ON HOTELLING'S WEIGHING PROBLEM[1]

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1. Summary. The paper contains some solutions of the weighing problems proposed by Hotelling [1]. The experimental designs are applicable to a broad class of problems of measurement of similar objects. The chemical balance problem (in which objects may be placed in either of the two pans of the balance) is almost completely solved by means of designs constructed from Hadamard matrices. Designs are provided both for a balance which has a bias and for one which has no bias.

The spring balance problem (in which objects may be placed in only one pan) is completely solved when the balance is biased. For an unbiased spring balance, designs are given for small numbers of objects and weighing operations. Also the most efficient designs are found for the unbiased spring balance, but it is shown that in some cases these cannot be used unless the number of weighings is as large as the binomial coefficient $\binom{p}{\frac{1}{2}p}$ or $\binom{p}{\frac{1}{2}(p+1)}$ where p is the number of objects.

It is found that when p objects are weighed in $N \geq p$ weighings, the variances of the estimates of the weights are of the order of σ^2/N in the chemical balance case (σ^2 is the variance of a single weighing), and of the order of $4\sigma^2/N$ in the spring balance case.

2. Introduction. The problem is fully discussed by Hotelling [1] and refers to the design of a certain class of simple experiments. We may consider the typical example of the class to be that of weighing several small objects on a chemical balance or other weighing device. Hotelling and Yates [2] have shown that the individual weights may be determined more accurately by weighing the objects in combinations rather than weighing each one separately. The designs are applicable to a great variety of problems of measurement, not only of weights, but of lengths, voltages and resistances, concentrations of chemicals in solutions, in fact any measurements such that the measure of a combination is a known linear function of the separate measures with numerically equal coefficients. The designs should be particularly useful in biological and chemical laboratories engaged in routine chemical analyses. We shall, however, in the interest of simplicity, discuss the problem in the language of weighing operations.

A particular design is denoted by a matrix. The three objects to be weighed in four weighing operations may be weighed by the following design:

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$$X = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{Vmatrix}$$

where the rows refer to weighing operations and the columns refer to the objects. In the above design the first two objects are weighed together in the first weighing operation, the first and third objects are weighed together in the second weighing operation, etc. From the four resulting weights the individual weights are estimated by the method of least squares. The design problem consists of finding matrices which will minimize the variances of these estimates.

There are two distinct though closely related problems here. One is to find efficient designs for the case in which the measure of a combination can only be the sum of the individual measures. This would be the case, for example, in weighing objects with a spring balance and we shall refer to it as the spring balance problem. The other problem is to find designs when an individual measurement may be either added or subtracted in a combination. This would be the case in weighting objects with a chemical balance (since an object may be put in either pan of the balance) and will be called the chemical balance problem. In the latter problem the design matrix may contain 0's, 1's, and -1's, whereas in the spring balance problem the matrix may contain only 0's and 1's.

We shall use Hotelling's notation. There are p objects with weights b_1, b_2, $\cdots$, b_p to be weighed in $N \geq p$ weighing operations. The design matrix is denoted by

(1) $$X = \| x_{\alpha i} \| \quad \alpha = 1, \cdots, N; i = 1, \cdots, p.$$

Denoting the transpose of X by X', let

(2) $$X'X = \| a_{ij} \| = \| a^{ij} \|^{-1}$$

(3) $$g_i = \sum_\alpha x_{\alpha i} y_\alpha$$

where y_α is the observed result of the α-th weighing operation. The least squares estimates of the b_i are

(4) $$\hat{b}_i = \sum_j a^{ij} g_j$$

and the variances of these estimates are

(5) $$\sigma_i^2 = a^{ii} \sigma^2$$

where σ^2 is the error variance of a single weighing operation. The a^{ii} will be called variance factors.

Hotelling's main theorem states that or any design, $a^{ii} \geq 1/N$, hence the best possible design is one such the inverse of the product of the design matrix by its transpose has its main diagonal elements equal to $1/N$. We shall call such a design an optimum design. Examples show that optimum designs do not exist for all values of N and p.

When an optimum design does not exist, the question arises as to how a best design shall be defined. In the present paper a design will be called best if the determinant of the matrix $\| a^{ij} \|$ is minimized. A best design in this sense is, therefore, a design which gives the smallest confidence region in the $\hat{b}_i$ $(i = 1, 2, \cdots, p)$ space for the estimates of the weights.

In certain situations, other definitions of best designs may conceivably be preferred. Thus, problems may arise in which one might prefer:

(a) to minimize the variance factors subject to the restriction that they be equal, (b) to minimize some function of the variance factors, or (c) to minimize only a certain subset of the a^{ii} on a minor of the matrix $\| a^{ij} \|$ as might be the case when one wanted only rough estimates of the weights of some of the objects, but accurate estimates of the others.

When an optimum design exists, the confidence regions are not only minimized, but, as Hotelling has shown, the variance factors are also minimized. It is not true in general, however, that a best design as here defined (minimum confidence regions) will also minimize the variance factors. Examples illustrating this point are given in the last part of section 6 and the first part of section 7.

3. Hadamard Matrices. The problem of finding the best designs is closely related to the Hadamard determinant problem. Hadamard [3] proved the following result: If the elements $x_{\alpha\beta}$ of a square matrix X are restricted to the range $-1 \leq x_{\alpha\beta} \leq 1$, the maximum possible value of the determinant of X is $N^{\frac{1}{2}N}$, and when this maximum is achieved all $x_{\alpha\beta} = \pm 1$ and the matrix is orthogonal in the sense that $X'X$ is a diagonal matrix; the non-zero elements of $X'X$ are all equal to N. A matrix X which satisfies these conditions will be denoted by H_N. Obviously if H_N exists for a given N, it is the solution of the design problem in the chemical balance case when $N = p$.

With regard to the existence of H_N, it is known that a necessary condition is

$$N \equiv 0 \pmod 4$$

with the exception of $N = 2$. It is not known however whether the above condition is sufficient, although it is known (Paley [4]) that H_{4k} exists for the range

$$0 < 4k \leq 100$$

with the possible exception of $4k = 92$. Paley and Williamson [5] give methods of constructing H_{4k} in the given range (excepting 92) based on the theory of finite fields.

When N is a power of two, H_N is easily constructed by taking direct products of

$$H_2 = \begin{Vmatrix} 1 & 1 \\ 1 & -1 \end{Vmatrix}$$

Thus

$$H_4 = H_2 \cdot H_2 = \begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{Vmatrix}$$

Sylvester [6] first studied this class of matrices and Kishen [7] has described weighing designs based on this subset of the H_N.

The following examples of Hadamard matrices may be found in the literature: Paley [4] exhibits an H_8, H_{12}, and H_{28}: Kishen gives an H_{16}. From these examples H_{24} and H_{32} may be constructed at once from the direct products $H_2 \cdot H_{12}$ and $H_2 \cdot H_{16}$. The following is an H_{20}:

$$\begin{Vmatrix}
+ & - & - & - & - & + & - & - & - & - & + & + & - & - & + & + & - & + & + & - \\
- & + & - & - & - & - & + & - & - & - & + & + & + & - & - & - & + & - & + & + \\
- & - & + & - & - & - & - & + & - & - & - & + & + & + & - & + & - & + & - & + \\
- & - & - & + & - & - & - & - & + & - & - & - & + & + & + & + & + & - & + & - \\
- & - & - & - & + & - & - & - & - & + & + & - & - & + & + & - & + & + & - & + \\
- & + & + & + & + & + & - & - & - & - & - & + & - & - & + & + & + & - & - & + \\
+ & - & + & + & + & - & + & - & - & - & + & - & + & - & - & + & + & + & - & - \\
+ & + & - & + & + & - & - & + & - & - & - & + & - & + & - & - & + & + & + & - \\
+ & + & + & - & + & - & - & - & + & - & - & - & + & - & + & - & - & + & + & + \\
+ & + & + & + & - & - & - & - & - & + & + & - & - & + & - & + & - & - & + & + \\
- & - & + & + & - & + & - & + & + & - & + & - & - & - & - & - & + & + & + & + \\
- & - & - & + & + & - & + & - & + & + & - & + & - & - & - & + & - & + & + & + \\
+ & - & - & - & + & + & - & + & - & + & - & - & + & - & - & + & + & - & + & + \\
+ & + & - & - & - & + & + & - & + & - & - & - & - & + & - & + & + & + & - & + \\
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- & + & - & - & + & - & - & + & + & - & + & - & - & - & - & + & - & - & - & - \\
+ & - & + & - & - & - & - & - & + & + & - & + & - & - & - & - & + & - & - & - \\
- & + & - & + & - & + & - & - & - & + & - & - & + & - & - & - & - & + & - & - \\
- & - & + & - & + & + & + & - & - & - & - & - & - & + & - & - & - & - & + & - \\
+ & - & - & + & - & - & + & + & - & - & - & - & - & - & + & - & - & - & - & +
\end{Vmatrix}$$

where the signs represent ± 1. This example was constructed by Williamson's method [5]. Thus examples of H_{4k} for the range $4 \leq 4k \leq 32$ are immediately available and methods of construction exist for the range $36 \leq 4k \leq 88$.

4. Chemical Balance Problem. When $N = 0 \pmod 4$ an optimum design exists if H_N exists and is obtained by using any p columns of H_N. When $N \not\equiv 0 \pmod 4$ we may construct very efficient designs as follows: If $N \equiv 1$ we may add a row of ones to H_{N-1}; if $N \equiv 2$ we may add two rows of ones or a row of H_2's to H_{N-2}; and if $N \equiv 3$ we may delete one row from H_{N+1}. The worst of these designs will be obtained when two rows of ones are added to an H_{N-2}, and in this case the variance factors are

$$a^{ii} = \frac{1}{N-2} \frac{N+2p-4}{N+2p-2} < \frac{1}{N-2}. \tag{6}$$

Since it is known that these factors must be greater than $1/N$ for the best possible design in this case, the above design will be quite near the best design for large N.

For small values of N we shall consider only the case $N = p$, since if one wanted to make $N > p$ weighings, he would normally choose N to be a multiple of four because the gain in efficiency by using optimum designs is rather large for small N. In general more than p weighings would be required because σ^2 is not usually known. Thus several additional weighings may be made in order to obtain several degrees of freedom for estimating σ^2.

When H_N does not exist we have already defined the best design as one which minimizes the confidence region for estimating the weights; that is equivalent to maximizing $|a_{ij}|$ or minimizing $|a^{ij}|$. There may be several designs with the same minimum, but we shall not give all of them. Thus when $p = 3$ the best designs are

$$X = \begin{Vmatrix} + & + & 0 \\ + & - & + \\ - & + & + \end{Vmatrix}, \quad \begin{Vmatrix} + & + & + \\ + & - & + \\ - & + & + \end{Vmatrix} \text{ and } \begin{Vmatrix} + & + & - \\ + & - & + \\ - & + & + \end{Vmatrix}$$

all of which have $A = 16$ (which is considerably smaller than the value 27 that A would have if an optimum design existed). Using the notation

$$(a^{ii}) = (a^{11}, a^{22}, \cdots, a^{pp}),$$

the first of the above designs for $p = 3$ gives

$$(a^{ii}) = (\tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{2})$$

while the second and third give

$$(a^{ii}) = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}).$$

For $N = p = 5$, two best designs are

$$X = \begin{Vmatrix} + & + & + & + & - \\ + & + & + & - & + \\ + & + & - & + & + \\ + & - & + & + & + \\ - & + & + & + & + \end{Vmatrix} \text{ and } \begin{Vmatrix} + & - & - & - & - \\ + & + & + & - & - \\ + & - & + & - & + \\ + & - & + & + & - \\ + & + & - & + & + \end{Vmatrix}$$

both of which have

$$A = 3^2 2^8 \text{ and } (a^{ii}) = (2/9, 2/9, 2/9, 2/9, 2/9)$$

For $N = p = 6$, a best design is

$$X = \begin{Vmatrix} + & - & - & - & - & - \\ + & - & - & - & + & + \\ + & - & - & + & + & - \\ + & - & + & + & - & + \\ + & + & + & - & + & - \\ + & + & - & + & - & + \end{Vmatrix}$$

which has

$$A = 5^2 2^{10} \text{ and all } a^{ii} = 1/5.$$

For $N = p = 7$, a best design is

$$X = \begin{Vmatrix} + & - & - & - & - & - & - \\ + & + & + & + & - & - & - \\ + & + & - & - & - & + & + \\ + & + & - & - & + & + & - \\ + & - & - & + & + & - & + \\ + & - & + & + & - & + & - \\ + & - & + & - & + & - & + \end{Vmatrix}$$

which has

$$A = 2^{12}3^4 \text{ and all } a^{ii} = 1/6.$$

These designs were constructed by a method due to Williamson [8] which will be described in sections 5 and 7. It is interesting to note that no minor of an H_8 is a best design for $N = p = 7$, for any minor of an H_8 gives $A = 2^{18} < 2^{12}3^4$ and all $a^{ii} = \frac{1}{4}$.

5. Spring Balance Problem. $N = p = 4k + 3$. When $N = p$ and $N \equiv 3 \pmod 4$ the best possible design for the spring balance case is determined by H_{N+1} if it exists. Let K_{N+1} denote a matrix formed from H_{N+1} by adding or subtracting the elements of the first row of H_{N+1} from the corresponding elements of the other rows in such a way as to make the first element of each of the remaining rows zero. Obviously

$$| K_{N+1} | = \pm | H_{N+1} |.$$

and excepting the first row, the elements of K_{N+1} are 0 and ± 2 with the signs of the non-zero elements the same for elements in the same row. Let L_N be the matrix obtained by omitting the first row and column of K_{N+1}, by changing all non-zero elements to $+1$, and by permitting two rows if necessary to make the determinant of L_N positive. Then

$$| H_{N+1} | = 2^N | L_N |$$

and it is clear that, given L_N, one could reverse the procedure and determine an H_{N+1}. In the same manner, there is a correspondence in general between square matrices with elements ± 1 and square matrices of one less order with elements 0 and 1. The ratio of the values of corresponding determinants is always 2^N if their determinants do not vanish; hence the 0, 1 determinant will have its maximum value when its corresponding $+1$ determinant has a maximum value. Thus $| L_N |$ is the maximum value possible for a determinant of 0's and 1's of order N, and the value is

$$| L_N | = (N + 1)^{\frac{1}{2}(N+1)}/2^N. \tag{7}$$

The variance factors are

$$a^{ii} = 4N/(N + 1)^2.$$

We knew in advance, of course, that the a^{ii} would be greater than $1/N$ since an optimum design cannot exist unless the design matrix has its elements equal to ± 1, and we must here restrict the design to have only 0 and 1 as its elements. Since L_N is a best possible design for the spring balance case, it follows that designs for the spring balance problem can be no more than about $\frac{1}{4}$ as efficient as designs for the chemical balance problem.

6. Spring Balance $N > p$. When $N > p$ the device used in the chemical balance case to get optimum designs cannot be used. For if we select p columns from an L_N we may get rows of zeros which would waste weighing operations. A different approach is necessary and a clue is given by the designs L_N. In these designs p is odd and the objects are weighed $\frac{1}{2}(p + 1)$ at a time in each weighing operation. We shall show in general that objects should be weighed $\frac{1}{2}(p + 1)$ at a time when p is odd, and we shall obtain a corresponding result for p even.

Let P_r be a matrix whose rows are all the arrangements of r ones and $p - r$ zeros $(0 \leq r \leq p)$. (The symbol should also have a subscript p but that is omitted because any specific value for p will always be clear from the context.) The matrix will have p columns and $\binom{p}{r}$ rows. Let Q be a matrix made up of matrices P_r arranged in vertical order. Let n_r be the number of times P_r is used in constructing Q. Q is a weighing design for p objects and

$$N = \sum_r n_r \binom{p}{r}$$

weighing operations. The matrix $Q'Q$ will have diagonal elements

$$a = \sum n_r \binom{p-1}{r-1} \tag{9}$$

and non-diagonal elements

$$b = \sum n_r \binom{p-2}{r-2}. \tag{10}$$

The determinant of $Q'Q$ is

$$A = (a - b)^{p-1}[a + (p - 1)b$$

and we may write A in the form

$$A = c^{p-1}d$$

where

$$c = a - b, \text{ and } d = a + (p - 1)b. \tag{11}$$

We shall prove the following theorem:

If $p = 2k - 1$ where k is a positive integer, and if N contains the factor $\binom{p}{k}$, then A will be maximized when $n_k = N \Big/ \binom{p}{k}$ and all other $n_r = 0$.

We shall demonstrate this statement by showing that if any n_s $(s \neq k)$ is decreased and n_k is increased in such a way that N remains unchanged, then A will be increased. Let n_s be reduced by an amount m so chosen that

$$m' = m\binom{p}{s}\Big/\binom{p}{k}$$

is an integer; we may then increase n_k by m' leaving N unchanged. It is readily found that these changes in n_s and n_k produce the following changes in c and d:

$$\Delta c = m\binom{p}{s}\frac{(k-s)(k-s-1)}{p(p-1)}$$

$$\Delta d = m\binom{p}{s}\frac{(k-s)(k+s)}{p}$$

both of which are positive on zero when $s < k$ and A is necessarily increased.

When $s > k$, Δc is positive but Δd is negative and it must be shown that the net effect of these changes is to increase A, we shall assume now that $n_r = 0$ when $r < k$.

$$\Delta A = (c + \Delta c)^{p-1}(d + \Delta d) - c^{p-1}d < [c^{p-1} + (p-1)c^{p-2}\Delta c](d + \Delta d)$$
$$- c^{p-1}d < c^{p-2}[c\Delta d + (p-1)d\Delta c + (p-1)\Delta c\Delta d]$$

where in the second line we have omitted terms in Δc of higher order than the first. These terms are all positive since all their fctors are positive. The bracket in the last expression on substituting from (9), (10), and (11), may be reduced to

$$m\binom{p}{s}\Bigg[(k-s)\sum_{r\geq k} n_r\left\{\binom{p-2}{r-1} + (k-s-1)\binom{p-2}{r-1}\right.$$
$$\left. + \frac{m}{p^2}\binom{p}{s}(k-s)^2(k+s)(k-s-1)\right],$$

and then to

$$m\binom{p}{s}\Bigg[\sum_{r\geq k} n_r\binom{p-1}{r-1}(k-s)\frac{r(k-s) + (k+s+1-2r)}{p-1}$$
$$+ \frac{m}{p^2}\binom{p}{s}(k-s)^2(k+s)(k-s-1)\Bigg].$$

Each term of the sum in the bracket is greater than or equal to zero when $k > 1$, $r \geq k$, $s > k$ since the fraction is readily seen to be negative or zero under these circumstances. The fraction vanishes only when $k = 2$, $r = k$, $s = k + 1$. The other term in the bracket is negative but it is dominated by the term in the sum for which $r = s$, as may be shown as follows: The two terms in question may be written

$$n_s \binom{p-1}{s-1}(k-s)\frac{s(k-s)+k-s+1}{p-1} + \frac{m}{p}\binom{p-1}{s-1}\frac{(k-s)^2(k+s)(k-s-1)}{s}$$

and since $n_s \geq m$, this expression is less than or equal to

$$m\binom{p-1}{s-1}(k-s)\left[\frac{s(k-s)+k-s+1}{p-1} + \frac{(k^2-s^2)(k-s-1)}{ps}\right]$$

which is positive for $s > k$ since the bracket is negative as may be seen by factoring out $\frac{1}{p(p-1)s}$ and putting the result in the form

$$(k-s+1)(s^2+(p-1)k^2) - pk(p-s) + (2s+1)(k-s).$$

Thus ΔA has been shown to be positive and the theorem is proved.

The above argument has shown that P_k or repetitions of P_k give more efficient designs than any other combination of the designs $P_1, P_2, \cdots, P_k$. The question now arises as to whether these are the best possible designs. We shall show that they are by considering the matrices L_N of section 5 which are known to give the greatest efficiency in the spring balance case. Let $p = 4t + 3$ and let $N = \binom{p}{2t+2}$, and suppose L_p exists (i.e. H_{p+1} exists). Using P_{2t} as the weighing design we find the a_{ii} are

$$a_{ii} = 2N(t+1)/p$$

$$a_{ij} = N(t+1)/p \qquad i \neq j.$$

A single application of the design L_p gives

$$a_{ii} = 2(t+1)$$

$$a_{ij} = t+1 \qquad i \neq j$$

and N/p repetitions of L_p gives an a_{ij} matrix with elements equal to N/p times the given elements for one application of the design. The two designs are therefore equivalent and P_{2t} is a best design.

The variance factors for repetitions of the design P_k are

$$a^{ii} = \frac{4}{N}\frac{p^2}{(p-1)^2} \qquad N \equiv 0 \text{ Mod } \binom{p}{k} \tag{12}$$

and these are minimum variance factors[2] as may be shown by an argument entirely analogous to that used in proving the theorem. Thus P_k is a design which not only minimizes the confidence region for estimating the weights, but also minimizes the individual variance factors.

Efficient sub-matrices of the P_k have not been studied except for small p, but we may point out that square sub-matrices of order p which are as efficient as P_k do not exist unless H_{p-1} exists, for by the argument of section 4, it is possible to construct H_{p+1} from such sub-matrices. Hence we cannot obtain variance factors as small as those given by equation (12) when $N = p$ unless H_{p+1} exists.

The situation here is analogous to that in the chemical balance case. By a proper selection of N we can obtain a design with the maximum possible efficiency for any odd value of p. But here we are much more restricted in our choice of N. In the chemical balance case N could be any multiple of 4 for which an H_N existed; in the present case N must be a multiple of p even in the most favorable instance ($p = 4t + 3$), and for some values of p it may be necessary that N be a multiple of $\binom{p}{\frac{1}{2}(p+1)}$.

We now turn to the case in which p is even. The theorem corresponding to the one given at the beginning of this section is:

If $p = 2k$ where k is a positive integer, and if N contains the factor $\binom{p+1}{k+1}$, then A will be maximimized when

$$n_k = n_{k+1} = N \bigg/ \binom{p+1}{k+1}$$

and all other $n_r = 0$.

We shall not prove this theorem in detail. By arguments analogous to those used earlier, it may be shown that A is increased when either n_s ($s < k$) is decreased and n_k is increased, or n_s ($s > k + 1$) is decreased and n_{k+1} is increased with N fixed. This done, we may put all $n_r = 0$ except n_k and n_{k+1} and then maximize A with respect to these two variables subject to the condition that

$$n_k \binom{p}{k} + n_{k+1} \binom{p}{k+1} = N.$$

The values of n_k and n_{k+1} which maximize A may be found by treating them as continuous variables and using the calculus.

The variance factors for these designs are

$$a^{ii} = \frac{4}{N} \frac{p}{p+2} \qquad N \equiv 0 \bmod \binom{p+1}{k+1}, \tag{13}$$

[2] The author is indebted to a referee for suggesting this property of the design, and for several other valuable suggestions and corrections to the paper.

but these are not minimum variance factors. In fact one can obtain smaller variance factors than these by using only P_k in the design (omitting P_{k+1} entirely). In this case

$$(14) \qquad a^{ii} = \frac{4}{N}\,\frac{(p-1)^2+1}{p^2} \qquad N \equiv 0 \bmod \binom{p}{k}$$

and

$$\frac{(p-1)^2+1}{p^2} < \frac{p}{p+2} \quad \text{when} \quad p > 2.$$

We have not found explicitly the design which minimizes the variance factors for p even, but it appears that the design would be made up largely from P_k with a small proportion of the design devoted to P_{k+1}. Thus (14) is very nearly the minimum possible variance factor.

7. Spring Balance Designs for Small p. When $p = 2$, each object may be weighed r times by itself, and the two objects may be weighed together s times to give

$$\| a_{ij} \| = \left\| \begin{matrix} r+s & s \\ s & r+s \end{matrix} \right\|$$

and if A is maximized subject to $2r + s = N$ we find

$$r = s = N/3$$

$$a^{ii} = 2/N$$

provided N is a multiple of 3. The most efficient basic design is therefore

$$X = \left\| \begin{matrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{matrix} \right\|$$

in accordance with the previous section. When N is not a multiple of 3 the best design is obtained by using the first row of X for the odd weighing when $N = 3t + 1$, and the last two rows when $N = 3t + 2$.

The case $p = 2$ is notable in that there is almost nothing to be gained by weighing the objects in combination. For the variance factors $2/N$ would be obtained by simply weighing each object separately $N/2$ times. The advantage of weighing in combination is only that square confidence regions in the b_1, b_2 space are replaced by ellipses with somewhat smaller area. If $a^{ii} = (r + s)/(r^2 + 2rs)$ is minimized subject to $2r + s = N$, we find

$$r = N(3 - \sqrt{3})/3, \; a^{ii} = 1.866/N$$

so that the a^{ii} are reduced slightly from $2/N$ but at the expense of increasing the area of the elliptical confidence regions.

For $p = 3$ the most efficient design when $N = 3$ is

$$X = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix}$$

as given by L_3 or P_2. It is easily shown that for $N > 3$, the most efficient design is given by repeating X even when $N \not\equiv 0 \pmod 3$. Thus for $N = 4$ we would repeat one row of X, for $N = 5$ we would repeat two rows of X, and so forth. The variance factors are

$$\begin{aligned} a^{ii} &= \frac{9}{4N} & N &= 3t \\ &= \frac{9(N+1)}{4(N-1)(N+2)} & N &= 3t+1 \\ &= \frac{9(N+1)}{4(N-2)(N+1)} & N &= 3t+2. \end{aligned}$$

For $p = 4$ we may attempt to find by trial and error a sub-matrix of the design given by using P_2 once and P_3 once, but this would be a tedious process and the labor would soon become prohibitive for larger values of p. Hence another method must be found for obtaining the best designs when $N = p$ except when L_p exists. A method is provided by Williamson [8]. Let D_p be the best design for $N = p$. Williamson shows that when $p < 7$, D_{p-1} is a minor of D_p, hence D_p may be found by adding a row and column of variables to D_{p-1} and expanding the determinant of the result by the bordered expansion. For small values of p it is easy to determine by inspection what values the variables should have in order to maximize the resulting expansion. Williamson determined D_4 and D_5 by this method.

There are two types of D_4 which give a maximum value of $A = 9$

$$D_4 = \begin{Vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{Vmatrix} \text{ and } \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{Vmatrix}.$$

The variance factors are all 7/9 for the first of these, and for the second

$$(a^{ii}) = (7/9, 7/9, 7/9, 4/9).$$

When $N = 5$, $p = 4$, there are a number of designs which give a maximum A of 19. None of these however has all a^{ii} equal, and we shall give only one example:

$$X = \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{Vmatrix}$$

which has

$$(a^{ii}) = \left(\frac{12}{19}, \frac{12}{19}, \frac{13}{19}, \frac{8}{19}\right).$$

When $N = 6$, there appears to be no design superior to P_2. It has variance factors all equal to 5/12 and $A = 48$,—a very large gain in efficiency over $N = 5$ at the expense of one additional observation.

When $p = 5$ there are three types of D_5 which give A a maximum value of 25, none of which has all variance factors equal. An example is

$$D_5 = \begin{Vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{Vmatrix}$$

with

$$(a^{ii}) = \left(\frac{19}{25}, \frac{19}{25}, \frac{16}{25}, \frac{11}{25}, \frac{16}{25}\right).$$

For $p = 6$, an example of a D_6 with all a^{ii} equal which maximizes A is

$$D_6 = \begin{Vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{Vmatrix}$$

with $A = 81$ and $a^{ii} = 17/27$. This example was constructed by the bordered expansion method from D_5 and it turns out to be a sub-matrix of P_3. It is not as efficient as P_3, however, since substitution of $N = p = 6$ in equation (14) gives $a^{ii} = 13/27$. Hence we have shown that there does not exist a minor of P_3 (for $p = 6$) of order 6 which is as efficient as P_3 itself.

For $p = 7$, there is a most efficient design given by L_7.

$$L_7 = \begin{Vmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{Vmatrix}$$

with $A = 2^{10}$ and all $a^{ii} = 7/16$.

D_p for $p = 8$, 9, and 10 could presumably be constructed from L_7 in the same way and the designs for $p = 4$, 5, and 6 were constructed from L_3, but the computations become very tedious for these larger values of p.

The designs given in section 3 were constructed from the above designs by the method described in section 4.

8. Bias in Measuring Devices. In some kinds of experiments it may be necessary to estimate a bias in the measuring scale in order to estimate the measures of the objects. Such a bias may simply be regarded as an additional object to be measured except that it is an object which must be included in all the measuring operations. In the chemical balance case the bias presents no difficulty, for if an H_N exists, then there exists an H_N with a column whose elements are all $+1$. Such an H_N may be constructed from any given H_N by merely changing the signs of all elements in rows which begin with a minus sign. The result will be an H_N with $+1$'s in the first column and that column may be assigned to the bias. We note that the gain in efficiency by measuring objects in combinations is even greater in the case of a biased measuring scale than when there is no bias. For if the objects were measured separately, their measures would be estimated by the difference of two scale readings and would have variance $2\sigma^2$; hence the variance factors a^{ii} are to be compared with 2 (rather than 1) in the case of bias.

In the spring balance case, the additional restriction that all the elements of one column be one necessarily reduces the efficiency of the designs in the sense that the variance factors for p objects and a bias will be larger than the variance factors for $p + 1$ objects without bias. When the measures of p objects and a bias are to be estimated from $N = p + 1$ measuring operations, a best design may be obtained by adding a row of zeros and a column of ones (in that order) to the best design for $N = p$ without bias. This can be seen by recalling that there are two determinantal exressions for the volume of a simplex with one vertex at the origin in a Euclidean p space. (A simplex (Sommerville, [9]) is a polytope with $p + 1$ vertices bounded by $p + 1$ $(p - 1)$-dimensional hyperplanes.) The determinant of the best design for $N = p$ (without bias) is proportional to the volume of the largest simplex with one vertex at the origin and the other vertices restricted to be selected from the vertices of the unit cube. A determinant of order $p + 1$ with a column of ones and the other elements zero or one also gives the volume of a simplex with vertices selected from the vertices of the unit cube. Hence the two determinants (one of order p and one of order $p + 1$) must have the same maximum value, and as one of the vertices may be selected arbitrarily in the case of bias, we may select the origin.

In general, for $N \geq p$, similar geometrical reasoning will show that the best designs for the spring balance problem in the case of bias are easily constructed from Hadamard matrices as described in the following theorem:

If X is a best design for the chemical balance problem in the case of bias and if X contains a row of $+1$'s, then a best design for the spring balance problem in the case of bias is given by replacing the -1's in X by zeros.

We have seen that the best design in the chemical balance case is obtained from a Hadamard matrix with a column of $+1$'s. Obviously the matrix may be also made to contain a row of $+1$'s by changing the signs of certain columns. The design X consists of the column of ones together with any other p columns. The determinant of $X'X$ is $1/p!^2$ times the sum of squares of the volumes of a set of simplexes in a p space. There are $\binom{N}{p+1}$ of these simplexes determined by the different combinations of the rows of X taken $p + 1$ at a time, and the coordinates of their vertices are the last p elements of the rows of X. The vertices are therefore selected from the vertices of a cube in the p space which has its edges parallel to the coordinate axes, the origin at its center, and the lengths of its edges equal to two. Since X is a best design, the vertices are selected so as to maximize the sum of squares of the volumes of the simplexes. Now in the spring balance case we must maximize the sum of squares of the volumes of a set of simplexes which have their vertices selected from the vertices of the unit cube. Obviously this may be done by selecting vertices corresponding to the selection given by X. Thus it is necessary only to set up a correspondence beteen the vertices of the two cubes. Since X contains the vertex $(1, 1, 1, 1, \cdots, 1)$ which is common to both cubes, the natural correspondence which identifies a vertex such as $(1, -1, -1, 1, -1, 1, \cdots)$ with $(1, 0, 0, 1, 0, 1, \cdots)$ may be used.

The variance factors for these spring balance designs are $4/N$ (for any $p \leq N$) when N is a multiple of four and H_N exists; when N is not a multiple of four and modifications of H_N as described in section 3 are used, the variance factors will differ from $4/N$ by terms of order $1/N^2$.

9. Addendum. After this paper was written, the paper of Plackett and Burman on "The Design of Multifactorial Experiments" appeared in *Biometrika*. Volume 33 (1946), pages 305–325. A part of this paper discusses Hadamard matrices much more completely than we have done in section 3. In particular Plackett and Burman have constructed all Hadamard matrices of order less than or equal to 100 (excepting 92).

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