# DISTRIBUTION OF A ROOT OF A DETERMINANTAL EQUATION

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**1. Summary.** S. N. Roy [2] obtained in 1943 the distribution of the maximum, minimum and any intermediate one of the orots of certain determinantal equations based on covariance matrices of two samples on the null hypothesis of equal covariance matrices in the two populations. The present paper gives a different method of working out the distribution of any of these roots under the same hypothesis. The distribution of the largest, smallest and any intermediate root when the roots are specified by their position in a monotonic arrangement has been derived for $p = 2, 3, 4$, and $5$ by the new method. The method is applicable for obtaining the distribution of the roots of an equation of any order, when the distributions of the roots of lower order equations have been worked out.

**2. Introduction.** If $x = || x_{ij} ||$ and $x^* = || x^*_{ij} ||$ are two $p$-variate sample matrices with $n_1$ and $n_2$ degrees of freedom respectively, and $S = xx'/n_1$ and $S^* = x^*x^{*\prime}/n_2$ are the covariance matrices which under the null hypothesis are independent estimates of the same population covariance matrix, then the joint distribution of the roots of the determinantal equation $|A - \theta(A + B)| = 0$ where $A = n_1 S$ and $B = n_2 S^*$ has been obtained by Hsu [1] in 1939. The distribution densty is

$$R(l, \mu, \nu) = \frac{\pi^{l/2} \cdot \prod_{i=1}^{l} \Gamma\left(\dfrac{l + \mu + \nu + i - 2}{2}\right)}{\prod_{i=1}^{l} \Gamma\left(\dfrac{\mu + i - 1}{2}\right) \Gamma\left(\dfrac{\nu + i - 1}{2}\right) \Gamma(i/2)} \tag{1}$$
$$\cdot \prod_{i=1}^{l} \theta_i^{(\mu/2)-1} \prod_{i=1}^{l} (1 - \theta_i)^{(\nu/2)-1} \prod_{i<j} (\theta_i - \theta_j),$$

$$(0 \leqq \theta_l \leqq \theta_{l-1} \leqq \cdots \theta_1 \leqq 1),$$

where $l = \min.\ (p, n_1)$, $\mu = | p - n_1 | + 1$, and $\nu = n_2 - p + 1$.

This formula also gives the joint distribution of the squares of canonical correlations on the null hypothesis, that the two sets of variates are independent [1]. If

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1N} \\ x_{21} & \cdots & x_{2N} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ x_{p1} & \cdots & x_{pN} \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} w_{11} & \cdots & w_{1N} \\ w_{21} & \cdots & w_{2N} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ w_{q1} & \cdots & w_{qN} \end{bmatrix}$$

are the observations on the two sets of canonical variates and the $x$'s are normally distributed, independently of the $w$'s, then the equation for the canonical roots is $| V_{xw}V_{ww}^{-1}V_{wx} - \theta V_{xx} | = 0$, where $\theta_i = r_i^2$ and $V_{xw} = XW$ etc. .... It is observed that $V_{xw}V_{ww}^{-1}V_{wx}$ is like $A$ with $n_1 = q$ and $V_{xx} - V_{xw}V_{ww}^{-1}V_{wx}$ is like $B$ with $n_2 = N - q - 1$ and the above equation is reduced to the form $| A - \theta(A + B) | = 0$. It is under this condition that $R(l, \mu, \nu)$ gives the joint distribution density of $r_1^2, r_2^2, \cdots, r_l^2$, where $l = \min.(p, q)$, $\mu = | p - q | + 1$, and $\nu = N - p - q$.

**3. Notation and preliminaries.**

(a). Let

$$\prod_{i<j} (\theta_i - \theta_j) = \{1, 2, 3, \cdots, l\}.$$

It is known that the value of the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_l \\ \theta_1^2 & \theta_2^2 & \theta_3^2 & \cdots & \theta_l^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \theta_1^{l-1} & \theta_2^{l-1} & \theta_3^{l-1} & \cdots & \theta_l^{l-1} \end{vmatrix}$$

is equal to $\prod_{i>j} (\theta_i - \theta_j) = (-1)^l\{1, 2, 3, \cdots, l\}$.

Then

$$\begin{vmatrix} 1 & 1 & 1 \\ \theta_1 & \theta_2 & \theta_3 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \end{vmatrix} = (\theta_2 - \theta_1)(\theta_3 - \theta_2)(\theta_3 - \theta_1) = -\{1, 2, 3\},$$

but the determinant can also, by expansion in minors of the first row, be expressed as

$$-[\theta_1\theta_2\{1, 2\} + \theta_2\theta_3\{2, 3\} + \theta_3\theta_1\{3, 1\}]$$

where

$$\theta_1 - \theta_2 = \{1, 2\}.$$

Hence

$$\{1, 2, 3\} = \theta_1\theta_2\{1, 2\} + \theta_3\theta_1\{3, 1\} + \theta_2\theta_3\{2, 3\}. \tag{2}$$

Similarly

$$\begin{aligned} \{1, 2, 3, 4\} = \theta_1\theta_2\theta_3\{1, 2, 3\} - \theta_4\theta_1\theta_2\{4, 1, 2\} \\ + \theta_3\theta_4\theta_1\{3, 4, 1\} - \theta_2\theta_3\theta_4\{2, 3, 4\}, \end{aligned} \tag{3}$$

and

(4) $$\{1, 2, 3, 4, 5\} = \theta_1\theta_2\theta_3\theta_4\{1, 2, 3, 4\} + \theta_5\theta_1\theta_2\theta_3\{5, 1, 2, 3\} + \theta_4\theta_5\theta_1\theta_2\{4, 5, 1, 2\} + \theta_3\theta_4\theta_5\theta_1\{3, 4, 5, 1\} + \theta_2\theta_3\theta_4\theta_5\{2, 3, 4, 5\}.$$

It is seen that in the successive terms the $\theta$'s are present in a decreasing order.

(b). Let

$$(a, b; m, n) = y^m(1 - y)^n \Big|_a^b = b^m(1 - b)^n - a^m(1 - a)^n,$$

and

$$(a, 1, b; m, n) = \int_a^b y^m(1 - y)^n \, dy;$$

then

(4, a) $$(a, 1, b; m + 1, n) = -\frac{(a, b; m + 1, n + 1)}{m + n + 2} + \frac{m + 1}{m + n + 2}(a, 1, b; m, n),$$

by a combination of the transformations obtained by partial integration and by breaking up $(1 - y)^{n+1}$ into $(1 - y)^n - y(1 - y)^n$.

(c) Let

$$(a, 2, 1, b; m, n) = \int_{a<\theta_2<\theta_1<b} (\theta_1\theta_2)^m(1 - \theta_1)^n(1 - \theta_2)^n\{1, 2\}\, d\theta_1\, d\theta_2$$

$$(a, 2, b, 1, c; m, n) = \int_{a<\theta_2<b<\theta_1<c} (\theta_1\theta_2)^m(1 - \theta_1)^n(1 - \theta_2)^n\{1, 2\}\, d\theta_1\, d\theta_2,$$

and

$$(a, 3, b, 2, c, 1, d; m + 1, n) = \int_{a<\theta_3<b<\theta_2<c<\theta_1<d} (\theta_1\theta_2\theta_3)^{m+1}(1 - \theta_1)^n(1 - \theta_2)^n(1 - \theta_3)^n\{1, 2, 3\}\, d\theta_1\, d\theta_2\, d\theta_3.$$

(d) Let

$$T_a^{b;m,n} g(y) = \int_a^b y^m(1 - y)^n g(y)\, dy,$$

then

$$T_a^{b;m,n}(0, y; k, l) = (a, 1, b; m + k; n + l), \qquad (k > 0)$$

and

$$T_a^{b;m,n}(b, 1, c; k, l) = (a, 1, b; m, n)(b, 1, c; k, l).$$

With these preliminaries we proceed to derive the distribution of the roots.

**4. Distribution of the largest root.** Let us suppose that the roots are arranged in decreasing order such that for $l$ roots we have

$$0 < \theta_l < \theta_{l-1} < \theta_{l-2}, \cdots, < \theta_2 < \theta_1 < 1.$$

If the distribution density $R(l, \mu, \nu)$ given by (1) be expressed as

$$R(l, m, n) = C(l, m, n) \prod_{i=1}^{l} \theta_i^m \prod_{i=1}^{l} (1 - \theta_i)^n \prod_{i<j} \theta_i - \theta_j),$$

then the distribution of the largest root in the general case would be given by

$$Pr(\theta_1 \leqq x) = C(l, m, n)(0, l, l - 1, \cdots, 2, 1, x; m, n).$$

Now we shall derive the distribution of the largest root for $l = 2, 3, 4,$ and 5.

(a) $l = 2$.

$$Pr(\theta_1 \leqq x) = C(2; m, n)(0, 2, 1, x; m, n).$$

$$\begin{aligned}(0, 2, 1, x; m, n) &= \int_{0<\theta_2<\theta_1<x} (\theta_1\theta_2)^m(1 - \theta_1)^n(1 - \theta_2)^n\{1, 2\}\, d\theta_1\, d\theta_2 \\ &= \int_{0<\theta_2<\theta_1<x} \theta_2^m(1 - \theta_2)^n\theta_1^m(1 - \theta_1)^n\{1, 2\}\, d\theta_1\, d\theta_2 \\ &= \int_{0<\theta_2<\theta_1<x} \theta_2^m(1 - \theta_2)^n\theta_1^{m+1}(1 - \theta_1)^n\, d\theta_1\, d\theta_2 \\ &\quad - \int_{0<\theta_1<\theta_2<x} \theta_2^m(1 - \theta_2)^n\theta_1^{m+1}(1 - \theta_1)^n\, d\theta_1\, d\theta_2 .\end{aligned}$$

The limits in the successive integrals are to be so adjusted as to keep the integrand same. Then using the notation given in section 3($d$) and equation (4, $a$).

$$(5)\quad (0, 2, 1, x; m, n) = T_0^{x;m,n}(y, 1, x; m + 1, n) - T_0^{x;m,n}(0, 1, y; m + 1, n)$$

or

$$\begin{aligned}(0, 2, 1, x; m, n) = T_0^{x;m,n}\Big[&-\frac{(y, x; m + 1, n + 1)}{m + n + 2} + \frac{m + 1}{m + n + 2}(y, 1, x; m, n) \\ &+ \frac{(0, y; m + 1, n + 1)}{m + n + 2} - \frac{(m + 1)}{m + n + 2}(0, 1, y; m, n)\Big].\end{aligned}$$

Now by a change in the order of integration,

$$T_0^{x;m,n}[(0, 1, y; m, n) - (y, 1, x; m, n)] = 0.$$

Therefore

$$\begin{aligned}(m + n + 2)(0, 2, 1, x; m, n) &= T_0^{x;m,n}[2(0, y; m + 1, n + 1) \\ &\quad - (0, y; m + 1, n + 1)] \\ &= 2(0, 1, x; 2m + 1, 2n + 1) \\ &\quad - (0, x; m + 1, n + 1)(0, 1, x; m, n).\end{aligned}$$

Hence

$Pr(\theta_1 \leqq x)$

$$= \frac{C(2, m, n)}{m + n + 2} [2(0, 1, x; 2m + 1, 2n + 1) - (0, x; m + 1, n + 1)(0, 1, x; m, n)]$$

$$= C(2, m, n) \left\{ \frac{2}{m + n + 2} \int_0^x y^{2m+1} (1 - y)^{2n+1} \, dy - \frac{x^{m+1}(1 - x)^{n+1}}{m + n + 2} \int_0^x y^m (1 - y)^n \, dy \right\}.$$

(b) $l = 3$. For this case we need certain results for $l = 2$ which can be easily obtained and are given below:

$$(6) \quad (a, 2, 1, b; m, n) = \frac{2}{m + n + 2} (a, 1, b; 2m + 1, 2n + 1)$$

$$- \frac{1}{m + n + 2} [(0, a; m + 1, n + 1) + (0, b; m + 1, n + 1)] \times (a, 1, b; m, n)$$

and

$$(a, 2, b, 1, c; m, n) = \frac{1}{m + n + 2} [-(0, a; m + 1, n + 1)(b, 1, c; m, n)$$

$$+ (0, b; m + 1, n + 1)(a, 1, c; m, n) - (0, c; m + 1, n + 1)(a, 1, b; m, n)].$$

Now

$(0, 3, 2, 1, x; m, n)$

$$= \int_{0<\theta_3<\theta_2<\theta_1<x} (\theta_1 \theta_2 \theta_3)^m (1 - \theta_1)^n (1 - \theta_2)^n (1 - \theta_3)^n \{1, 2, 3\} \, d\theta_1 \, d\theta_2 \, d\theta_3$$

$$= \int_{0<\theta_3<\theta_2<\theta_1<x} (\theta_1 \theta_2 \theta_3)^m (1 - \theta_1)^n (1 - \theta_2)^n (1 - \theta_3)^n [\theta_1 \theta_2 \{1, 2\} + \theta_3 \theta_1 \{3, 1\} + \theta_2 \theta_3 \{2, 3\}] \, d\theta_1 \, d\theta_2 \, d\theta_3$$

(using equation (2))

$$= \int_{0<\theta_3<\theta_2<\theta_1<x} \theta_3^m (1 - \theta_3)^n (\theta_1 \theta_2)^m (1 - \theta_1)^n (1 - \theta_2)^n \{1, 2\} \, d\theta_1 \, d\theta_2$$

$$+ \int_{0<\theta_1<\theta_3<\theta_2<x} + \int_{0<\theta_2<\theta_1<\theta_3<x},$$

or

$$(0, 3, 2, 1, x; m, n) = T_0^{x; m, n}(y, 2, 1, x; m + 1, n)$$

$$+ T_0^{x; m, n}(0, 1, y, 2, x; m + 1, n)$$

$$+ T_0^{x; m, n}(0, 2, 1, y; m + 1, n),$$

but the $\theta$'s are to be always arranged in the same order, hence

$$\begin{aligned}(0, 3, 2, 1, x; m, n) &= T_0^{x;m,n}(y, 2, 1, x; m+1, n)\\ &\quad - T_0^{x;m,n}(0, 2, y, 1, x; m+1, n)\\ &\quad + T_0^{x;m,n}(0, 2, 1, y; m+1, n).\end{aligned}$$

Using equations (6) and (7), we have

$$\begin{aligned}&(0, 3, 2, 1, x; m, n)\\ &= \frac{T_0^{x;m,n}}{m+n+3}\{2(y, 1, x; 2m+3, 2n+1) - (y, 1, x; m+1, n)\\ &\quad \times [(0, y; m+2, n+1) + (0, x; m+2, n+1)]\\ &\quad - (0, 1, x; m+1, n)(0, y; m+2, n+1) + (0, 1, y; m+1, n)(0, x; m+2, n+1)\\ &\quad + 2(0, 1, y; 2m+3, 2n+1) - (0, 1, y; m+1, n)(0, y; m+2, n+1)\}\\ &= \frac{T_0^{x;m,n}}{m+n+3}\{2(y, 1, x; 2m+3, 2n+1) + (0, 1, y; 2m+3, 2n+1)\\ &\quad - (0, y; m+2, n+1)[(0, 1, x; m+1, n) + (0, 1, y; m+1, n)\\ &\quad + (y, 1, x; m+1, n)] - (0, x; m+2, n+1)\\ &\quad [(y, 1, x; m+1, n) - (0, 1, y; m+1, n)]\}\\ &= \frac{T_0^{x;m,n}}{m+n+3}\{2(0, 1, x; 2m+3, 2n+1)\\ &\quad - 2(0, y; m+2, n+1)(0, 1, x; m+1, n)\\ &\quad - (0, x; m+2, n+1)[(y, 1, x; m+1, n) - (0, 1, y; m+1, n)]\}.\end{aligned}$$

Using equation (5), we have

$$\begin{aligned}(0, 3, 2, 1, x; m, n) = \frac{1}{(m+n+3)}\{&2(0, 1, x; 2m+3, 2n+1)(0, 1, x; m, n)\\ &- 2(0, 1, x; 2m+2, 2n+1)(0, 1, x; m+1, n)\\ &- (0, x; m+2, n+1)(0, 2, 1, x; m, n)\}.\end{aligned}$$

Hence

$$\begin{aligned}Pr(\theta_1 \leqq x) = \frac{C(3, m, n)}{(m+n+3)}\{&2(0, 1, x; 2m+3, 2n+1)(0, 1, x; m, n)\\ &- 2(0, 1, x; 2m+2, 2n+1)(0, 1, x; m+1, n)\\ &- (0, x; m+2, n+1)(0, 2, 1, x; m, n)\}.\end{aligned} \tag{8}$$

(c) $l = 4$. In order to determine $(0, 4, 3, 2, 1, x; m, n)$ we need the values of

$(a, 3, 2, 1, b; m, n)$, $(a, 3, b, 2, 1, c; m, n)$ and $(a, 3, 2, b, 1, c; m, n)$, which are obtained according to the procedure given above.

Now

$$(0, 4, 3, 2, 1, x; m, n) = \int_{0<\theta_4<\theta_3<\theta_2<\theta_1<x} \theta_4^m(1-\theta_4)^n(\theta_1\theta_2\theta_3)^m$$

$$\cdot (1-\theta_1)^n(1-\theta_2)^n(1-\theta_3)^n\{1, 2, 3, 4\}\, d\theta_1\, d\theta_2\, d\theta_3\, d\theta_4$$

$$= \int_{0<\theta_4<\theta_3<\theta_2<\theta_1<x} \theta_4^m(1-\theta_4)^n(\theta_1\theta_2\theta_3)^m$$

$$\cdot (1-\theta_1)^n(1-\theta_2)^n(1-\theta_3)^n[\theta_1\theta_2\theta_3\{1, 2, 3\}$$

$$- \theta_4\theta_1\theta_2\{4, 1, 2\} + \theta_3\theta_4\theta_1\{3, 4, 1\} - \theta_2\theta_3\theta_4\{2, 3, 4\}]\, d\theta_1\, d\theta_2\, d\theta_3\, d\theta_4$$

$$= \int_{0<\theta_4<\theta_3<\theta_2<\theta_1<x} \theta_4^m(1-\theta_4)^n[(\theta_1\theta_2\theta_3)^{m+1}$$

$$\cdot (1-\theta_1)^n(1-\theta_2)^n(1-\theta_3)^n]\{1, 2, 3\}$$

$$- \int_{0<\theta_1<\theta_4<\theta_3<\theta_2<x} + \int_{0<\theta_2<\theta_1<\theta_4<\theta_3<x} - \int_{0<\theta_3<\theta_2<\theta_1<\theta_4<x}$$

$$= T_0^{x;m,n}(y, 3, 2, 1, x; m+1, n) - T_0^{x;m,n}(0, 1, y, 3, 2, b; m+1, n)$$

$$+ T_0^{x;m,n}(0, 2, 1, y, 3, x; m+1, n) - T_0^{x;m,n}(0, 3, 2, 1, y; m+1, n)$$

$$= T_0^{x;m,n}(y, 3, 2, 1, x; m+1, n) - T_0^{x;m,n}(0, 3, y, 2, 1, b; m+1, n)$$

$$+ T_0^{x;m,n}(0, 3, 2, y, 1, x; m+1, n) - T_0^{x;m,n}(0, 3, 2, 1, y; m+1, n).$$

Using the results of $(a, 3, 2, 1, b; m, n)$, $(a, 3, b, 2, 1, c; m, n)$ and $(a, 3, 2, b, 1, c; m, n)$, we have $Pr(\theta_1 \leqq x)$ equal to

$$C(4, m, n)(0, 4, 3, 2, 1, x; m, n) = \frac{C(4, m, n)}{m+n+4}$$

$$\cdot\Big\{2(0, 1, x; 2m+5, 2n+1)(0, 2, 1, x; m, n)$$

$$- \frac{2(0, 1, x; 2m+4, 2n+1)}{(m+n+3)}[2(0, 1, x; 2m+2, 2n+1) \tag{9}$$

$$- (0, x; m+2, n+1)(0, 1, x; m, n) + (m+2)(0, 2, 1, x; m, n)]$$

$$+ 2(0, 1, x; 2m+3, 2n+1)(0, 2, 1, x; m+1, n)$$

$$- (0, x; m+3, n+1)(0, 3, 2, 1, x; m, n)\Big\}.$$

(d) $l = 5$. In the evaluation of the distribution of the largest root for $l = 5$; the following parts need to be calculated:

$(a, 4, 3, 2, 1, b; m, n)$, $(a, 4, b, 3, 2, 1, c; m, n)$, $(a, 4, 3, b, 2, 1, c; m, n)$, $(a, 4, 3, 2, b, 1, c; m, n)$.

Proceeding along the lines indicated in the previous sections we get

$$
\begin{aligned}
Pr(\theta_1 \leqq x) = \frac{C(5, m, n)}{(m+n+5)} \Big[ & 2(0, 1, x; 2m+7, 2n+1)(0, 3, 2, 1, x; m, n) \\
& - \frac{2(0, 1, x; 2m+6, 2n+1)}{(m+n+4)} \{2(0, 1, x; 2m+4, 2n+1)(0, 1, x; m, n) \\
& - 2(0, 1, x; 2m+3, 2n+1)(0, 1, x; m+1, n) \\
& \qquad - (0, x; m+3, n+1)(0, 2, 1, x; m, n) \\
& + (m+3)(0, 3, 2, 1, x; m, n)\} + \frac{2(0, 1, x; 2m+5, 2n+1)}{(m+n+4)} \\
& \cdot \Big\{ 2(0, 1, x; 2m+5, 2n+1)(0, 1, x; m, n) \\
& \qquad - 2(0, 1, x; 2m+3, 2n+1)(0, 1, x; m+2, n) \\
& - \frac{(0, x; m+3, n+1)}{(m+n+3)} [2(0, 1, x; 2m+2, 2n+1) \\
& \qquad - (0, x; m+2, n+1)(0, 1, x; m, n) \\
& + (m+2)(0, 2, 1, x; m, n)] \Big\} \\
& \qquad - 2(0, 3, 2, 1, x; m+1, n)(0, 1, x; 2m+4, 2n+1) \\
& - (0, x; m+4, n+1)(0, 4, 3, 2, 1, x; m, n) \Big].
\end{aligned} \tag{10}
$$

It is evident now that the above method can be used to derive the distribution for any value of $l$.

**5. Distribution of the smallest root.** Let $Pr[\theta_1 \leqq x/\mu, \nu] = P(x/\mu, \nu)$ where $\theta_1$ is the largest root. Let us make the following transformations in the $R(l, \mu, \nu)$ distribution:

$$
\begin{aligned}
r_1 &= 1 - \theta_l \\
r_2 &= 1 - \theta_{l-1} \\
&\cdot \\
&\cdot \\
&\cdot \\
r_l &= 1 - \theta_1 ;
\end{aligned}
$$

then since $0 < \theta_l < \theta_{l-1} < \cdots < \theta_1 < 1$, we have $0 < r_l < r_{l-1} < r_{l-2} \cdots <$

$r_1 < 1$, and thus the domain of integration does not change. Hence the joint distribution of the $r$'s can be expressed as

$$C(l, \nu, \mu) \prod_{i=1}^{l} (r_i)^{(\nu/2)-1} \prod_{i=1}^{l} (1 - r_i)^{(\mu/2)-1} \prod_{i<j} (r_i - r_j), \qquad 0 < r_l < \cdots < r_1 < 1.$$

Thus the $r$'s have the same distribution as the $\theta$'s, but $\mu$ and $\nu$ are interchanged. Therefore

$$\begin{aligned} Pr(\theta_l \leq x) &= Pr(1 - r_1 \leqq x) = 1 - Pr(r_1 \leqq 1 - x) \\ &= 1 - P(1 - x/\nu, \mu). \end{aligned}$$

Hence, for getting the distribution of the smallest root, we have to change $x$ into $1 - x$ and interchange $m, n$ in the distributions of the largest roots and subtract the resultant probability from 1. The distributions for the smallest root are given below for $l = 2, 3, 4$ and 5.

(i) $l = 2$.

$$\begin{aligned} Pr(\theta_2 \leq x) &= 1 - Pr(\theta_1 \leqq 1 - x/n, m) \\ &= 1 - \frac{C(2, n, m)}{m + n + 2} \{2(0, 1, \overline{1 - x}, 2n + 1, 2m + 1) \\ &\qquad - (0, \overline{1 - x}, n + 1, m + 1)(0, 1, \overline{1 - x}, n, m)\}. \end{aligned} \tag{11}$$

(ii) $l = 3$.

$$\begin{aligned} P_r(\theta_3 \leqq x) = 1 - \frac{C(3, n, m)}{m + n + 3} &\{2(0, 1, \overline{1 - x}; 2n + 3, 2m + 1) \\ &\cdot (0, 1, \overline{1 - x}; n, m) \\ &- 2(0, 1, \overline{1 - x}; n + 1, m)(0, 1, \overline{1 - x}; 2n + 2, 2m + 1) \\ &- (0, \overline{1 - x}; n + 2, m + 1)(0, 2, 1, \overline{1 - x}; n, m)\}. \end{aligned} \tag{12}$$

(iii) $l = 4$.

$$\begin{aligned} Pr(\theta_4 \leqq x) = 1 - \frac{C(4, n, m)}{m + n + 4} &\Big\{2(0, 1, \overline{1 - x}; 2n + 5, 2m + 1) \\ &\quad (0, 2, 1, \overline{1 - x}; n, m) \\ &- \frac{2(0, 1, \overline{1 - x}, 2n + 4, 2m + 1)}{(m + n + 3)} [2(0, 1, \overline{1 - x}; 2n + 2, 2m + 1) \\ &- (0, \overline{1 - x}; n + 2, m + 1)(0, 1, \overline{1 - x}; n, m) \\ &+ (n + 2)(0, 2, 1, \overline{1 - x}; n, m)] \\ &+ 2(0, 1, \overline{1 - x}; 2n + 3, 2m + 1)(0, 2, 1, \overline{1 - x}; n + 1, m) \\ &- (0, \overline{1 - x}; n + 3, m + 1)(0, 3, 2, 1, \overline{1 - x}; n, m)\Big\}. \end{aligned} \tag{13}$$

(iv) $l = 5$.

$$Pr(\theta_5 \leqq x) = 1 - \frac{C(5, n, m)}{(m + n + 5)}\Bigg[2(0, 1, \overline{1 - x}; 2n + 7, 2m + 1) \cdot (0, 3, 2, 1, \overline{1 - x}; n, m)$$
$$- \frac{2(0, 1, \overline{1 - x}; 2n + 6, 2m + 1)}{(m + n + 4)}\{2(0, 1, \overline{1 - x}; 2n + 4, 2m + 1) \cdot (0, 1, \overline{1 - x}; n, m)$$
$$- 2(0, 1, \overline{1 - x}; 2n + 3, 2m + 1)(0, 1, \overline{1 - x}; n + 1, m)$$
$$- (0, \overline{1 - x}; n + 3, m + 1)(0, 2, 1, \overline{1 - x}; n, m)$$
$$+ (n + 3)(0, 3, 2, 1, \overline{1 - x}; n, m)\}$$
$$+ \frac{2(0, 1, \overline{1 - x}; 2n + 5, 2m + 1)}{(m + n + 4)}\Bigg\{2(0, 1, \overline{1 - x}; 2n + 5, 2m + 1) \tag{14}$$
$$(0, 1, \overline{1 - x}; n, m)$$
$$-[2(0, 1, \overline{1 - x}; 2n + 3, 2m + 1)(0, 1, \overline{1 - x}; n + 2, m)$$
$$- \frac{(0, \overline{1 - x}; n + 3, m + 1)}{(m + n + 3)}[2(0, 1, \overline{1 - x}; 2n + 2, 2m + 1)$$
$$- (0, \overline{1 - x}; n + 2, m + 1)(0, 1, \overline{1 - x}; n, m)$$
$$+ (n + 2)(0, 2, 1, \overline{1 - x}; n, m)]\Bigg\} - 2(0, 3, 2, 1, \overline{1 - x}; n + 1, m)$$
$$\cdot (0, 1, \overline{1 - x}; 2n + 4, 2m + 1)$$
$$- (0, \overline{1 - x}; n + 4, m + 1)(0, 4, 3, 2, 1, \overline{1 - x}; n, m)\Bigg].$$

**6. Distribution of any intermediate root.**

(i) $l = 3$.

$$Pr(\theta_2 \leqq x) = Pr(0 < \theta_3 < \theta_2 < \theta_1 < x) + Pr(0 < \theta_3 < \theta_2 < x < \theta_1)$$
$$= C(3, m, n)[(0, 3, 2, 1, x; m, n) + (0, 3, 2, x, 1; m, n)]$$

as the two probabilities are independent, or

$$Pr(\theta_2 \leqq x) = C(3, m, n)[(0, 3, 2, 1, x; m, n) + (0, 3, 2, x, 1, z; m, n)], \text{ where } z = 1$$
$$= \frac{C(3, m, n)}{m + n + 3}\Bigg\{2(0, 1, x; 2m + 3, 2n + 1)(0, 1, x; m, n)$$
$$- 2(0, 1, x; m + 1, n)(0, 1, x; 2m + 2, 2n + 1)$$
$$- (0, x; m + 2, n + 1)(0, 2, 1, x; m, n)$$
$$+ (x, 1, z; m, n)[2(0, 1, x; 2m + 3, 2n + 1) \tag{15}$$
$$- (0, x; m + 2, n + 1)(0, 1, x; m + 1, n)]$$
$$- \frac{(x, z; m + 2, n + 1)}{m + n + 2}[2(0, 1, x; 2m + 1, 2n + 1)$$
$$- (0, x; m + 1, n + 1)(0, 1, x; m, n)]$$
$$+ (x, 1, z; m + 1, n)(0, 1, x; m, n)(0, x; m + 2, n + 1)$$
$$- 2(x, 1, z; m + 1, n)(0, 1, x; 2m + 2, 2n + 1)\Bigg\}.$$

(ii) $l = 4$.

$$\begin{aligned} Pr(\theta_2 \leqq x) &= Pr(0 < \theta_4 < \theta_3 < \theta_2 < \theta_1 < x; m, n) \\ &\quad + Pr\ (0 < \theta_4 < \theta_3 < \theta_2 < x < \theta_1\ ; m, n) \\ &= C(4, m, n)[(0, 4, 3, 2, 1, x; m, n) + (0, 4, 3, 2, x, 1; m, n)] \end{aligned}$$

and

$$\begin{aligned} Pr(\theta_3 \leqq x) &= Pr(0 < \theta_4 < \theta_3 < \theta_2 < \theta_1 < x; m, n) \\ &\quad + Pr(0 < \theta_4 < \theta_3 < \theta_2 < x < \theta_1\ ; m, n) \\ &\quad + Pr(0 < \theta_4 < \theta_3 < x < \theta_2 < \theta_1\ ; m, n) \\ &= C(4, m, n)[(0, 4, 3, 2, 1, x; m, n) + (0, 4, 3, 2, x, 1; m, n) \\ &\quad + (0, 4, 3, x, 2, 1; m, n)]. \end{aligned}$$

The different parts of these probabilities can be evaluated as indicated in section 4(d). Thus the method already indicated to obtain the distribution of the largest root also gives the distribution of any one of the roots.

**7. Further problems.** It is intended to prepare the probability distribution tables for small values of $l$. The results obtained in this paper are found to be useful in finding the distribution of the sum of the roots when the numbers of canonical variates in two sets differ by one. This problem is, however, being investigated further.

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