NOTE ON DIFFERENTIATION UNDER THE EXPECTATION SIGN IN THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

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Let z be any chance variable and $z_1, z_2, z_3, \cdots$ a sequence of independent chance variables, each with the same distribution as z. Let $Z_N = z_1 + z_2 + \cdots + z_N$. Let $\phi(t) = Ee^{zt}$ for all complex t for which the latter exists. Let $S_1, S_2, \cdots$ be a sequence of mutually exclusive events such that S_j depends only on $z_1, z_2, \cdots, z_j$, and $\sum_{j=1}^{\infty} P(S_j) = 1$. Let the chance variable n be defined as $n = j$ when S_j occurs. Blackwell and Girshick [1], generalizing a result of Wald [2], showed that if there is a positive constant M such that

$$(1) \qquad |Z_N| < M \text{ when } n > N$$

then the identity

$$(2) \qquad E\{e^{Z_n t}(\phi(t))^{-n}\} = 1$$

holds for all complex t for which $\phi(t)$ exists and $|\phi(t)| \geq 1$. Wald [3] established conditions, including the existence of $\phi(t)$ for all real t, under which (2) may be differentiated under the expectation sign an unlimited number of times.

Without assuming the existence of $\phi(t)$ for a real t-interval the following result holds: *If (1) is true and if $E(z^k)$ and $E(n^k)$ are both finite, k a positive integer, then*

$$(3) \qquad E\left\{\frac{d^k}{ds^k}\left[e^{Z_n is}(\phi(is))^{-n}\right]_{s=0}\right\} = 0$$

where $i = \sqrt{-1}$ and s is real. Certain identities, obtained by differentiating (2) and putting $t = 0$, can also be obtained from (3). For example, if $En = 0$, and if En^2 and Ez^2 both exist then $EZ_n^2 = Ez^2En$.

Let $P_N = P(n \leq N)$; $p_N = P(n = N)$. Let $H(j, Z_j)$ and $F(N, Z_N)$ be the conditional cumulatives of Z_j and Z_N for $n = j$ and $n > N$ respectively. Now (2) was derived by Wald [2], p. 285, from a relation, valid whenever $\phi(t)$ exists, which in the present notation becomes

$$(4) \qquad \sum_{j=1}^{N} p_j \int_{-\infty}^{\infty} (\phi(t))^{-j} e^{Z_j t}\, dH(j, Z_j) + \frac{(1 - P_N)}{(\phi(t))^N} \int_{-\infty}^{\infty} e^{Z_N t}\, dF(N, Z_N) = 1.$$

Examination of Wald's derivation of (4) shows it to be valid under the present hypotheses. Now the finiteness of $E(z^k)$ clearly implies that of $E(Z_j^k \mid n = j)$. Also, since $F(N, Z_N)$ is constant outside the interval $[-M, M]$, the integral $\int_{-\infty}^{\infty} Z_N^k\, dF(N, Z_N)$ is finite. Hence we may set $t = is$ in (4) and differentiate

k times, obtaining for all real s

$$(5)\quad \sum_{j=1}^{N} p_j \int_{-\infty}^{\infty} \frac{d^k}{ds^k}[(\phi(is))^{-j} e^{Z_j i s}] \, dH(j, Z_j) + (1 - P_N) \sum_{r=0}^{k} \binom{k}{r} \frac{d^r}{ds^r} [(\phi(is))^{-N}] \cdot \int_{-\infty}^{\infty} (iZ_N)^{k-r} e^{Z_N is} \, dF(N, Z_N) = 0.$$

The derivatives of $(\phi(is))^{-N}$ are sums of terms of the form $Q(N) \cdot (\phi(is))^{-N-r}$ times terms independent of N, where $Q(N)$ is a polynomial in N of degree $\leq k$. For any $r \leq k$,

$$\lim_{N\to\infty} | (1 - P_N)N^r | = \lim_{N\to\infty} \left| N^r \sum_{j=N+1}^{\infty} p_j \right| \leq \lim_{N\to\infty} \left| \sum_{j=N+1}^{\infty} j^k p_j \right| = 0,$$

since En^k is finite. Hence $\lim (1 - P_N)Q(N) = 0$. Because of (1) the integrals in the second term of (5) are bounded as $N \to \infty$. Now set $s = 0$ in (5) and then let $N \to \infty$. Since $\phi(0) = 1$, the second term of (5) approaches 0 and the limit of the first term is just the left side of (3).

For the case of a Wald sequential process, Stein [4] has shown that all moments of n are finite. In this case (3) holds whenever Ez^k is finite.

REFERENCES

[1] David Blackwell and M. A. Girshick, "On functions of sequences of independent chance vectors, with applications to the problem of the random walk in k dimensions," *Annals of Math. Stat.*, Vol. 17 (1946), p. 310.
[2] Abraham Wald, "On cumulative sums of random variables," *Annals of Math. Stat.*, Vol. 15 (1944), p. 283.
[3] Abraham Wald, "Differentiation under the expectation sign in the fundamental identity of sequential analysis," *Annals of Math. Stat.*, Vol. 17 (1946), p. 493.
[4] Charles Stein, "A note on cumulative sums," *Annals of Math. Stat.*, Vol. 17 (1946), p. 498.

A UNIQUENESS THEOREM FOR UNBIASED SEQUENTIAL BINOMIAL ESTIMATION

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In a recent note [1], J. Wolfowitz extended some of the results of a paper by Girshick, Mosteller and Savage [2] on sequential binomial estimation. The present note carries one of Wolfowitz's ideas somewhat further. The nomenclature of [1] and [2] will be used freely. The concept of "doubly simple region" introduced in [1] and assumed there only in the hypothesis of Theorem 3, will here be shown to be unnecessarily restrictive. In so doing, we find that sim-

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