SOLUTION OF EQUATIONS BY INTERPOLATION

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Introduction and summary. The present paper deals with the numerical solution of equations by the combined use of Newton's method and inverse interpolation. In Part I the case of one equation in one unknown is discussed. The methods described here were developed by Aitken [1] and Neville [2], but do not seem as widely known as they should be, perhaps because the original papers are not readily available. (A short summary of Aitken's work will be found in a recent paper by Womersley [3].) Mention should also be made of an interesting paper by Spoerl [4], which treats the same problem from a somewhat different viewpoint.

In Part II these methods are extended to sets of simultaneous equations.

PART I. EQUATIONS IN ONE UNKNOWN

1. Nature of the problem. We first consider the problem of locating, to any desired degree of accuracy, a real root x_0 of an equation of the form

$$y(x) = 0 \tag{1}$$

where $y(x)$ is assumed to be analytic in an interval containing the root in question. Since we shall not be concerned here with the necessary preliminary work of separating the roots, etc., we may suppose that x_0 is known to lie within a given interval that contains no zeros of $y'(x)$. (Multiple roots are thus excluded; but of course any such root is a simple root of an equation obtained from (1) by differentiation, and the methods described below can be applied to this equation.)

2. Aitken's method of interpolation. The method to be described, which may be regarded as a generalization of Newton's, depends on the use of inverse interpolation. It is therefore desirable to recall a few points from the theory of interpolation before proceeding further.

Let f be a function such that $f(t)$ is known for $t = t_1, t_2, \cdots, t_n$. Then the Lagrange interpolating polynomial $f_{12\cdots n}(t)$ is defined by

$$\begin{aligned} f_{12\cdots n}(t) = f(t_1) & \frac{(t - t_2)(t - t_3)\cdots(t - t_n)}{(t_1 - t_2)(t_1 - t_3)\cdots(t_1 - t_n)} \\ & + f(t_2) \frac{(t - t_1)(t - t_3)\cdots(t - t_n)}{(t_2 - t_1)(t_2 - t_3)\cdots(t_2 - t_n)} \\ & + \cdots + f(t_n) \frac{(t - t_1)(t - t_2)\cdots(t - t_{n-1})}{(t_n - t_1)(t_n - t_2)\cdots(t_n - t_{n-1})}. \end{aligned} \tag{2}$$

We note that

$$
(3)\qquad
\begin{aligned}
f_{12}(t) &= f(t_1)\,\frac{t - t_2}{t_1 - t_2} + f(t_2)\,\frac{t - t_1}{t_2 - t_1} = \frac{\begin{vmatrix} f(t_1) & t - t_1 \\ f(t_2) & t - t_2 \end{vmatrix}}{t_1 - t_2}, \\
f_{123}(t) &= \frac{\begin{vmatrix} f_{12}(t) & t - t_1 \\ f_{23}(t) & t - t_3 \end{vmatrix}}{t_1 - t_3}, \cdots, f_{123\cdots n}(t) = \frac{\begin{vmatrix} f_{123\cdots n-1}(t) & t - t_1 \\ f_{23\cdots n}(t) & t - t_n \end{vmatrix}}{t_1 - t_n},
\end{aligned}
$$

so that $f_{123\cdots n}(t)$ can be evaluated for any given value t_0 of t by a succession of linear interpolations. It is convenient to arrange the work in a table like the following ($n = 4$):

TABLE Ia

t	$f(t)$	*I*	*II*	*III*	*Parts*
t_1	$f(t_1)$				t_0-t_1
		$f_{12}(t_0)$			
t_2	$f(t_2)$		$f_{123}(t_0)$		t_0-t_2
		$f_{23}(t_0)$		$f_{1234}(t_0)$	
t_3	$f(t_3)$		$f_{234}(t_0)$		t_0-t_3
		$f_{34}(t_0)$			
t_4	$f(t_4)$				t_0-t_4

This form is well adapted for machine computation, for each denominator $t_i - t_j = (t_0 - t_j) - (t_0 - t_i)$ automatically appears in one set of counters when the corresponding numerator is obtained in the other.

If $f'(t)$ is known at one or more of the given points, this information is readily fitted into the scheme. For we see that

$$
(4)\qquad f_{11}(t) \equiv \lim_{t_2 \to t_1} f_{12}(t) = f(t_1) + (t - t_1)f'(t_1)
$$

and all that is necessary is to repeat certain entries in Table Ia and to fill in column I by using (4) as indicated in Table Ib. The extension to higher derivatives is obvious.

TABLE Ib

t	$f(t)$	*I*	*II*	*III*	*Parts*
t_1	$f(t_1)$				t_0-t_1
		$f_{11}(t_0)$			
	$f(t_1)$		$f_{112}(t_0)$		t_0-t_1
		$f_{12}(t_0)$		$f_{1123}(t_0)$	
t_2	$f(t_2)$		$f_{123}(t_0)$		t_0-t_2
		$f_{23}(t_0)$		$f_{1233}(t_0)$	
t_3	$f(t_3)$		$f_{233}(t_0)$		t_0-t_3
		$f_{33}(t_0)$			
	$f(t_3)$				t_0-t_3

In applying the above to obtaining the root x_0 of (1), we must suppose that $y(x)$ is tabulated or can be computed for a set of values of x in the neighborhood of x_0. What we do not know is the value of x corresponding to $y = 0$. It is therefore convenient to regard x as a function of y whose value is known at certain points and then interpolate to get $x_0 = x(0)$. That is, we let y take the place of t and x that of $f(t)$ in the preceding discussion, while 0 replaces t_0. The work is slightly simplified by the fact that the column of "parts" becomes identical with the left-hand column which contains the y's and can therefore be omitted.

3. Application to an example. The procedure will be most clearly indicated by an example. Consider the equation

$$y = x^4 + 2x^3 - 5x^2 - 8x + 1 = 0, \tag{5}$$

which has a root between 0 and 1. (If the root were located elsewhere, it would be desirable to shift it to this interval in order to simplify the computation of y.)

The work of evaluating this root to ten places is summarized in Table II, and explained below. In the first column, the numbers in parentheses are values of $\frac{dy}{dx}$, and the other numbers are values of y, corresponding to the values of x in the second column.

TABLE II

y	x	*I*	*II*	*III*
1.000 000 000 000	0.000			
(−8.000 000 000)	0.000	0.125 000 000 00		
0.152 100 000 000	0.100	0.117 938 436 13	0.116 671 702 00	
−0.001 054 385 279	0.117	6 882 964 17	884 075 87	
(−9.081 459 548)	0.117	3 896 94	3 890 52	0.116 883 877 01
0.008 022 855 936	0.116	3 842 98	90 67	90 68
(−9.073 020 416)	0.116	4 254 15	90 74	90 68

$x_0 = 0.116\ 883\ 890\ 7$

The procedure is as follows. Taking $x = 0$ as a first approximation to x_0, we find that $y(0) = 1$, $y'(0) = -8$, and record this data in the y and x columns of the table. Note that for convenience, the value of $y'(0)$ takes the place of a blank entry in Table Ib. We now apply (4), which here takes the form

$$x_{11}(0) = x\Big|_{y=1} + (0 - 1)\frac{dx}{dy}\Big|_{y=1} = 0 + \frac{-1}{\left.\frac{dy}{dx}\right|_{x=0}} = 0 + \frac{-1}{-8} = 0.125 \tag{6}$$

and enter the result in column I. Note that this is equivalent to one step of Newton's method.

In view of (6) we take $x = 0.1$ for our next approximation and apply (3) to obtain the second entry in column I and the first in column II. This last suggests $x = 0.117$ for our next trial value. (We do not compute $y'(0.1)$, as little would be gained by doing so, and the time is better spent in going ahead as indicated.) Finding $y(0.117)$ and filling in the table gives us the root to six places.

4. Employment of tables. Continuing in the same line, it would seem natural to take $x = 0.116884$ at the next step; and doing so would lead to the most rapid convergence. But another consideration enters. Up to this point the values of y were computed with the aid of the WPA Table of Powers, which is limited to three places in the argument. Rather than going to the extra labor of evaluating $y(.116884)$, we proceed as indicated in the table, using $y'(.117)$, $y(.116)$ and $y'(.116)$, and stopping when the values of x in the last column agree to the desired number of places.

This point has been dwelt on because it is likely to arise whenever tables are used in evaluating $y(x)$. In the example just given, to be sure, we had a certain freedom of choice; but if $y(x)$ is not algebraic, direct computation may be quite impractical. It may be noted that in such cases the method of inverse interpolation is not only faster than the simple Newton's method but is capable of giving more accurate results.

The error in the final result can be estimated from the standard formula for the error of interpolation, but this may be awkward because it requires the evaluation of higher derivatives of x with respect to y. In practice it is generally safe to rely on agreement of different interpolated values, and of course the result may be checked by substitution in the original equation. One simple point is worth noting, however—if the error in the original column of x's is $O(\epsilon)$, that in the successive columns to the right is $O(\epsilon^2)$, $O(\epsilon^3)$, etc.

5. Applicability of the method. Although the example we have presented is algebraic, the method is, of course, equally applicable to transcendental equations. Moreover, it can be used, theoretically at least, to yield complex as well as real roots. The sole difficulty is that the numerical work becomes cumbersome in this case; how serious it is depends on the type of computing machines used. If the equation is algebraic, Bernoulli's [5], [6] and Graeffe's [7] methods are applicable. In fact, they are likely to be the most effective since they do not require prior knowledge of a first approximation to the root. If the alternative procedure of replacing the equation by two simultaneous equations for the real and imaginary parts of the root is decided upon, the methods described in the next section may prove useful.

PART II. SETS OF SIMULTANEOUS EQUATIONS

6. Two equations; general considerations. It is natural to take up next the problem of finding the simultaneous solutions of two equations in two unknowns. Let these equations be

$$u(x, y) = 0, \qquad v(x, y) = 0, \tag{7}$$

where u and v are analytic functions of x and y.

If we had a general method of interpolation of functions of two independent variables, the problem could be solved in a fashion similar to that used in the preceding section. That is, u and v would be computed for values of x and y near the desired ones; then x and y would be regarded as functions of u and v and interpolations would be performed to obtain the values corresponding to $u = v = 0$.

It is easy to set up interpolating functions in a variety of ways, but the author has found none that are satisfactory for the problem in hand. Note that what is required is to determine the value of a function at any point in the plane, given its values at a set of fixed points. The most obvious idea is to use polynomials of the least possible degree for this purpose, as is done in the case of a single variable. In this case, however, the coefficients of a polynomial of the nth degree are determined by its values not at $n + 1$ but at $\frac{(n+1)(n+2)}{2}$ points; thus if a function is given at 5 points, no unique quadratic interpolating polynomial can be constructed. What is worse, even if a function is given at 6 points, say, the quadratic polynomial determined will in general have large coefficients and take on unreasonable values if all the points happen to lie close to a common conic. Other schemes considered by the author have similar drawbacks, though the possibility of course remains of finding a suitable one by further research.

The problem can also be handled, at least in principle, by eliminating one of the variables; but, apart from the difficulty of carrying this out in practice, the resulting single equation is likely to be more complicated in form than the original two. If so, solving it may require more computation than would be involved in attacking the original equations directly by the methods described below. So far is this true that even when a single equation is given in the first place it may be advantageous to replace it by a set of simpler equations.

7. Newton's Method. Although a direct extension of the method of inverse interpolation is not presently available, Newton's method may be suitably generalized for this case.

Starting with equations (7), we set up the auxiliary variables

$$X = uv_y - vu_y, \qquad Y = uv_x - vu_x, \tag{8}$$

where the subscripts denote partial derivatives; $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, etc.

We have

$$\begin{aligned} \frac{\partial X}{\partial x} &= u_x v_y - v_x u_y + uv_{xy} - vu_{xy}, \qquad & \frac{\partial X}{\partial y} &= uv_{yy} - vu_{yy}, \\ \frac{\partial Y}{\partial x} &= uv_{xx} - vu_{xx}, & \frac{\partial Y}{\partial y} &= u_y v_x - v_y u_x + uv_{xy} - vu_{xy}. \end{aligned} \tag{9}$$

For $u = 0, v = 0$, equations (8), (9) reduce to

$$X = Y = \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} = 0, \qquad \frac{\partial X}{\partial x} = -\frac{\partial Y}{\partial y} = J, \tag{10}$$

where J is the Jacobian of u and v with respect to x and y.

Equations (10) will hold approximately for values of x and y near those satisfying equations (7). That is, in the neighborhood of a solution X can be regarded as a function of x alone and Y as a function of y alone. Then if $x = x_0$, $y = y_0$ is the desired solution, (x_1, y_1) is a point in its neighborhood, and $x_1 = X(x_1, y_1)$, $Y_1 = Y(x_1, y_1)$, $J_1 = J(x_1, y_1)$, we have

$$x_0 \sim x_1 - \frac{X_1}{J_1}, \qquad y_0 \sim y_1 + \frac{Y_1}{J_1}. \tag{11}$$

Also if (x_2, y_2) is another point near (x_0, y_0),

$$x_0 \sim \frac{x_1 X_2 - x_2 X_1}{X_2 - X_1}, \qquad y_0 \sim \frac{y_1 Y_2 - y_2 Y_1}{Y_2 - Y_1}. \tag{12}$$

Relations (11) and (12) can be used to obtain successive approximations to the solution. Use of these relations corresponds to employing Newton's method and linear interpolation for the solution of one equation in one unknown.

As a first example we consider the equations

$$\begin{aligned} u &\equiv x^2 + xy + y^2 - 3 = 0 \\ v &\equiv x^2 y + y^2 - 1 = 0. \end{aligned} \tag{13}$$

We have

$$\begin{aligned} u_x &= 2x + y \qquad & u_y &= x + 2y \\ v_x &= 2xy \qquad & v_y &= x^2 + 2y. \end{aligned} \tag{14}$$

Drawing a rough graph indicates a solution near (1, 1). We evaluate u, v, etc., at this point as shown in Table III. Using (11) we get (2, 0) for a second approximation, and proceed as before. We can now use both (12) and (11) to get new approximations; they are (1.33, 0.57) and (1.25, 0.50), and are entered in the last two columns of the table. We therefore try (1.3, 0.5) next, and continue in this fashion until the desired accuracy is attained. Both (11) and (12) are used at each step and the values obtained are entered in the last two columns. The entrics in the numbered rows are obtained by using (11), the others by using (12). The number of places to take in each succeeding step is judged from the agreement shown.

Table IV indicates the process of finding a second solution of (11) by the same method. The convergence is very rapid in this case, mainly because the first guess is fairly close.

TABLE III

	x	y	u	v	u_x	v_x	u_y	v_y
1	1.0	1.0	0.0	1.0	3.0	2.0	3.0	3.0
2	2.0	0.0	1.0	−1.0	4.0	0.0	2.0	4.0
3	1.3	0.5	−.41	.095	3.1	1.3	2.3	2.69
4	1.5	0.4	.01	.06	3.4	1.2	2.3	3.05
5	1.51	0.37	−.0243	−.019463	3.39	1.1174	2.25	3.0201
6	1.5138	0.3750	$-.0_{3}10956$	$-.0_{4}28585$	3.4026	1.13535	2.2638	3.04159044
7	1.5138345	0.37499651	$-.0_{7}698799749$	$-.0_{7}3062799647$	3.40266551	1.1353653084	2.26382752	3.0416879134

	J	X	Y	(x)	(y)
1	3.0	−3.0	−3.0	2.0	0.0
				1.333	0.5714
2	8.0	6.0	4.0	1.250	0.500
				1.4263	.4143
3	5.349	−1.3214	−.8275	1.54704	.34530
				1.51771	.36979
4	7.61	−.1075	−.192	1.51413	.37477
				1.513799	.375046
5	7.723989	−.02959668	.03882675	1.51383179	.37503677
				1.51383479	.374996509
6	7.779110301	$-.0_{3}2685259256$	$-.0_{4}27125625$	1.51383451188	.374996513
				1.51383451841017	.374996513197826
7	7.779575324	$-.0_{6}1432165737$	$.0_{7}2487752800$	1.5138345184093048	.3749965131978003

$x_0 = 1.5138345184093$
$y_0 = 0.3749965131978$

TABLE IV

	x	y	u	v	u_x	v_x	u_y	v_y
1	−1.0	−1.0	0.0	−1.0	−3.0	2.0	−3.0	−1.0
2	−.7	−1.3	.09	.053	−2.7	1.82	−3.3	−2.11
3	−.699	−1.274	$.0_2 2203$	$.0_3 598326$	−2.672	1.781052	−3.247	−2.059399
4	−.69877	−1.27351	$-.0_5 21843$	$-.0_5 1122373279$	−2.67105	1.7797811654	−3.24579	2.0587404871

	J	X	X	(x)	(y)
1	9.0	−3.0	−3.0	−.667 −.6984	−1.33 −1.2721
2	11.703	−.015	.3069	−.69872 −.69879	−1.27378 −1.273524
3	11.28578997	$-.0_2 2594091475$	$.0_2 5522384628$	−.698770145 −.698770075686	−1.27351068 −1.27351061019
4	11.27579469	$-.0_6 8539188805$	$-.0_5 6885491146$	−.69877007573026	−1.27351061064354

$x_0 = -0.69877007573$
$y_0 = -1.27351061064$

8. Inverse interpolation. In the preceding section, attention was drawn to the difficulty that may arise when tables, necessarily limited to a certain number of places in the argument, are used in the computation. In the example just discussed the values of u and v were easily computed directly to the number of places wanted. But a glance at the work will show that if we had been limited in computing u and v to values of x and y having, say, two decimal places, the solutions could have been carried to four places only.

The device adopted in the preceding section was to use quadratic and cubic interpolates to secure greater accuracy, and it might occur to us to try the same idea here. But for such an interpolation to be strictly valid, equations (10) would have to hold identically. Since they hold only approximately, an error is introduced which, in general, is of the same order of magnitude as the error in linear interpolation. Thus continuing the interpolation would not improve the results.

However, this very situation suggests a way out. For suppose we give x a constant value x_1, and compute X and Y for a number of values of y. For $x = x_1$, both X and Y can be regarded as functions of y alone; or we can regard X and y as functions of Y. Doing so, we can interpolate to any number of stages to find values of X and y corresponding to $Y = 0$; call these X_1, y_1. Assigning x other constant values $x_2, x_3, \cdots, x_m$, we repeat the process, getting a set of values $X_2, \cdots X_m$ and $y_2, \cdots, y_m$, all corresponding to $Y = 0$. Now along the curve $Y = 0$ we can regard x and y as functions of X; performing one more interpolation, we obtain the desired values of x, y corresponding to $X = Y = 0$. The error in the final result can be estimated from the errors in the interpolations, and is of the same order of magnitude as the greatest of these.

It will be noted that we did not refer to the definitions of X and Y in describing this procedure. Any pair of independent (analytic) functions X' and Y' having the property that $X' = Y' = 0$ when $u = v = 0$ could be used. However, it is convenient to choose them so that $\dfrac{\partial X'}{\partial y}$ and $\dfrac{\partial Y'}{\partial x}$ are small. Probably the simplest course is to set

$$X' = a_1u + b_1v, \qquad Y' = a_2u + b_2v,$$

where a_1, a_2, b_1, b_2 are constants such that

$$\frac{a_1}{b_1} \sim -\frac{v_y}{u_y}, \qquad \frac{a_2}{b_2} \sim -\frac{v_x}{u_x}.$$

Let us apply this procedure to the example we have already worked (Table III). Suppose we wish to use values of x and y having not more than two decimal places. Within this restriction, we can still carry through the first few steps indicated in Table III to ascertain that $x_0 \sim 1.514$, $y_0 \sim 0.375$ where (x_0, y_0) is the desired solution. At the point (1.51, 0.37) we have

$$X = 3.0201u - 2.25v, \quad Y = 1.1174u - 3.39v.$$

TABLE V

x	y	u	v	X'	Y'	y	y_I	y_{II}	y_{III}
1.50	0.36	−.0804	−.0604	−.1404	.1008	0.36			
	0.37	−.0581	−.0306	−.1406	.0337	0.37	.3750223546.9		
	0.38	−.0356	−.0006	−.1406	−.0338	0.38	49925925.9	.3759999662.8	
	0.39	−.0129	.0296	−.1404	−.1017	0.39	50220913.1	49999345.9	.3750000007.3
1.51	0.36	−.0467	−.049564	−.038108	.101992	0.36			
	0.37	−.0243	−.019463	−.038811	.034089	0.37	.3750202494.7		
	0.38	−.0017	.010838	−.039314	−.034124	0.38	49908495.9	.3749982346.4	
	0.39	−.0211	.041339	−.039617	−.102917	0.39	50200136.8	81060.2	.3749981706.2
1.52	0.36	−.0128	−.038656	.064768	.103168	0.36			
	0.37	.0097	−.008252	.063566	.034456	0.37	.3750145534.9		
	0.38	.0324	.022352	.062544	−.034656	0.38	49855307.3	.3749928285.4	
	0.39	.0553	.053156	.061732	−.104168	0.39	50143860.0	27029.2	.3749927660.3
1.53	0.36	.0213	−.027676	.168228	.104328	0.36			
	0.37	.0439	.003033	.166501	.034801	0.37	.3750053935.8		
	0.38	.0667	.033942	.164974	−.035126	0.38	49767614.7	.3749839733.9	
	0.39	.0897	.065051	.163647	−.105453	0.39	50053322.3	38506.9	.3749839123.7

Y'	X'	X'_I	X'_{II}	X'_{III}
.1008	−.1404			
.0337	−.1406	−.1407004470.9		
−.0338	−.1406	6000000.0	−.1406252237.1	
−.1017	−.1404	6995581.7	47792.5	−.1406250024.7
.101992	−.038108			
.034089	−.038811	−.03916392353.7		
−.034124	−.039314	06203973.4	−.03908763223.7	
−.102917	−.039617	16310641.4	718653.1	−.03908741039.0
.103168	.064768			
.034456	.063566	.06294823611.5		
−.034656	.062544	305146428.9	.06302550744.8	
−.104168	.061732	294883185.6	595425.3	.06302572977.3
.104328	.168228			
.034801	.166501	.1656365685.3		
−.035126	.164974	7410485.2	.1657147318.6	
−.105453	.163647	6367924.1	51796.2	.1657149545.4

X'_{III}	y_{III}	$y_{I'}$	$y_{II'}$	$y_{III'}$
−.1406250024.7	.3750000007.3			
−.03908741039.0	.3749981706.2	.3749974661.1		
.06302572977.3	.3749927660.3	61018.2	.3749965240.4	
.1657149545.4	.3749839123.7	81999.8	65022.6	.3749965140.4

X'_{III}	x_{III}	$x_{I'}$	$x_{II'}$	$x_{III'}$
−.1406250024.7	1.50			
−.03908741039.0	1.51	1.5138495506.5		
.06302572977.3	1.52	278531.3	1.5138345680.7	
.1657149545.4	1.53	624787.5	44615.8	1.5138345191.9

Noting the ratios of the coefficients of u and v, we select

$$X' = 4u - 3v, \quad Y' = u - 3v.$$

Next we evaluate X' and Y' for the 16 points having x-coordinates 1.50, 1.51, 1.52, 1.53 and y-coordinates 0.36, 0.37, 0.38, 0.39, as shown in Table V. Starting with the four points for which $x = 1.50$, we interpolate to find the values of y and X' corresponding to $Y' = 0$; they are $y_1 = .3750000007$, $X_1' = -.1406250025$. We proceed in the same way with the points corresponding to the other values of x; the results, as shown, are $y_2 = .3749981706$, $X_2' = -.03908741039$; $y_3 = .3749927660$, $X_3' = .06302572977$; $y_4 = .3749839124$, $X_4' = .1657149545$. (The extra digits given in Table V are to take care of rounding-off.) Finally, using these values, we interpolate to find the values of x and y corresponding to $X' = 0$, and get

$$x = 1.5138345192, \quad y = .3749965140.$$

Comparing these results with those obtained earlier, we see that they are in error by about 1 unit in the ninth place; a distinct improvement over the four correct places that could have been secured without using this device. Note that if we had not had our earlier results for comparison, a check could have been obtained by carrying through the interpolation in the reverse order; i.e., starting with fixed values of y and finding values of x and Y' corresponding to $X' = 0$.

As in the case of one equation in one unknown, derivatives could be brought into the interpolation scheme, permitting greater accuracy with fewer points. But the derivatives needed would be $\frac{\partial x}{\partial X'}, \frac{\partial x}{\partial Y'}, \frac{\partial^2 x}{\partial X' \partial Y'}$,etc., and the general setup would be rather awkward, so that extra labor would probably be required.

9. Three or more equations. The methods discussed in this section are readily extended to the solution of three or more simultaneous equations in an equal number of unknowns. For example, if we are given three equations of the form

$$u(x, y, z) = 0, \quad v(x, y, z) = 0, \quad w(x, y, z) = 0,$$

we define new variables

$$X = \begin{vmatrix} u & v & w \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix}, \qquad Y = \begin{vmatrix} u_x & v_x & w_x \\ u & v & w \\ u_z & v_z & w_z \end{vmatrix}, \qquad Z \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u & v & w \end{vmatrix}$$

which are analogous to the X and Y of (8); from this point on the work is practically the same as before.

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