NOTES

This section is devoted to brief research and expository articles on methodology and other short items.

ON SEQUENTIAL BINOMIAL ESTIMATION

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The present note, written after a reading of the very interesting paper by Girshick, Mosteller, and Savage [1], is for the purpose of adding a few remarks in the nature of a supplement. For the sake of brevity the notation and terminology of [1] are adopted in toto.

Theorem 1 below generalizes Theorem 1 of [1]. In Theorem 2′ we formulate explicitly the fact which lies at the basis of the GSM method of estimation. Parts of the proofs of Theorems 3 and 4 of [1] are simply proofs of special cases of this (e.g., equation (2) of [1]). We then use this fact repeatedly in proving Theorem 3, which states that the Girshick-Mosteller-Savage estimate is the only proper unbiased estimate for sequential tests defined by regions which we shall call doubly simple.

A doubly simple region is defined precisely below. Intuitively we may describe such a region as the one between two curves $y = f_1(x)$ and $x = f_2(y)$, where $f_1(x)$ is defined and monotonically non-decreasing for all non-negative x, $f_2(y)$ is defined and monotonically non-decreasing for all non-negative y, $f_1(0) > 0$, $f_2(0) > 0$. If the two curves intersect, the region is finite, and the values of the functions f_1 and f_2 beyond the point of intersection are of no interest. This description is of course purely heuristic, because in actual fact only integral values of the variables come into play, and intersection of the curves, for example, is not needed to make the region finite. Since the question of finite regions is completely settled by [1], Theorem 7, only non-finite regions remain to be discussed, and the precise definition given below is such as to imply that the region is not finite. It seems to the present writer that at least many of the non-finite sequential tests which may be developed for meaningful statistical problems will require doubly simple regions. The Wald sequential binomial test [2] defines such a region, which also falls within the scope of Theorem 6 of [1]. It is easy to see that there exist closed regions which are doubly simple and do not satisfy the conditions of this theorem.

By a "proper" estimate $p(\alpha)$ we shall mean an estimate such that $0 \leq p(\alpha) \leq 1$ for every α. It is difficult to see how any estimate which is not proper can make much sense.

THEOREM 1. *A sufficient condition that a region R be closed is that* $\liminf_{n\to\infty} \frac{A(n)}{\sqrt{n}} < \infty$, *where $A(n)$ is the number of accessible points of index n.*

PROOF: The hypothesis of the theorem implies that there exist a positive number H and an increasing sequence of positive integers $n_1, n_2, n_3, \cdots$, with the following properties:

a) $n_{i+1} > 2n_i$ ($i = 1, 2, \cdots$ ad inf.)

b) $A(n_i) < H\sqrt{n_i}$.

For n_i sufficiently large, the conditional probability of reaching the accessible points on $x + y = n_{i+1}$, when an accessible point on $x + y = n_i$ has been reached, is $< K < 1$ by the normal approximation to the binomial distibution, where K is constant (and depends on H). Hence the probability of passing through accessible points on all members of the set $x + y = n_i$ ($i = 1, 2, \cdots, L$) approaches zero as $L \rightarrow \infty$, so that the region is closed.

THEOREM 2. *Let R be any region, B its boundary, and $t = (a, b)$; any accessible point in R. Let $l_t(\alpha)$ be the number of paths from t to $(x, y) = \alpha \epsilon B$. Let $Q(t)$ be the conditional probability that a path, which has reached t, will reach the boundary B. Then*

$$\sum_{\alpha \epsilon B} l_t(\alpha)p^y q^x = Q(t)p^b q^a.$$

THEOREM 2'. (Corollary to Theorem 2)
If R is closed, then

$$\sum_{\alpha \epsilon B} l_t(\alpha)p^y q^x = p^b q^a. \tag{1}$$

PROOF: Let $k(t)$ be the number of paths in R from the origin to t. The probability of reaching $\alpha \epsilon B$ by a path which passes through t is $k(t)l_t(\alpha)p^y q^x$. The probability of reaching t from the origin is $k(t)p^b q^a$, and hence the probability of reaching the boundary via t is $Q(t)k(t)p^b q^a$. From this the desired result follows.

We now define a doubly simple region. The boundary of the region consists of the two infinite sequences of points

$$(0, a_0), (1, a_1), (2, a_2), \cdots$$

and

$$(b_0, 0), (b_1, 1), (b_2, 2), \cdots$$

where $a_0, a_1, a_2, \cdots$ and $b_0, b_1, b_2, \cdots$ are two infinite non-decreasing sequences of positive integers. The accessible points of the region are all points which can be reached by a path from the origin which does not contain a boundary point. (It is to be noted that since a boundary point is, by definition, a point not in the region which can be reached by a path in the region, the above definition implies that a doubly simple region is not finite. The reason for making this so has been given above.)

THEOREM 3. Let R be a closed doubly simple region. Then $\hat{p}(\alpha)$ is the unique proper unbiased estimate of p.

PROOF: Suppose there were two proper unbiased estimates $p_1(\alpha)$ and $p_2(\alpha)$. Writing $m(\alpha) = p_1(\alpha) - p_2(\alpha)$, we would have

$$\sum_{\alpha \in B} m(\alpha)k(\alpha)p^y q^x = 0 \tag{2}$$

with

$$| m(\alpha) | \leq 1 \tag{3}$$

First we prove

LEMMA 1. *If* $a_0 > 1$, *then* $m(b_0, 0) = 0$.

PROOF: Let $k^*(\alpha)$ denote the number of paths in R from the point (0, 1) to the boundary point α. For all points $\alpha \in B$ except $(b_0, 0)$ we have

$$b_0 k^*(\alpha) \geq k(\alpha). \tag{4}$$

From (1), (2), (3), and (4) we have, since $k(b_0, 0) = 1$,

$$\begin{aligned} | m(b_0, 0) | q^{b_0} &= \Big| \sum_{\alpha \in B, \alpha \neq (b_0, 0)} m(\alpha)k(\alpha)p^y q^x \Big| \\ &\leq \sum_{\alpha \in B, \alpha \neq (b_0, 0)} k(\alpha)p^y q^x \leq b_0 \sum_{\alpha \in B} k^*(\alpha)p^y q^x = b_0 p. \end{aligned} \tag{5}$$

Now as $p \to 0$, the left member of the inequality (5) approaches $| m(b_0, 0) |$, and the right member approaches zero. This proves Lemma 1.

LEMMA 2. *For every* $z < a_0 - 1$, $m(b_z, z) = 0$.

PROOF: In view of Lemma 1 it is sufficient to prove the following:

If $Z \leq a_0 - 2$, and if $m(b_z, z) = 0$ for $z = 0, 1, \cdots, Z - 1$, then $m(b_Z, Z) = 0$. Let $k_{Z+1}(\alpha)$ denote the number of paths in R from $(0, Z + 1)$ to the boundary point α. For any point $\alpha \in B$ whose ordinate is $\geq Z + 1$ we have

$$b_0 b_1 \cdots b_Z k_{Z+1}(\alpha) \geq k(\alpha). \tag{6}$$

From (1), (2), (3), and (6) we have

$$\begin{aligned} | m(b_Z, Z) | k(b_Z, Z)p^Z q^{b_Z} &= | \Sigma m(\alpha)k(\alpha)p^y q^x | \leq \Sigma k(\alpha)p^y q^x \\ &\leq b_0 b_1 \cdots b_Z \Sigma k_{Z+1}(\alpha)p^y q^x = b_0 b_1 \cdots b_Z p^{Z+1} \end{aligned} \tag{7}$$

where the summations take place over all boundary points whose ordinates are $\geq Z + 1$. Hence

$$| m(b_Z, Z) | k(b_Z, Z)q^{b_Z} \leq b_0 b_1 \cdots b_Z p.$$

and letting $p \to 0$ we obtain the desired result.

LEMMA 3. $m(b_{a_0-1}, a_0 - 1) = 0$.

PROOF: Let s be the smallest integer such that (s, a_0) is an accessible point. We proceed as in Lemma 2, with (s, a_0) playing the role of $(0, Z + 1)$, and eventually obtain the following inequality:

$$\begin{aligned} | m(b_{a_0-1}, a_0 - 1) | k(b_{a_0-1}, a_0 - 1)p^{a_0-1} q^{b_{a_0-1}} &= | \textstyle\sum_0 m(\alpha)k(\alpha)p^y q^x | \\ &\leq p^{a_0} \left(\sum_{i=0}^{s-1} k(i, a_0)q^i + b_0 b_1 \cdots b_{a_0-1} q^s \right), \end{aligned} \tag{8}$$

where Σ_0 denotes summation over all boundary points with ordinate $\geq a_0$. The desired result follows.

LEMMA 4. *Let $h(\geq a_0)$ be the smallest ordinate for which at least one boundary point (w^*, h) exists such that $m(w^*, h) \neq 0$ (If no such h exists the theorem is proved). Of all such points let w be the one with the smallest abscissa. Then the point (w, h) is a member of the sequence*

$$(0, a_0)\ (1, a_1),\ (2, a_2),\ \cdots$$

PROOF: If the lemma is not true, then for all boundary points α with ordinate h, $m(\alpha) = 0$, except that $m(b_h, h) \neq 0$. Let W be that accessible point of R whose ordinate is $h + 1$ and whose abscissa v is a minimum. Let $k_w(\alpha)$ be the number of paths in R from W to the boundary point α. For boundary points α accessible from W we have

$$b_0 b_1 \cdots b_h k_W(\alpha) \geq k(\alpha). \tag{9}$$

From (1), (2), (3), and (9) we have

$$\begin{aligned} | m(b_h, h) | k(b_h, h) p^h q^{b_h} = | \Sigma_1 (m(\alpha) k(\alpha) p^y q^x | \leq \Sigma_2 k(\alpha) p^{h+1} q^x \\ + b_0 b_1 \cdots b_h p^{h+1} q^v = K^* p^{h+1}, \end{aligned} \tag{10}$$

where:

a) Σ_1 denotes summation over all $\alpha \in B$ for which $y > h$

b) Σ_2 denotes summation over all boundary points α of ordinate $h + 1$ and abscissa $< v$.

c) K^* denotes a constant.

From this it easily follows that $m(b_h, h) = 0$, in contradiction to the definition of h. This proves Lemma 4.

PROOF OF THEOREM 3: Let (w, h) be as defined in the statement of Lemma 4. From Lemma 4 it follows that, if any other boundary points with abscissa w exist, they must be members of the sequence $(b_0, 0)$, $(b_1, 1)$, $(b_2, 2)$, $\cdots$ and hence their ordinates are $< h$. From the definition of (w, h) and from Lemma 4 it follows that for any $\alpha \in B$ whose abscissa is $< w$, $m(\alpha) = 0$.

Now in the proofs of Lemmas 1–4 the roles of x and y are not symmetrical. However, symmetry of course exists, and analogous lemmas follow. In particular, the analogue to Lemma 4 has as a consequence that, since w is the smallest abscissa such that $m(\alpha) = 0$ when abscissa of $\alpha < w$, and $m(w, h) \neq 0$, there exists a boundary point (w, h'), such that $m(w, h') \neq 0$ and (w, h') is a member of $(b_0, 0)$, $(b_1, 1)$, $(b_2, 2)$, ... Then $h' < h$. But this contradicts the definition of h and proves the theorem.

It is easy to see that, if the boundary points of a closed region constitute a single "curve" instead of two "curves" as in a doubly simple region, the estimate $\hat{p}(\alpha)$ will be the only proper unbiased estimate of p.

It is interesting to consider some of the consequences of Theorem 3 for all unbiased estimates (not necessarily proper) for doubly simple regions. An examination of the proof of Theorem 3 shows that it would go through with little change if equation (3) were replaced by the requirement that $| m(\alpha) |$ be bounded. We therefore obtain the following result: If for a doubly simple region there exists an unbiased estimate $p(\alpha)$ of p, not identically equal to $\hat{p}(\alpha)$, then not only is $p(\alpha)$ not proper, but also, no matter how large M, there exists a boundary point α such that $| p(\alpha) | > M$. The uselessness of such an estimate is manifest.

The author is of the opinion that freedom from bias is not necessarily an indispensable characteristic of an optimum estimate. In general there is no reason for requiring the first moment of the estimate rather than any other moment to be the unknown parameter. The justification in any particular case must be based on special conditions of the problem.

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REFERENCE

[1] M. A. Girshick, Frederick Mosteller, and L. J. Savage, "Unbiased estimates for certain binomial sampling problems with applications," *Annals of Math. Stat.*, Vol. 17 (1946). pp. 13–23.

[2] A. Wald, "Sequential tests of statistical hypotheses," *Annals of Math. Stat.*, Vol. 16 (1945), pp. 117–186.

DIFFERENTIATION UNDER THE EXPECTATION SIGN IN THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

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1. Introduction. Let $\{z_\alpha\}$ ($\alpha = 1, 2, \cdots$, ad inf.) be a sequence of random variables which are independently distributed with identical distributions. Let a be a positive, and b a negative constant. For each positive integral value m, let Z_m denote the sum $z_1 + \cdots + z_m$. Denote by n the smallest integral value for which Z_n does not lie in the open interval (b, a). For any random variable u, let the symbol $E(u)$ denote the expected value of u. The following identity, which plays a fundamental role in sequential analysis, has been proved in [1].

$$E[e^{Z_n t}\varphi(t)^{-n}] = 1, \tag{1.1}$$

where

$$\varphi(t) = E(e^{zt}) \tag{1.2}$$

and the distribution of z is equal to the common distribution of $z_1, z_2, \cdots$, etc. Identity (1.1) holds for all points t in the complex plane for which $\varphi(t)$ exists and $| \varphi(t) | \geq 1$.