precise statement of these hypotheses in terms of Neyman-Pearson likelihood ratio terminology will be found in Part II.

It should be noted that H_{mvc} is a comprehensive hypothesis which specifies equality of means, equality of variances and equality of covariances and would be tested if one is interested in all of these quantities as a system. On the other hand H_{vc} refers only to equality of variances and equality of covariances regardless of what values the means may have. H_{vc} would be tested if one is only concerned with equality of variances and equality of covariances. H_m is a more restrictive hypothesis than either H_{mvc} or H_{vc}, for it refers to equality of means *under the assumption* that H_{vc} is true. In other words, H_m can only be tested accurately when H_{vc} is true; H_m would be a generalization of the Behrens-Fisher problem [1] when H_{vc} is false.

1.3. The sample test criteria. The three hypotheses H_{mvc}, H_{vc} and H_m are to be tested on the basis of a sample O_n from Π consisting of the following values of the x's: $x_{i\alpha}$, $i = 1, 2, \cdots, k$; $\alpha = 1, 2, \cdots, n$.

The criteria for testing H_{mvc}, H_{vc}, and H_m depend on the following quantities to be determined from the sample:

$$\bar{x}_i = \frac{1}{n}\sum_{\alpha=1}^{n} x_{i\alpha}, \qquad \bar{x} = \frac{1}{k}\sum_{i=1}^{k} \bar{x}_i, \tag{1.1}$$

$$s_{ij} = \frac{1}{n}\sum_{\alpha=1}^{n} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j) = \frac{1}{n}\sum_{\alpha=1}^{n} x_{i\alpha}x_{j\alpha} - \bar{x}_i\bar{x}_j \tag{1.2}$$

$$s^2 = \frac{1}{k}\sum_{i=1}^{k} s_{ii}, \qquad s^2 r = \frac{1}{k(k-1)}\sum_{i\neq j=1}^{k} s_{ij}. \tag{1.3}$$

The sample criteria, based on the method of likelihood ratios, for testing H_{mvc}, H_{vc} and H_m are respectively, as follows:

$$L_{mvc} = L_{vc}\cdot L_m^{k-1} \tag{1.4}$$

$$L_{vc} = \frac{|s_{ij}|}{(s^2)^k(1-r)^{k-1}(1+(k-1)r)} \tag{1.5}$$

$$L_m = \frac{s^2(1-r)}{s^2(1-r) + \frac{1}{k-1}\sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2} \tag{1.6}$$

where $|s_{ij}|$ is the determinant of sample variances and covariances.

The range of values of each of the three criteria is from 0 to 1. A necessary and sufficient condition for each criterion to have the value 1 is that the hypothesis for which the criterion is a test be (accidentally) identically supported by the sample. If the hypothesis (any one of the three being considered) is true, the average value of the corresponding criterion will be less than 1, but this average value will be nearer 1 than when the hypothesis is false.

If H_{mvc} is true (i.e., found to be supported by the sample on the basis of the test L_{mvc}) then there will be three parameters which characterize Π, namely, a (the common mean), σ^2 (the common variance), and ρ (the common correlation coefficient). The best estimates of these three parameters are, respectively:

$$\begin{aligned} \bar{x} &= \frac{1}{k}\sum_{i=1}^{k}\bar{x}_i, \\ s_0^2 &= s^2 + \frac{1}{k}\sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2 \\ r_0 &= \left[s^2 r - \frac{1}{k(k-1)}\sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2\right] \Big/ s_0^2. \end{aligned} \tag{1.7}$$

If H_{vc} is true (i.e , found to be supported by the sample on the basis of the test L_{vc}) there will be $k + 2$ parameters which characterize Π, namely the means

$a_1, a_2, \cdots, a_k, \sigma^2$ (the common variance) and ρ (the common correlation coefficient). The best estimates of these parameters are, respectively

$$\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k, s^2, \text{ and } r. \tag{1.8}$$

In order to be able to use the three sample criteria L_{mvc}, L_{vc} and L_m for testing the hypotheses H_{mvc}, H_{vc}, H_m, it is necessary to have their distribution functions under the assumptions that the respective hypotheses H_{mvc}, H_{vc} and H_m are true.

1.4. Sampling theory of the test criteria. The moments of the exact sampling distributions of L_{mvc} and L_{vc} when H_{mvc} and H_{vc} are true respectively, have been determined for all values of k (number of variables) and all values of n (sample size) for which such distributions exist; i.e., for $k \geq 2$ and $n > k$. The g-th moments of the distributions of the two criteria are as follows:

$$M_g(L_{mvc}) = (k-1)^{g(k-1)} \prod_{i=2}^{k} \frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))} \cdot \frac{\Gamma(\frac{1}{2}(k-1)n)}{\Gamma(\frac{1}{2}(k-1)(n-1)+g(k-1))} \tag{1.9}$$

and

$$M_g(L_{vc}) = (k-1)^{g(k-1)} \prod_{i=2}^{k} \frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))} \cdot \frac{\Gamma(\frac{1}{2}(k-1)(n-1))}{\Gamma(\frac{1}{2}(k-1)(n-1)+g(k-1))}. \tag{1.10}$$

For the cases of $k = 2$ and $k = 3$, these moments simplify so that the distribution functions of L_{mvc} and L_{vc} can be readily inferred. They turn out to be as follows:

For $k = 2$:

$$dF(L_{mvc}) = \tfrac{1}{2}(n-2)(L_{mvc})^{\frac{1}{2}(n-4)} d\, L_{mvc} \tag{1.11}$$

$$dF(L_{vc}) = \frac{\Gamma(\frac{1}{2}(n-1))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}(n-2))} L_{vc}^{\frac{1}{2}(n-4)} (1 - L_{vc})^{-\frac{1}{2}}\, dL_{vc}. \tag{1.12}$$

For $k = 3$:

$$dF(L_{mvc}) = \frac{\Gamma(n)}{2\Gamma(n-3)} (\sqrt{L_{mvc}})^{n-4} (1 - \sqrt{L_{mvc}})^2\, d\sqrt{L_{mvc}} \tag{1.13}$$

$$dF(L_{vc}) = \frac{\Gamma(n-1)}{\Gamma(n-3)} (\sqrt{L_{vc}})^{n-4} (1 - \sqrt{L_{vc}})\, d\sqrt{L_{vc}}. \tag{1.14}$$

The distribution function of L_m when the hypothesis H_m is true has been found to be

$$(1.15)\qquad dF(L_m) = \frac{\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}(n-1)(k-1))\Gamma(\frac{1}{2}(k-1))} \cdot L_m^{\frac{1}{2}(n-1)(k-1)-1}(1-L_m)^{\frac{1}{2}(k-1)-1}\,dL_m .$$

Details of the derivation of these distribution functions will be found in Part II.

In a paper published elsewhere in the present issue of the *Annals of Mathematical Statistics*, Tukey and Wilks [2] show how the probability integrals of L_{mvc} and L_{vc} and of other statistical critéria having moments of a rather general class can be fitted by Incomplete Beta Functions in such a way that all moments of the fitted distribution agree with those of the actual distribution up to and including terms of order $\frac{1}{n}$.

It will be noted that the probability integrals of L_{mvc} and L_{vc} for $k = 2$, those of $\sqrt{L_{mvc}}$ and $\sqrt{L_{vc}}$ for $k = 3$, and that of L_m for any value of k, are Incomplete Beta Functions [3], with the following values of p and q:

k	criterion	p	q
2	L_{mvc}	$\frac{1}{2}(n-2)$	1
2	L_{vc}	$\frac{1}{2}(n-2)$	$\frac{1}{2}$
3	$\sqrt{L_{mvc}}$	$n-3$	3
3	$\sqrt{L_{vc}}$	$n-3$	2
k	L_m	$\frac{1}{2}(n-1)(k-1)$	$\frac{1}{2}(k-1)$

Percentage points[2] of the distributions of these criteria for the cases mentioned in this table can therefore be read from Thompson's [4] tables of per cent points for the Incomplete Beta Function. 5% and 1% points for L_{mvc} and L_{vc} for $k = 2$ and 3 are given in Table I for certain values of n. Table II shows 5% and 1% points of L_m for certain values of n for $k = 2, 3, 4, 5$ and 6.

1.5. The equivalence of L_m and an analysis of variance test for a k by n layout. One can set up a Snedecor F ratio for testing hypothesis H_m by setting

$$(1.16)\qquad F = \frac{\frac{1}{2}(n-1)(k-1)(1-L_m)}{\frac{1}{2}(k-1)L_m}$$

and entering the F tables with $n_1 = k - 1$ and $n_2 = (n-1)(k-1)$ degrees of

[2] The $100\epsilon\%$ point, say L_ϵ, of a given criterion L (any of those being considered) having distribution $dF(L)$ is given by $\int_0^{L_\epsilon} dF(L) = \epsilon$.

TABLE I

5% and 1% points of L_{mvc} and L_{vc} for $k = 2$ and $k = 3$

	$k = 2$					$k = 3$			
n	L_{mvc}		L_{vc}		n	L_{mvc}		L_{vc}	
	5%	1%	5%	1%		5%	1%	5%	1%
3	0.0025	.0001	0.0062	.0002	4	0.00029	0.00001	0.00064	0.00003
4	.0500	.0100	.0975	.0199	5	.0095	.0018	.0183	.0035
5	.1357	.0464	.2285	.0808	6	.0358	.0112	.0618	.0198
6	.2236	.1000	.3416	.1588	7	.0736	.0300	.1174	.0493
7	.3017	.1585	.4307	.2352	8	.1165	.0559	.1749	.0866
8	.3684	.2154	.5005	.3039	9	.1603	.0860	.2297	.1272
9	.4249	.2683	.5559	.3637	10	.2028	.1181	.2802	.1682
10	.4729	.3162	.6007	.4154	11	.2432	.1508	.3259	.2079
11	.5139	.3594	.6375	.4601	12	.2808	.1829	.3670	.2457
12	.5493	.3981	.6682	.4989	13	.3157	.2141	.4040	.2811
13	.5800	.4329	.6943	.5328	14	.3480	.2439	.4373	.3141
14	.6070	.4642	.7165	.5626	15	.3778	.2722	.4674	.3448
15	.6307	.4924	.7358	.5889	16	.4052	.2990	.4946	.3732
16	.6518	.5180	.7528	.6124	17	.4306	.3243	.5193	.3996
17	.6707	.5411	.7675	.6334	18	.4540	.3482	.5418	.4240
18	.6877	.5623	.7807	.6522	23	.5484	.4482	.6293	.5230
19	.7030	.5817	.7925	.6693	33	.6660	.5811	.7326	.6470
20	.7169	.5995	.8031	.6848	63	.8135	.7591	.8549	.8029
21	.7294	.6159	.8126	.6989	∞	1.0000	1.0000	1.0000	1.0000
22	.7411	.6310	.8213	.7119					
23	.7518	.6450	.8292	.7237					
24	.7616	.6579	.8365	.7347					
25	.7707	.6700	.8431	.7448					
26	.7791	.6813	.8493	.7542					
27	.7869	.6918	.8549	.7629					
28	.7942	.7017	.8602	.7710					
29	.8010	.7110	.8651	.7786					
30	.8074	.7197	.8697	.7857					
31	.8133	.7279	.8739	.7924					
32	.8190	.7356	.8779	.7987					
42	.8609	.7943	.9073	.8454					
62	.9050	.8577	.9375	.8945					
122	.9513	.9261	.9684	.9460					
∞	1.0000	1.0000	1.0000	1.0000					

TABLE II

5% and 1% points of L_m

n	k = 2 5%	k = 2 1%	n	k = 3 5%	k = 3 1%	n	k = 4 5%	k = 4 1%	n	k = 5 5%	k = 5 1%
2	0.0062	0.0002	2	.0500	0.0100	2	.0973	.0328	2	.1354	.0589
3	.0975	.0199	3	.2236	.1000	3	.2960	.1698	3	.3426	.2221
4	.2285	.0808	4	.3684	.2154	4	.4372	.3002	4	.4793	.3566
5	.3416	.1588	5	.4729	.3162	5	.5340	.4019	5	.5709	.4560
6	.4307	.2352	6	.5493	.6033	6	.6033	.4800	6	.6356	.5302
7	.5005	.3039	7	.6070	.4642	7	.6550	.5409	7	.6837	.5872
8	.5559	.3637	8	.6518	.5180	8	.6950	.5895	8	.7206	.6321
9	.6007	.4154	9	.6877	.5623	9	.7267	.6290	11	.7933	.7232
10	.6375	.4601	10	.7169	.5995	10	.7525	.6617	16	.8559	.8043
11	.6682	.4989	11	.7411	.6310	11	.7739	.6892	31	.9246	.8961
12	.6943	.5328	12	.7616	.6579	21	.8788	.8290	∞	1.0000	1.0000
13	.7165	.5626	13	.7791	.6813	41	.9372	.9101			
14	.7358	.5889	14	.7942	.7017	∞	1.0000	1.0000			
15	.7527	.6124	15	.8074	.7197						
16	.7675	.6334	16	.8190	.7356						
17	.7807	.6522	21	.8609	.7943						
18	.7925	.6693	31	.9050	.8577						
19	.8031	.6848	61	.9513	.9261						
20	.8126	.6989	∞	1.0000	1.0000						
21	.8213	.7119									
22	.8292	.7237									
23	.8365	.7347									
24	.8431	.7448									
25	.8493	.7542									
26	.8549	.7629									
27	.8602	.7710									
28	.8651	.7786									
29	.8697	.7857									
30	.8739	.7924									
31	.8779	.7987									
41	.9073	.8454									
61	.9375	.8945									
121	.9684	.9460									
∞	1.0000	1.0000									

freedom. Making use of the definition of s^2, s_0^2, r and r_0 in L_m, one finds that F can be written as

$$(1.17) \qquad F = \frac{S_1}{(k-1)} \Big/ \frac{S_2}{(n-1)(k-1)}$$

where $S_1 = n \sum_{i=1}^{k} (\bar{x}_i - \bar{x})^2$, and $S_2 = \sum_{\alpha=1}^{n} \sum_{i=1}^{k} (x_{i\alpha} - \bar{x}'_\alpha - \bar{x}_i + \bar{x})^2$ and $\bar{x}'_\alpha = \frac{1}{k} \sum_{\alpha=1}^{n} x_{i\alpha}$. Thus, the use of L_m as a criterion for testing H_m is equivalent to an analysis of variance test for testing "row" effects in a k by n rectangular layout when rows are associated with the k variables in the multivariate population and columns are associated with the n individuals in the sample.

1.6. Approximate sampling theory of the test criteria for large samples. In the case of large samples, it follows from a theorem [5] concerning the distribution of likelihood ratio criteria for large samples that $-n \ln L_{mvc}$, $-n \ln L_{vc}$, and $-n(k-1) \ln L_m$ are approximately distributed according to chi-square distributions with $\frac{1}{2}k(k+3) - 3$, $\frac{1}{2}k(k+1) - 2$, and $k - 1$ degrees of freedom respectively. Approximate 5% and 1% points of these three quantities taken from Thompson's [6] tables of the percentage points of the chi-square distribution are given in Table III.

Table IV is given in order to furnish some idea of how the accuracy of the approximations provided by Table III depend on n. It will be noted that the approximate values exceed the exact values in every case, differences occuring in the third decimal place in almost every case in which n exceeds 60. The approximate percentages to which the approximate per cent points correspond are given by the numbers in the parentheses in Table IV. These numbers in each case were obtained by linear interpolation from the exact 5% and 1% points.

1.7. Comparison of L_{vc} with Mauchly's "sphericity" test. The criterion L_{vc} for testing hypothesis H_{vc} is, in a sense, an extension of a test developed by Mauchly [7] for testing the hypothesis of "sphericity" of a normal multivariate distribution. Mauchly's test was designed for testing the hypothesis that all variances are equal, and that all covariances are equal to *zero* irrespective of the values of the population means. The likelihood criterion for testing this hypothesis of "sphericity" is

$$(1.18) \qquad L_s = \frac{|s_{ij}|}{(s^2)^k}$$

which should be compared with L_{vc}. Actually, Mauchly used $\sqrt{L_s}$ as the test criterion, which, of course, is equivalent to using L_s. The g-th moment of L_s when the hypothesis of sphericity is true is given by

$$(1.19) \qquad k^{gk} \cdot \prod_{i=1}^{k} \left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))} \right] \cdot \frac{\Gamma(\frac{1}{2}k(n-1))}{\Gamma(\frac{1}{2}k(n-1)+gk)}$$

TABLE III

Approximate 5% and 1% points for $-n \ln L_{mvc}$, $-n \ln L_{vc}$, *and* $-n(k-1) \ln L_m$ *for* $k = 2, 3, 4, 5, 6$.

k	$-n \ln L_{mvc}$			$-n \ln L_{vc}$			$-n(k-1) \ln L_m$		
	d.f.	5%	1%	*d.f.*	5%	5%	*d.f.*	5%	1%
2	2	5.99147	9.21034	1	3.84146	6.63490	1	3.84146	6.63490
3	6	12.5916	16.8119	4	9.48773	13.2767	2	5.99147	9.21034
4	11	19.6751	24.7250	8	15.5073	20.0902	3	7.81473	11.3449
5	17	27.5871	33.4087	13	22.3621	27.6883	4	9.48773	13.2767
6	24	36.4151	42.9798	19	30.1435	36.1908	5	11.0705	15.0863

TABLE IV

Table indicating the accuracy of the approximate 5% and 1% points of L_{mvc}, L_{vc} *and* L_m *provided by Table III*

criterion	k	n	5%		1%	
			exact	approx.	exact	approx.
L_{mvc}	2	30	0.8074	0.8190 (5.53)*	0.7197	0.7357 (1.73)*
L_{mvc}	2	62	.9050	.9079 (5.25)	.8577	.8619 (1.36)
L_{mvc}	2	122	.9513	.9521 (5.13)	.9261	.9273 (1.19)
L_{mvc}	3	33	.6660	.6828 (5.79)	.5811	.6008 (1.88)
L_{mvc}	3	63	.8135	.8188 (5.40)	.7591	.7658 (1.49)
L_{vc}	2	30	.8697	.8799 (5.49)	.7857	.8016 (1.76)
L_{vc}	2	62	.9375	.9399 (5.22)	.8945	.8985 (1.37)
L_{vc}	2	122	.9684	.9690 (5.11)	.9460	.9471 (1.20)
L_{vc}	3	33	.7326	.7501 (5.82)	.6470	.6688 (2.01)
L_{vc}	3	63	.8549	.8602 (5.41)	.8029	.8100 (1.55)
L_m	2	31	.8779	.8835 (5.28)	.7987	.8073 (1.43)
L_m	2	61	.9375	.9389 (5.13)	.8945	.8969 (1.20)
L_m	2	121	.9684	.9688 (5.07)	.9460	.9467 (1.13)
L_m	3	31	.9050	.9079 (5.25)	.8577	.8619 (1.36)
L_m	3	61	.9513	.9521 (5.10)	.9261	.9273 (1.14)
L_m	4	41	.9372	.9385 (5.19)	.9101	.9119 (1.26)
L_m	5	31	.9246	.9264 (5.25)	.8961	.8984 (1.32)

*The numbers in the parentheses are approximate percentages (obtained by linear interpolation) to which the approximate percent points correspond.

which should be compared with the g-th moment of L_{vc}. Stated in other words, Mauchly's criterion L_s is a test for the hypothesis that contours of equal probability density in the multivariate normal population distribution are spheres, while L_{vc} is a test for the hypothesis that the contours of equal probability are k-dimensional ellipsoids with $k - 1$ equal axes in general shorter than the k-th axis which is equally inclined to the k coordinate axes of the distribution function.

1.8. Illustrative Example. As an example to illustrate the use of the test criteria L_{mvc}, L_{vc}, L_m, we shall consider data on three forms of a subtest in verbal aptitude, and inquire as to whether the data are consistent with the hypothesis of the three forms being "parallel forms".

A procedure[3] was used for partitioning the first 60 of an entire test of 80 items into three sets of 20 items each by using only a "difficulty' and a "validity" index on each of the items. A random sample of 100 test booklets was selected from those in which the first 60 items had been attempted. Total scores were obtained on each of the three subtests selected in this manner. The question is this: Does this procedure of selecting items produce "parallel" subtests? In other words considering the three scores on the three subtests in each of the 100 test booklets as a sample of 100 items from a trivariate normal population is the sample consistent with the hypothesis H_{mvc} of equal means, equal variances and equal covariances? If not, is the sample consistent with the hypothesis H_{vc} of equal variances and equal covariances irrespective of means? If the answer to this question is no, then the failure of the tests to be parallel is at least partially attributable to differences in variances and/or differences in covariances. If the answer to the question is yes, we test H_m, the hypothesis of equal means, assuming equal variances and equal covariances. If the sample is not consistent with H_m, then the subtests fail to be parallel because of significant differences in means.

If we denote the three subtests by T_1, T_2, T_3, and the scores on the α-th individual in the sample on the three tests by $x_{1\alpha}$, $x_{2\alpha}$, $x_{3\alpha}$ respectively ($\alpha = 1, 2, \cdots, 100$), the information in the sample needed for computing L_{mvc}, L_{vc} and L_m and testing H_{mvc}, H_{vc} and H_m is as follows:

$$
\begin{aligned}
\bar{x}_1 &= 10.9900 & s^2 &= 17.5558 \\
\bar{x}_2 &= 10.9300 & s_0^2 &= 17.5764 \\
\bar{x}_3 &= 11.2600 & r &= .7963 \\
s_{11} &= 16.8451 & r_0 &= .7948 \\
s_{22} &= 18.1099 & |s_{ij}| &= 545.5308 \\
s_{33} &= 17.7124 & & \\
s_{12} &= 13.5493 & & \\
s_{13} &= 14.5826 & & \\
s_{23} &= 13.8056 & &
\end{aligned}
$$

[3] Devised by Mr. L. R. Tucker of the College Entrance Examination Board. The author is indebted to Mr. Tucker for the data used in the illustrative example.

Using formulas (1.4), (1.5), and (1.6), for $k = 3$, for calculating the values of L_{mvc}, L_{vc} and L_m, we find

$$L_{mvc} = .9209$$
$$L_{vc} = .9370$$
$$L_m = .9914$$

It will be seen from Table III that the 5 % point of $-n \ln L_{mvc}$ for $k = 3$ is 12.5912. Setting $-100 \ln L_{mvc} = 12.5912$ and solving we find the approximate 5% point of L_{mvc} to be .8817 which is considerably less than the observed value of L_{mvc}, namely .9209. Hence, the sample is consistent with H_{mvc}. As a matter of fact the observed value .9209 lies at approximately the 25% point of L_{mvc}.

In practice, there would be no point in proceeding to test H_{vc} or H_m, because if L_{mvc} is non-significant there is a high probability (not certainty) that both L_{vc} and L_m will be non-significant. But for illustrative purposes, it is perhaps useful to consider L_{vc} and L_m anyway.

The 5% point of $-n \ln L_{vc}$ for $k = 3$ is 9.48773 (See Table III). Setting $-100 \ln L_{vc} = 9.48773$ and solving, we get .9095 as the approximate 5% point of L_{vc}, which is considerably less than the observed value .9370, thus indicating that L_{vc} is not significant at the 5% level. In fact the observed value .9370 lies between the 25% and 10% point of L_{vc}.

The 5% point of $-n(k - 1) \ln L_m$ for $k = 3$ is 5.99147. Setting $-200 \ln L_m = 5.99147$ and solving we get .9704 as the approximate 5% point. Since the observed value of L_m exceeds .9704, we find L_m not significant at the 5% level. In fact, the observed value .9914 lies between the 50% and 25% points.

II. Derivation of Results

In this part we shall derive the criteria L_{mvc}, L_{vc} and L_m for testing H_{mvc}, H_{vc} and H_m by the Neyman-Pearson method of likelihood ratios, and determine the distribution theory of the criteria.

2.1. The test L_{mvc} for H_{mvc}, the hypothesis of equality of means, equality of variances and equality of covariances.

2.1.1 *Derivation of the criterion L_{mvc}.* Let Π be a normal k-variate population, in which $x_1, x_2, \cdots, x_k$ are variables, such that a_i is the mean of x_i, σ_i^2 the variance of x_i and $\rho_{ij}\sigma_i\sigma_j$ the covariance (ρ_{ij} the correlation coefficient) between x_i and x_j. The distribution law of $x_1, x_2, \cdots, x_k$ in the population, is

$$\frac{|A_{ij}|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}k}} \exp\left[-\tfrac{1}{2}\sum_{i,j=1}^{k} A_{ij}(x_i - a_i)(x_j - a_j)\right] \tag{2.1}$$

where $\| A_{ij} \|$ is symmetric and is the inverse of the variance-covariance matrix, i.e. $\| A_{ij} \|^{-1} = \| \rho_{ij}\sigma_i\sigma_j \|$, $(\rho_{ii} = 1)$.

Now suppose O_n is a random sample of n individuals from population Π, and let $x_{i\alpha}$ be the value of the x_i for the αth individual in the sample. Then, the probability function for the entire sample (likelihood function) is

$$P = \frac{|A_{ij}|^{\frac{1}{2}n}}{(2\pi)^{\frac{1}{2}nk}} \exp\left[-\tfrac{1}{2}\sum_{\alpha=1}^{n}\sum_{i,j=1}^{k} A_{ij}(x_{i\alpha} - a_i)(x_{j\alpha} - a_j)\right]. \tag{2.2}$$

The hypothesis which we wish to test is that the population means a_1, a_2, $\cdots$, a_k are equal, the variances σ_1^2, σ_2^2, $\cdots$, σ_k^2 are all equal and the covariances $\rho_{12}\sigma_1\sigma_2$, $\rho_{13}\sigma_1\sigma_3$, $\cdots$, $\rho_{k-1,k}\sigma_{k-1}\sigma_k$ are all equal, the test to be made on the basis of the sample of values $x_{i\alpha}$. In other words, we wish to test the hypothesis that

$$\begin{cases} a_1 = a_2 = \cdots = a_k = a \\ \begin{Vmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1k}\sigma_1\sigma_k \\ \rho_{21}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2k}\sigma_2\sigma_k \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \rho_{k1}\sigma_1\sigma_k & \rho_{k2}\sigma_2\sigma_k & \cdots & \sigma_k^2 \end{Vmatrix} = \begin{Vmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \cdots & \rho\sigma^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{Vmatrix}. \end{cases} \tag{2.3}$$

Testing the hypothesis that (2.3) holds is equivalent to testing the hypothesis that

$$\begin{cases} a_1 = a_2 = \cdots = a_k = a \\ \begin{Vmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & & \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ A_{k1} & & & A_{kk} \end{Vmatrix} = \begin{Vmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ B & B & \cdots & A \end{Vmatrix} \end{cases} \tag{2.4}$$

where

$$A = \frac{1 + (k-2)\rho}{\sigma^2(1-\rho)(1+(k-1)\rho)}, \qquad B = \frac{-\rho}{\sigma^2(1-\rho)(1+(k-1)\rho)}. \tag{2.5}$$

To obtain the likelihood criterion L_{mvc} for testing the hypothesis H_{mvc} we maximize the likelihood (2.2) under two conditions, for the given sample O_n, and take the ratio of the two resulting maxima. First, we maximize (2.2) over the set Ω of admissible values of the parameters, i.e. with respect to all means a_i and all variances and covariances $\rho_{ij}\sigma_i\sigma_j$, denoting the resulting maximum of (2.2) by P_Ω. Secondly, we maximize (2.2) over the set of values ω of the parameters which satisfy the hypothesis H_{mvc}; that is, we replace in (2.2) each mean a_1, a_2, $\cdots$, a_k by a, and each of the variances σ_1^2, σ_2^2, $\cdots$, σ_k^2 by σ^2 and each of the covariances $\rho_{ij}\sigma_i\sigma_j$, $(i \neq j)$, by $\rho\sigma^2$ and then maximize (2.2) with respect to a, σ^2, and ρ, denoting the resulting maximum by P_ω.

Maximizing (2.2) under the first set of conditions is equivalent to maximizing it with respect to the a_i, and the A_{ij}, while maximizing (2.2) under the second set of conditions is equivalent to imposing condition (2.4) and maximizing it with respect to a, A and B.

The values of the a_i and A_{ij} which maximize (2.2) under the first set of conditions are given by solving the following $(k^2 + 3k)/2$ equations.

$$(2.6) \qquad \frac{\partial P}{\partial a_i} = 0, \qquad i = 1, 2, \cdots, k$$

$$(2.7) \qquad \frac{\partial P}{\partial A_{ij}} = 0, \qquad i, j = 1, 2, \cdots, k, \quad (i \leq j).$$

Expressions for these equations are

$$(2.8) \qquad \left[n \sum_{j=1}^{k} A_{ij}(\bar{x}_j - a_j) \right] P = 0, \qquad i = 1, 2, \cdots, k$$

$$(2.9) \qquad \left[\frac{n}{2} A^{ij} - \tfrac{1}{2} \sum_{\alpha=1}^{n} (x_{i\alpha} - a_i)(x_{j\alpha} - a_j) \right] P = 0, \quad i, j, = 1, 2, \cdots, k, (i \leq j),$$

where A^{ij} is the element in the ith row and jth column of $\| A_{ij} \|^{-1}$, i.e. $A^{ij} = \rho_{ij}\sigma_i\sigma_j$, and $\bar{x}_j = \frac{1}{n}\sum_{\alpha=1}^{n} x_{j\alpha}$.

The solution of (2.8) and (2.9) is

$$(2.10) \qquad \begin{aligned} a_j &= \bar{x}_j, & j &= 1, 2, \cdots, k \\ A^{ij} &= s_{ij}, \text{ or } A_{ij} = s^{ij}, & i, j &= 1, 2, \cdots, k, \quad (i \leq j) \end{aligned}$$

where $s_{ij} = \frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)$, and where $\| s^{ij} \| = \| s_{ij} \|^{-1}$. Inserting the values of (2.10) in (2.2) and noting that the exponent in (2.2) reduces to $-\frac{n}{2}\sum_{i,j=1}^{k} s^{ij}s_{ij}$, which in turn reduces to $-\frac{1}{2}kn$, since $\sum_{i=1}^{k} s^{ij}s_{ij} = 1$ for each value of j, we obtain

$$(2.11) \qquad P_\Omega = \frac{e^{-\frac{1}{2}kn}}{| s_{ij} |^{\frac{1}{2}n}(2\pi)^{\frac{1}{2}kn}}.$$

In order to obtain P_ω, we specialize the a_i and the matrix $\| A_{ij} \|$ in (2.2) in accordance with (2.4), noting that the determinant $| A_{ij} |$ reduces to $(A - B)^{k-1}(A + (k - 1)B)$, thus obtaining the following specialized form of (2.2)

$$(2.12) \qquad P' = \frac{[(A - B)^{k-1}(A + (k - 1)B)]^{\frac{1}{2}n}}{(2\pi)^{\frac{1}{2}nk}} \exp\left\{ -\tfrac{1}{2}\left[A \sum_{\alpha=1}^{n}\sum_{i=1}^{k}(x_{i\alpha} - a)^2 + B \sum_{\alpha=1}^{n}\sum_{i\neq j=1}^{k}(x_{i\alpha} - a)(x_{j\alpha} - a) \right] \right\}.$$

The values of a, A and B which maximize P' are given by solving the following three equations

$$\text{(2.13)} \qquad \frac{\partial P'}{\partial a} = 0, \qquad \frac{\partial P'}{\partial A} = 0, \qquad \frac{\partial P'}{\partial B} = 0.$$

These equations are respectively

$$\text{(2.14)} \begin{cases} \left[(A - B)\sum_{\alpha=1}^{n}\sum_{i=1}^{k}(x_{i\alpha} - a) + B\sum_{\alpha=1}^{n}\left(\sum_{i=1}^{k}(x_{i\alpha} - a)\right)\right]P' = 0 \\ \left[\frac{\frac{1}{2}n(k-1)}{A-B} + \frac{\frac{1}{2}n}{A+(k-1)B} - \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{i=1}^{k}(x_{i\alpha} - a)^2\right]P' = 0 \\ \left[\frac{-\frac{1}{2}n(k-1)}{A-B} + \frac{\frac{1}{2}n(k-1)}{A+(k-1)B} - \sum_{\alpha=i}^{n}\sum_{i\neq j=1}^{k}(x_{i\alpha} - a)(x_{j\alpha} - a)\right]P' = 0. \end{cases}$$

Replacing a_i by $\bar{x}_i$ in (2.15) putting $\frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j) = s_{ij}$, and setting

$$\text{(2.15)} \begin{cases} \bar{x} = \frac{1}{nk}\sum_{\alpha=1}^{n}\sum_{i=1}^{k} x_{i\alpha} \\ s_{0ij} = \frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \bar{x})(x_{j\alpha} - \bar{x}) = s_{ij} + (\bar{x}_i - \bar{x})(\bar{x}_j - \bar{x}) \\ r_0 = \sum_{i\neq j=1}^{k} s_{0ij}\Big/(k-1)\sum_{i=1}^{k} s_{0ii} \\ \quad = \left[\sum_{i\neq j=1}^{k} s_{ij} - \sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2\right]\Big/(k-1)\left[\sum_{i=1}^{k} s_{ii} + \sum_{i=1}^{k}(\bar{x}_i - x)^2\right] \\ s_0^2 = \sum_{i=1}^{k} s_{0ii}/k = \frac{1}{k}\left[\sum_{i=1}^{k} s_{ii} + \sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2\right] \end{cases}$$

we obtain as solutions of (2.14)

$$\text{(2.16)} \qquad \begin{aligned} a &= \bar{x} \\ A &= \frac{1 + (k-2)r_0}{s_0^2(1 - r)(1 + (k-1)r_0)} \\ B &= \frac{-r_0}{s_0^2(1 - r_0)(1 + (k-1)r_0)}. \end{aligned}$$

Substituting these in (2.12) we obtain

$$\text{(2.17)} \qquad P_\omega = \frac{e^{-\frac{1}{2}kn}}{[(s_0^2)^k(1 - r_0)^{k-1}(1 + (k-1)r_0)]^{\frac{1}{2}n}(2\pi)^{\frac{1}{2}kn}}.$$

The likelihood ratio λ_{mvc} for testing hypothesis H_{mvc} is given by

$$\lambda_{mvc} = \frac{P_\omega}{P_\Omega}.$$

It will be convenient to use the $\frac{2}{n}$ th root of λ_{mvc} as the test criterion for H_{mvc}. Denoting this criterion by L_{mvc}, we have

$$L_{mvc} = \frac{| s_{ij} |}{(s_0^2)^k(1 - r_0)^{k-1}(1 + (k - 1)r_0)}. \tag{2.18}$$

The use of L_{mvc} as a test criterion is obviously equivalent to the use of λ_{mvc}.

It will be seen that L_{mvc} is equal to unity when and only when the sample means $\bar{x}_i$ are all equal, the sample variances s_{ij} are all equal and when the sample covariances s_{ij}, $(i \neq j)$, are all equal. The greater the departure of sample means from equality, sample variances from equality and sample covariances from equality, the smaller will be the value of L_{mvc}, its value, of course, always remaining between 0 and 1.

2.1.2. *Approximate distribution of* $-n \ln L_{mvc}$ *in large samples.* In order to make use of L_{mvc} as a criterion for testing hypothesis H_{mvc} we must find its sampling distribution under the assumption that H_{mvc} is true, i.e. that our sample has, in fact, been drawn from a k-variate normal population having equal means, equal variances and equal covariances. In the case of large samples, it follows from a theorem on asymptotic distributions of likelihood ratios [5] that $-2 \ln \lambda_{mvc}$ (i.e. $-n \ln L_{mvc}$) is approximately distributed according to the chi-square law with $\frac{1}{2}k(k + 3) - 3$ degrees of freedom (obtained by taking the difference between the number of parameters used in maximizing P to obtain P_Ω and that used in maximizing P' to obtain P_ω).

Thus, to apply the test, one computes the value of $-n \ln L_{mvc}$ for the given sample, and sees whether the obtained value is significant at the given probability level (5% or 1%) using the chi-square table for $\frac{1}{2}k(k + 3) - 3$ degrees of freedom.

To make a study of how closely the chi-square distribution approximates the exact distribution of $-n \ln L_{mvc}$ for various values of k and n would be an arduous task in computation. But existing experience with approximations to large sample distributions indicates that the approximation in the present problem would be satisfactory for small values of k (say not more than 5) and values of n not less than 50. Some light is thrown on this question for $k = 2$ and 3 by Table IV.

2.1.3. *Moments of the exact distribution of* L_{mvc}. In Section 2.1.2 an approximation is given to the distribution of $-n \ln L_{mvc}$ for large samples. As a matter of fact, one can find expressions for the moments of the exact distribution of L_{mvc}, which for the cases of $k = 2$ and $k = 3$ yield simple expressions for the exact distribution of L_{mvc}.

To find the moments of L_{mvc} it will be noted that if one sets

$$ns_{ij} = a_{ij}$$

$$ns_{0ij} = a_{0ij}$$

in expression (2.18) for L_{mvc}, the following expression is obtained for L_{mvc}.

$$L_{mvc} = \left[\frac{|a_{ij}|}{R_0^{k-1} S_0}\right] \tag{2.19}$$

where

$$\begin{aligned} R_0 &= \frac{1}{k}\sum_{i=1}^{k} a_{0ii} - \frac{1}{k(k-1)} \sum_{i \neq j=1}^{k} a_{0ij} \\ S_0 &= \frac{1}{k}\left(\sum_{i=1}^{k} a_{0ii} + \sum_{i \neq j=1}^{k} a_{0ij}\right). \end{aligned} \tag{2.20}$$

It will be seen that L_{mvc} depends on the $\bar{x}_i$ and the a_{ij}. In the case of a sample from a general normal multivariate population, we know the a_{ij} to be distributed according to the Wishart [8] distribution function

$$W_{n-1,k}(a_{ij}; A_{ij}) = \frac{|A_{ij}|^{\frac{1}{2}(n-1)} |a_{ij}|^{\frac{1}{2}(n-k-2)} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{k} A_{ij}a_{ij}\right]}{2^{\frac{1}{2}k(n-1)} \pi^{\frac{1}{4}k(k-1)} \prod_{i=1}^{k} \Gamma(\frac{1}{2}(n-i))} \tag{2.21}$$

and the means $\bar{x}_i$ to be independently distributed according to the normal distribution

$$f(\bar{x}_i) = \frac{n^{\frac{1}{2}k} |A_{ij}|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}k}} \exp\left[-\frac{n}{2}\sum_{i,j=1}^{k} A_{ij}(\bar{x}_i - a_i)(\bar{x}_j - a_j)\right] \tag{2.22}$$

where the A_{ij} and a_i were defined in (2.1).

We now define a function $\varphi(g, u, v)$ as the mean value of $|a_{ij}|^g e^{uR_0+vS_0}$ when H_{mvc} is true, i.e.,

$$\varphi(g, u, v) = E(|a_{ij}|^g e^{uR_0+vS_0}) \tag{2.23}$$

where the right hand side denotes multiplication of (2.21) by (2.22) (after imposing condition (2.4)) by $|a_{ij}|^g e^{uR_0+vS_0}$ and then integration with respect to the a_{ij} and $\bar{x}_i$. This yields

$$\begin{aligned} \varphi(g, u, v) &= 2^{gk} \prod_{i=1}^{k}\left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))}\right] \\ &\times \frac{(A-B)^{\frac{1}{2}n(k-1)}(A+(k-1)B)^{\frac{1}{2}(n-1)}}{\left(A - B - \frac{2u}{k-1}\right)^{\frac{1}{2}(k-1)(n+g)} (A+(k-1)B-2v)^{\frac{1}{2}(n-1)+g}}. \end{aligned} \tag{2.24}$$

Now the gth moment $M_g(L_{mvc})$ of L_{mvc} is defined by

$$(2.25)\qquad M_g(L_{mvc} = E[(L_{mvc})^g]$$

and is obtained by evaluating the partial derivative

$$(2.26)\qquad \frac{\partial^{r(k-1)+s}}{\partial u^{r(k-1)}\,\partial v^s}(\varphi)$$

at $u = v = 0$, and then putting $r = -g$ and $s = -g$. The validity of this operation for the range of values of g in which we are interested can be established by an argument based on analytic continuation. Alternatively, the same result can be achieved by taking the indefinite integral of φ $r(k - 1)$ times successively with respect to n, and s times successively with respect to v (the lower limit of integration being $-\infty$ in every case) and then evaluating the final result at $u = v = 0$. Accordingly, we obtain for the gth moment of L_{mvc}, when hypothesis H_{mvc} is true, the following expression

$$(2.27)\qquad \begin{aligned} Mg(L_{mvc}) &= \prod_{i=1}^{k}\left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))}\right] \\ &\times (k-1)^{g(k-1)}\frac{\Gamma(\frac{1}{2}(n-1)\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}(n-1)+g)\Gamma(\frac{1}{2}n(k-1)+g(k-1))}. \end{aligned}$$

2.1.4. *Distribution of L_{mvc} for $k = 2$ and 3.* For $k = 2$, the criterion L_{mvc} can be expressed as

$$(2.28)\qquad L_{mvc} = \frac{\begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}}{\begin{vmatrix} \frac{1}{2}(s_{11}+s_{22})+\frac{1}{4}(\bar{x}_1-\bar{x}_2)^2 & s_{12}-\frac{1}{4}(\bar{x}_1-\bar{x}_2)^2 \\ s_{21}-\frac{1}{4}(\bar{x}_1-\bar{x}_2)^2 & \frac{1}{2}(s_{11}+s_{22})+\frac{1}{4}(\bar{x}_1-\bar{x}_2)^2 \end{vmatrix}}.$$

The gth moment of L_{mvc} for $k = 2$ (obtained by putting $k = 2$ in (2.26) is

$$(2.29)\qquad M_g(L_{mvc}) = \frac{\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(n-2)+g)}{\Gamma(\frac{1}{2}n+g)\Gamma(\frac{1}{2}(n-2))} = \frac{(\frac{1}{2}(n-2))}{(\frac{1}{2}(n-2)+g)},$$

and the distribution function of L_{mvc} is found to be

$$(2.30)\qquad dF(L_{mvc}) = \tfrac{1}{2}(n-2)L_{mvc}^{\frac{1}{2}(n-4)}\,dL_{mvc}, \qquad (0 \leq L_{mvc} \leq 1).$$

For $k = 3$, L_{mvc} can be written as

$$(2.31)\qquad L_{mvc} = \frac{\begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix}}{(s_0^2)^3(1-r_0)^2(1+2r_0)}$$

where s_0^2 and r_0 are defined in (2.15) for $k = 3$. Putting $k = 3$ in (2.26) we find the gth moment of L_{mvc} for this case to be

$$(2.32) \qquad M_g(L_{mvc}) = 2^{2g} \frac{\Gamma(\frac{1}{2}(n-2)+g)\Gamma(\frac{1}{2}(n-3)+g)\Gamma(n)}{\Gamma(\frac{1}{2}(n-2))\Gamma(\frac{1}{2}(n-3))\Gamma(n+2g)}.$$

By using the fact that

$$\Gamma(t+\tfrac{1}{2})\Gamma(t+1) = \frac{\sqrt{\pi}\,\Gamma(2t+1)}{2^{2t}},$$

it is seen that $M_g(L_{mvc})$ reduces to

$$(2.33) \qquad M_g(L_{mvc}) = \frac{\Gamma(n)\Gamma(n-3+2g)}{\Gamma(n+2g)\Gamma(n-3)},$$

from which we deduce the distribution of L_{mvc} to be

$$(2.34) \qquad dF(L_{mvc}) = \frac{\Gamma(n)}{\Gamma(3)\Gamma(n-3)} (\sqrt{L_{mvc}})^{n-4}(1-\sqrt{L_{mvc}})^2\, d\sqrt{L_{mvc}}, \qquad (0 \le L_{mvc} \le 1).$$

For values of $k > 3$, the exact distribution of L_{mvc} seems to be too complicated to lend itself to ready computation.

Thus, relatively simple exact tests of significance of L_{mvc} can be set up for $k = 2$ and $k = 3$ by using distribution functions (2.30) and (2.34) respectively. For large values of n we have pointed out that the significance of L_{mvc} can be tested by making use of the fact that $-n \ln L_{mvc}$ is approximately distributed according to a chi-square law with $\frac{1}{2}k(k+3) - 3$ degrees of freedom when H_{mvc} is true.

For $k = 2$, L_{mvc} is essentially a criterion for simultaneously testing, on the basis of a sample, the hypothesis of equality of means and equality of variances of a normal bivariate population.

It should be noted that if H_{mvc} is true, or more realistically, is supported by the sample as a result of applying test L_{mvc}, then population Π is characterized by the three parameters a, σ^2 and ρ in (2.3). The likelihood estimates of these parameters are $\bar{x}$, s_0^2 and r_0.

2.2. The test L_{vc} for H_{vc}, the hypothesis of equality of variances and equality of covariances, irrespective of the values of the means.

2.2.1. *Derivation of the criterion L_{vc}.* If, in testing hypothesis H_{mvc} by means of the criterion L_{mvc}, at a given level of significance, say ϵ, a non-significant value of L_{mvc} is obtained, one states that the sample is consistent with the hypothesis H_{mvc} that all the population means are equal, the variances are equal and the covariances are equal. Consideration of the Neyman-Pearson Type II error involved in this statement would be very arduous and involved and will not be attempted. On the other hand, if a significant value of L_{mvc} is obtained, one states that the sample contradicts the hypothesis H_{mvc} with probability ϵ of making a Neyman-Pearson Type I error. In this case it may be reasonable to inquire whether the sample would support the hypothesis if the variability due to the means were eliminated. In other words, we may inquire whether the sample supports the hypothesis H_{vc} of equal variances and equal covariances, irrespective of what values the population means may have. To obtain the likelihood ratio criterion L_{vc} for testing H_{vc} we maximize the likelihood (2.2) under the following two sets of conditions: First, with respect to the means a_i and the variances and covariances $\rho_{ij}\sigma_i\sigma_j$; and Secondly, with respect to the means a_i and A and B, where A and B are obtained by imposing the condition on the matrix $\| A_{ij} \|$ specified in (2.14). The maximum of (2.2) under the first condition is given by (2.11). Denoting the maximum of (2.2) under the second set of conditions by $P_{\omega'}$, it is found, by a procedure similar to that used in finding P_ω (given by (2.17), that $P_{\omega'}$ is given by

$$P_{\omega'} = \frac{e^{-\frac{1}{2}kn}}{[(s^2)^k(1-r)^{k-1}(1+(k-1)r)]^{\frac{1}{2}n}(2\pi)^{\frac{1}{2}kn}} \tag{2.35}$$

where

$$\begin{aligned} r &= \sum_{i\neq j=1}^{k} s_{ij} \Big/ (k-1)\sum_{i=1}^{k} s_{ii} \\ s^2 &= \sum_{i=1}^{k} s_{ii} \Big/ k. \end{aligned} \tag{2.36}$$

The likelihood ratio λ_{vc} for testing H_{vc} is given by

$$\lambda_{vc} = \left[\frac{|s_{ij}|}{(s^2)^k(1-r)^{k-1}(1+(k-1)r)}\right]^{\frac{1}{2}n}.$$

The test criterion which will be used for testing H_{vc} is L_{vc}, the $\frac{2}{n}$th root of λ_{vc}, i.e.,

$$L_{vc} = \frac{|s_{ij}|}{(s^2)^k(1-r)^{k-1}(1+(k-1)r)}. \tag{2.37}$$

2.2.2. *Approximate distribution of* $-n \ln L_{vc}$ *in large samples.*

In the case of large samples $-n \ln L_{vc}$ is approximately distributed according to the chi-square law with $\frac{1}{2}k(k+1) - 2$ degrees of freedom when hypothesis H_{vc} is true.

2.2.3. *Moments of the exact distribution of* L_{vc}. The moments of L_{vc} when H_{vc} is true can be found by a method similar to that used in Section 2.1.3 for determining the moments of L_{mvc}. For it will be noted that L_{vc} can be written as

$$L_{vc} = \left[\frac{|a_{ij}|}{R^{k-1}S}\right] \tag{2.38}$$

where

$$R = \frac{1}{k}\sum_{i=1}^{k} a_{ii} - \frac{1}{k(k-1)}\sum_{i\neq j=1}^{k} a_{ij} \tag{2.39}$$

$$S = \frac{1}{k}\left[\sum_{i=1}^{k} a_{ii} + \sum_{i\neq j=1}^{k} a_{ij}\right],$$

from which it is evident that L_{vc} depends only on the a_{ij}, whose distribution in the case of a general normal multivariate population is given by (2.21). We now define a function $\theta(g, y, z)$ as the mean value of $| a_{ij} |^g e^{yR+zS}$ under the assumption that H_{vc} is true, i.e.,

$$\theta(g, y, z) = E(| a_{ij} |^g e^{yR+zS}) \tag{2.40}$$

where the value of the right hand side is obtained by multiplying (2.21) by $| a_{ij} |^g e^{yR+zS}$, then imposing the condition on $|| A_{ij} ||$ stated in (2.4) and integrating with respect to the a_{ij}. Accordingly, we find

$$\theta(g, y, z) = 2^{gk}\prod_{i=1}^{k}\left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma\frac{1}{2}(n-i)}\right] \times \frac{(A-B)^{\frac{1}{2}(k-1)(n-1)}(A+(k-1)B)^{\frac{1}{2}(n-1)}}{\left(A-B-\frac{2y}{k-1}\right)^{\frac{1}{2}(k-1)(n-1+2g)}(A+(k-1)B-2z)^{\frac{1}{2}(n-1)+g}}. \tag{2.41}$$

The gth moment $M_g(L_{vc})$ of L_{vc} is obtained by evaluating the partial derivative

$$\frac{\partial^{r(k-1)+s}}{\partial y^{r(k-1)}\,\partial z^{s}}\,\theta \tag{2.42}$$

at $y = z = 0$, and then setting, $r = -g$ and $s = -g$. These operations yield

$$M_g(L_{vc}) = \prod_{i=1}^{k}\left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma\frac{1}{2}(n-i)}\right] \times (k-1)^{g(k-1)}\frac{\Gamma(\frac{1}{2}(n-1))\Gamma(\frac{1}{2}(k-1)(n-1))}{\Gamma(\frac{1}{2}(n-1)+g)\Gamma(\frac{1}{2}(k-1)(n-1)+g(k-1))}. \tag{2.43}$$

2.2.4. *Distribution of L_{vc} for $k = 2$ and 3.* For $k = 2$, L_{vc} can be expressed as follows:

$$L_{vc} = \frac{\begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}}{\begin{vmatrix} \frac{1}{2}(s_{11}+s_{22}) & s_{12} \\ s_{21} & \frac{1}{2}(s_{11}+s_{22}) \end{vmatrix}} \tag{2.44}$$

and the gth moment of L_{vc} is given by

$$M_g(L_{vc}) = \frac{\Gamma(\frac{1}{2}(n-1))\Gamma(\frac{1}{2}(n-2)+g)}{\Gamma(\frac{1}{2}(n-1)+g)\Gamma(\frac{1}{2}(n-2))} \tag{2.45}$$

from which the distribution of L_{vc} is deduced to be

$$dF(L_{vc}) = \frac{\Gamma(\frac{1}{2}(n-1))}{\sqrt{\pi}\Gamma(\frac{1}{2}(n-2))} L_{vc}^{\frac{1}{2}(n-4)}(1-L_{vc})^{-\frac{1}{2}}\, dL_{vc}, \qquad (0 \leq L_{vc} \leq 1). \tag{2.46}$$

For $k = 3$, L_{vc} can be expressed as

$$L_{vc} = \frac{\begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix}}{(s^2)^3(1-r)^2(1+2r)} \tag{2.47}$$

where s^2 and r are defined in (2.36) by setting $k = 3$. Setting $k = 3$ in (2.43), we find as the gth moment of L_{vc}

$$M_g(L_{vc}) = 2^{2g}\frac{\Gamma(\frac{1}{2}(n-2)+g)\Gamma(\frac{1}{2}(n-3)+g)\Gamma(n-1)}{\Gamma(\frac{1}{2}(n-2))\Gamma(\frac{1}{2}(n-3))\Gamma(n-1+2g)}. \tag{2.48}$$

Following the method by which (2.32) was reduced to (2.33), we find that the gth moment of L_{vc} reduces to

$$M_g(L_{vc}) = \frac{\Gamma(n-1)\Gamma(n-3+2g)}{\Gamma(n-1+2g)\Gamma(n-3)}, \tag{2.49}$$

and hence the distribution function of L_{vc} for $k = 3$ is

$$dF(L_{vc}) = \frac{\Gamma(n-1)}{\Gamma(n-3)}(\sqrt{L_{vc}})^{n-4}(1-\sqrt{L_{vc}})\, d\sqrt{L_{vc}}, \quad (0 \leq L_{vc} \leq 1). \tag{2.50}$$

For higher values of k the distribution of L_{vc} is apparently too complicated for ready computation. But distributions (2.46) and (2.50) provide relatively simple significance tests for the cases $k = 2$ and 3, respectively. For large samples, we remark again that a significance test for L_{vc} is provided by the fact $-2 \ln \lambda_{vc}$ (i.e., $-n \ln L_{vc}$) is approximately distributed according to the chi-square law with $\frac{1}{2}k(k+1) - 2$ degrees of freedom when H_{vc} is true.

For $k = 2$, L_{vc} is essentially a criterion for testing, on the basis of a sample, the hypothesis of equality of variances of a normal bivariate population.

If H_{vc} is true, Π will be characterized by the parameters $a_1, a_2, \cdots, a_k, \sigma^2$ and ρ. The maximum likelihood estimates of these parameters are $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k, s^2$ and r, respectively.

2.3. The test L_m for H_m, the hypothesis of equality of means, when the variances are equal and covariances are equal.

2.3.1. *Devivation of the criterion L_m.* Suppose L_{vc}, described in Section 2.2.1 for testing H_{vc}, the hypothesis of equal variances and equal covariances, does not have a significantly small value, thus indicating that the sample does not contradict the hypothesis H_{vc}. Then, assuming that the original test L_{mvc} of H_{mvc} turned out to have a significantly small value, we may inquire as to whether the significance of L_{mvc} is due to the inequality of the population means a_i. In this section we shall consider a criterion L_m for testing the hypothesis H_m that the means a_i are equal, assuming that the variances are equal and that the covariances are equal. In this hypothesis we maximize the likelihood (2.2) under the following two sets of conditions: First, with respect to the a_i, A and B, where A and B are defined by the condition on $\| A_{ij} \|$ given in (2.4); secondly, with respect to a, A and B where these parameters are specified by (2.4). The maxima of the likelihood (2.2) under these two conditions are $P_{\omega'}$, and P_ω, given by (2.35) and (2.17) respectively. The likelihood ratio λ_m is therefore

$$(2.51) \qquad \lambda_m = \frac{P_\omega}{P_{\omega'}} = \left[\frac{(s^2)^k(1 - r)^{k-1}(1 + (k - 1)r)}{(s_0^2)^k(1 - r_0)^{k-1}(1 + (k - 1)r_0)}\right]^{\frac{1}{2}n}.$$

Now it follows from the definitions of s^2, , s_0^2 and r_0, (2.15) and (2.36) that

$$s^2(1 + (k - 1)r) \equiv s_0^2(1 + (k - 1)r_0)$$

and hence we may write

$$(2.52) \qquad \lambda_m^{2/n} = \left[\frac{s^2(1 - r)}{s_0^2(1 - r_0)}\right]^{k-1}$$

We can also express $\lambda_m^{2/n}$ as

$$(2.53) \qquad \lambda_m^{2/n} = \left(\frac{R}{R_0}\right)^{k-1}$$

where R_0 and R are defined by (2.20) and (2.39) respectively.

It will be most convenient for our purposes to use L_m, the $[2/n(k - 1)]$-th root of λ_m. as the criterion for testing H_m, i.e.

$$(2.54) \qquad L_m = R/R_0 = \frac{s^2(1 - r)}{s_0^2(1 - r_0)} = \frac{s^2(1 - r)}{s^2(1 - r) + \frac{1}{k - 1}\sum_{i=1}^{k} (\bar{x}_i - \bar{x})^2}.$$

2.3.2. *Approximate distribution of $-n(k - 1) \ln L_m$ in large samples.*
In large samples $-2 \ln \lambda_m$ (i.e., $-n(k - 1) \ln L_m$) is approximately distributed according to the chi-square law with $k - 1$ degrees of freedom. However, the exact distribution of L_m is relatively simple and will be derived.

2.3.3. *Exact distribution of L_m when H_m is true.* We shall determine the distribution of L_m by first finding the gth moment of L_m when H_m is true. For this purpose we set up the function

$$(2.55) \qquad \psi(p, q) = E(e^{pR + qR_0})$$

where the mean value is taken when H_m is true, i.e., when the a_i and $\| A_{ij} \|$ satisfactory conditions (2.4). Now R and R_0 are functions of the a_{ij} and $\bar{x}_i$. Hence, to find $E(e^{pR+qR_0})$ we multiply (2.21) by (2.22) by e^{pR+qR_0} and impose conditions (2.4), then take the integral over the entire space of the a_{ij} and $\bar{x}_i$. These operations yield

$$\psi(p, q) = \frac{(A - B)^{\frac{1}{2}n(k-1)}}{\left(A - B - \frac{2(p+q)}{k-1}\right)^{\frac{1}{2}(n-1)(k-1)} \left(A - B - \frac{2q}{k-1}\right)^{\frac{1}{2}(k-1)}}. \tag{2.56}$$

The gth moment of L_m is obtained by performing the following differentiations

$$\left[\frac{\partial^h}{\partial q^h}\left\{\frac{\partial^g \psi}{\partial p^g}\right\}_{p=0}\right]_{q=0} \tag{2.57}$$

and then putting $h = -g$. These operations yield

$$M_g(L_m) = \frac{\Gamma(\frac{1}{2}(n-1)(k-1) + g)\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}(n-1)(k-1))\Gamma(\frac{1}{2}n(k-1) + g)} \tag{2.58}$$

from which the distribution of L_m (when H_m is true) is found to be

$$dF(L_m) = \frac{\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}(n-1)(k-1))\Gamma(\frac{1}{2}(k-1))} L_m^{\frac{1}{2}(n-1)(k-1)-1} \cdot (1 - L_m)^{\frac{1}{2}(k-1)-1} dL_m, \quad (0 \le L_m \le 1). \tag{2.59}$$

Thus, we are able to make an exact test of significance of L_m on the basis of the function (2.59)

2.4. Relations between L_{mvc}, L_{vc} and L_m.

It will be seen from the definitions of L_{mvc}, L_{vc} and L_m in (2.18), (2.37) and (2.54) (noting that $s^2(1 + (k-1)r) \equiv s_0^2(1 + (k-1)r_0)$) that

$$L_{mvc} = L_{vc} \cdot L_m^{k-1}.$$

Furthermore, it will be noted that when H_{mvc} is true, the gth moment of L_{mvc} given by (2.27) is equal to the product of the gth moment of L_{vc} given by (2.43) and the gth moment of L_m^{k-1} (obtained by replacing g by $g(k-1)$ in (2.58). Thus, when H_{mvc} is true λ_{mvc} is composed of the product of two independently distributed quantities, namely L_{vc} and L_m^{k-1}.

REFERENCES

[1] H. Scheffé, "A Note on the Behrens-Fisher Problem," *Annals. of Math. Stat.*, Vol. 15 (1944), pp. 430–432.

[2] John W. Tukey and S. S. Wilks, "Approximation of the distribution of the product of beta variables by a single beta variable," *Annals of Math. Stat.*, Vol. 17 (1946), pp. 318–324.

[3] K. Pearson, *Tables of the Incomplete Beta Function*, Cambridge University Press, 1932.

[4] Catherine M. Thompson, "Table of percentage points of the Incomplete Beta Function, *Biometrika*, Vol. 32, Part III (1941), pp. 151–181.
[5] S. S. Wilks, "The large-sample distribution of the likelihood ratio for testing composite hypotheses," *Annals of Math. Stat.*, Vol. 9 (1938), pp. 60–62.
[6] Catherine M. Thompson, "Table of percentage points of the χ^2 distribution," *Biometrika*, Vol. 32, Part II (1941), pp. 187–191.
[7] John W. Mauchly, "Significance test for sphericity of a normal multivariate distribution," *Annals of Math. Stat.*, Vol. 11 (1940), pp. 204–209.
[8] J. Wishart, "The generalized product moment distribution in samples from a normal multivariate population," *Biometrika*, Vol. 20A, pp. 32–52.