[3] E. T. Whittaker and G. N. Watson, *Modern Analysis*, Cambridge Univ. Press, p. 253.
[4] L. C. Hsu, "Some combinatorial formulas with applications to mathematical expectations and to differences of zero," *Annals of Math. Stat.*, Vol. 15 (1944).

AN INEQUALITY FOR KURTOSIS

By Louis Guttman

Cornell University

1. Summary. It is well known that, if the fourth moment about the mean of a frequency distribution equals the square of the variance, then the frequencies are piled up at exactly two points, namely, the two points that are one standard deviation away from the mean. In this paper is developed a general inequality which describes the piling up of frequency around these two points for the case where the fourth moment exceeds the square of the variance. In a sense, it is shown how "U-shaped" a distribution must be according to its second and fourth moments.

2. An inequality. Let x be a random variable whose distribution has the following moments:

$$\mu = E(x);\ \sigma^2 = E(x - \mu)^2;\ (\alpha^2 + 1)\sigma^4 = E(x - \mu)^4. \tag{1}$$

α^2 is non-negative for any distribution, and its positive square root will be denoted by α. Let

$$t = (x - \mu)/\sigma. \tag{2}$$

It will be shown that, if λ is an arbitrary positive number, then

$$\text{Prob}\ \{1 - \lambda\alpha \leqq t^2 \leqq 1 + \lambda\alpha\} > 1 - \lambda^{-2}. \tag{3}$$

If λ is chosen so as to make the left member in the braces positive, then t^2 is bounded away from zero, and (3) becomes:

$$\text{Prob}\ \left\{\sqrt{1 - \lambda\alpha} \leqq |\,t\,| \leqq \sqrt{1 + \lambda\alpha}\right\} > 1 - \lambda^{-2}, \qquad (\lambda\alpha < 1). \tag{4}$$

For example, if $\alpha = .5$ and $\lambda = \sqrt{2}$, then (4) shows that the probability is greater than .50 that t is either between .54 and 1.30, or between -1.30 and $-.54$. If $\alpha = .2$ and $\lambda = 3$, then (4) shows that the probability is greater than .88 that t is either between .63 and 1.27, or between -1.27 and $-.63$. In general, the smaller α is, the greater the probability that t is in a small interval around $+1$ or -1. In particular, if $\alpha = 0$, then λ may be taken arbitrarily large, so that (4) shows that the probability is unity that $t = \pm 1$; this is the well known case referred to above.

3. Derivation. Inequality (3) is a special case of a slightly more general inequality which follows very simply from that of Tchebychef. Consider the function $t^2 - 1 + c$, where c is an arbitrary real number. By using (1) and (2), it is seen that

$$E(t^2 - 1 + c)^2 = \alpha^2 + c^2. \tag{5}$$

Then, according to Tchebychef's inequality, if λ is an arbitrary positive number,

$$\text{Prob}\ \{(t^2 - 1 + c)^2 \leqq \lambda^2(\alpha^2 + c^2)\} > 1 - \lambda^{-2}, \tag{6}$$

or,

$$\text{Prob}\ \{1 - c - \lambda\sqrt{\alpha^2 + c^2} \leqq t^2 \leqq 1 - c + \lambda\sqrt{\alpha^2 + c^2}\} > 1 - \lambda^{-2}. \tag{7}$$

This is the general inequality that was to be shown.

Inequality (3) is obtained by setting $c = 0$ in (7).

Another special case is obtained by determining c so as to maximize the left member in the braces of (7). By differentiation, the maximizing value is found to be $c = -\alpha/\sqrt{\lambda^2 - 1}$, for which (7) becomes:

$$\text{Prob}\ \{1 - \alpha\beta \leqq t^2 \leqq 1 + \alpha(\beta^2 + 2)/\beta\} > 1 - 1/(\beta^2 + 1), \tag{8}$$

where β is used instead of the notation $\sqrt{\lambda^2 - 1}$, and denotes any positive number. For the same probability on the right, (8) has the advantage over (3) of having $1 - \alpha\beta$ greater than $1 - \lambda\alpha$, so that the former may be positive even though the latter is negative. Inequality (8) starts the positive interval for t as close to $+1$ as possible. On the hand, (3) provides the minimum size interval for t^2 from among all values of c that make the left member in the braces of (7) positive.

If it is desired to have the positive interval for t *end* as close to $+1$ as possible, then the right member in the braces of (7) is to be minimized. By differentiation, the minimizing value is found to be $c = \alpha/\sqrt{\lambda^2 - 1}$, and the minimum inequality is:

$$\text{Prob}\ \{1 - \alpha(\beta^2 + 2)/\beta \leqq t^2 \leqq 1 + \alpha\beta\} > 1 - 1/(\beta^2 + 1). \tag{9}$$

4. Distribution Around μ. If the left member in the braces of (7) is negative, then instead of giving information about the piling up of probability of t around $+1$ and -1, (7) provides a statement about the probability in an interval around μ. Alternatively, this may be regarded as a confidence interval for μ. The minimum interval is given by (9); actually, it holds regardless of the value of the left member in the braces; another way of stating it is:

$$\text{Prob}\ \{-\sqrt{1 + \alpha\beta} \leqq t \leqq \sqrt{1 + \alpha\beta}\} > 1 - 1/(\beta^2 + 1). \tag{10}$$