ON THE COMPOUND AND GENERALIZED POISSON DISTRIBUTIONS

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1. Summary. In this note we deduce several properties of the compound and generalized Poisson distributions; in particular their closure and divisibility properties. An infinite class of functions whose members are both compound and generalized Poisson distributions is exhibited, and several of the distributions of Neyman, Polya, etc. are identified. The present note stems from a paper by Feller [2].

2. The compound Poisson distribution. If $F(x \mid a)$ is a family of distribution functions depending on the parameter a, and $U(a)$ is a distribution function such that it assigns zero probability to any a domain for which $F(x|a)$ is undefined, then

$$G(x) = \int_{-\infty}^{\infty} F(x \mid a)\, dU(a)$$

is a distribution function. In particular if $F(x|a)$ is the Poisson distribution with mean a, and $U(0) = 0$, $G(x)$ is called the *compound Poisson distribution* associated with the distribution function $U(a)$; cf. Feller [2]. Clearly $G(x)$ is a step function over the non-negative integers, the saltus at the point $x = n$ being

$$\pi_n = \int_0^{\infty} e^{-a} \frac{a^n}{n!}\, dU(a), \qquad n = 0, 1, 2, \cdots .$$

It is convenient to introduce the factorial moment generating function (f.m.g.f.) for $G(x)$ as follows

$$\begin{aligned} \omega(z) &= E((1 + z)^x) = \sum_{n=0}^{\infty} \pi_n (1 + z)^n \\ &= \int_0^{\infty} e^{+az}\, dU(a) \\ &= \phi(z) \end{aligned}$$

where $\phi(z)$ is the ordinary moment generating function (m.g.f.) for $U(a)$. This gives a convenient relationship between the moments of $U(a)$ and its associated compound Poisson distribution.

On account of the multiplicative properties of $\omega(z)$ and $\phi(z)$ under the convolution of $G(x)$ and $U(a)$ respectively, it is seen that the compound Poisson distributions form a closed family, and if $G_1(x)$ and $G_2(x)$ are two compound Poisson distributions associated with $U_1(a)$ and $U_2(a)$ respectively then $G_1(x)_* G_2(x)$ is associated with $U_1(a)_* U_2(a)$. In addition, if $U(a)$ is infinitely divisible (cf. Cramér [1]) then $G(x)$ is also, since it can be factored into the convolution of arbitrarily many compound Poisson distributions.

Choosing in particular $U(a)$ as the Pearson type III distribution, the associated function is the Polya-Eggenberger distribution, and if $U(a)$ is a Poisson distribution the associated function is the Neyman contagious distribution of Type A.

3. The generalized Poisson distribution. If $F(x \mid a)$, defined for non-negative integers $a = 0, 1, 2, \cdots$, is the a-fold convolution of a given distribution $F(x)$ with itself, i.e. $F(x|a) = F(x)^{*a}$, and $U(a)$ is the Poisson distribution with parameter α, then the distribution function

$$G(x) = \int_0^\infty F(x \mid a)\, dU(a)$$

is called the *generalized Poisson distribution* associated with $F(x)$.

If $\Omega(z)$ is the f.m.g.f. of $U(a)$ then for the f.m.g.f. of $G(x)$ we have

$$\begin{aligned}\omega(z) &= \sum_{n=0}^{\infty} (\Omega(z))^n e^{-\alpha} \frac{\alpha^n}{n!} \\ &= e^{\alpha(\Omega(z)-1)}.\end{aligned}$$

It follows that $\omega(z)$ can be written as $\prod_{\nu=1}^{\infty} \omega_\nu(z)$ where $\omega_\nu(z)$ is a generalized Poisson distribution, and thus $\omega(z)$ belongs to the infinitely divisible family. Moreover, if $G_1(x)$ and $G_2(x)$ are two generalized Poisson distributions associated with $U_1(a)$ and $U_2(a)$ with parameters α_1 and α_2 respectively, then $G(x) = G_1(x)_* G_2(x)$ has for f.m.g.f.

$$\omega_1(z)\omega_2(z) = \exp\left\{(\alpha_1 + \alpha_2)\left(\frac{\alpha_1 \Omega_1(z) + \alpha_2 \Omega_2(z)}{\alpha_1 + \alpha_2} - 1\right)\right\},$$

and $G(x)$ is again a generalized Poisson distribution function associated with the distribution

$$U(a) = \frac{\alpha_1 U_1(a) + \alpha_2 U_2(a)}{\alpha_1 + \alpha_2}$$

and with the parameter $\alpha_1 + \alpha_2$. Thus the generalized Poisson distributions form a closed family. The analytic nature of the generalized Poisson distributions have been studied by Hartman and Wintner [3]. As noted by Feller [2] the various Neyman contagious distributions are generalized Poisson distributions.

4. Further remarks. From the above observations it is clear that a necessary and sufficient condition for a distribution to be a compound Poisson distribution is that its f.m.g.f. be of the form

$$\omega_1(z) = \phi(z) \tag{1}$$

where $\phi(z)$ is the ordinary m.g.f. of a non-negative random variable. Likewise a necessary and sufficient condition for $\omega(z)$ to be the f.m.g.f. of a generalized Poisson distribution is that it be of the form

$$\omega_2(z) = e^{\alpha(\Omega(z)-1)}, \qquad \alpha > 0, \tag{2}$$

where $\Omega(z)$ is the f.m.g.f. of an arbitrary distribution function $F(x)$. If we choose $\phi(z) = e^{\alpha(e^{cz}-1)}$ and $\Omega(z) = e^{cz}$, then $\omega_1(z) = \omega_2(z)$, and the distribution whose f.m.g.f. is $\omega_1(z)$ (the Neyman contagious distribution of Type A) is simultaneously a compound and a generalized Poisson distribution (cf. Feller [2]). We now show that there is an infinite class of distributions with this property.

First note that if $\phi(z)$ is the m.g.f. of an arbitrary distribution, then exp $\{\alpha(\phi(z) - 1)\}$ is also the m.g.f. of a d.f., and in fact is the m.g.f. of the generalized Poisson distribution associated with the distribution whose m.g.f. is $\phi(z)$. Now let $\phi(z)$ be the m.g.f. of an arbitrary non-negative random variable, and define

$$\omega(z) = \exp\{\alpha(\phi(z) - 1)\} \qquad \alpha > 0. \tag{3}$$

Then $\omega(z)$ is simultaneously of the forms (1) and (2), since $\phi(z)$ is, by (1), also the f.m.g.f. of a distribution function, i.e. the compound Poisson distribution associated with the distribution whose m.g.f. is $\phi(z)$. However, not every distribution which is both a compound and a generalized Poisson distribution can be generated in this manner. For example, the Polya-Eggenberger distribution is easily shown to be both a generalized and a compound Poisson distribution, yet its f.m.g.f.

$$\omega(z) = (1 - dz)^{-h/d}, \qquad d > 0, h > 0,$$

manifestly is not of the form (3), since this would imply $\phi(iz) = 1 - \frac{h}{\alpha d} \log (1 - diz)$ is a characteristic function. But $|\phi(iz)|$ is unbounded as $z \to \pm\infty$ and thus is not the characteristic function of a distribution.

REFERENCES

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[2] W. Feller, "On a general class of contagious distributions," *Annals of Math. Stat.*, Vol. 14 (1943), pp. 389–400.
[3] P. Hartman and A. Wintner, "On the infinitesimal generators of integral convolutions," *Am. Jour. of Math.*, Vol. 64 (1942), pp. 272–279.

ON CONFIDENCE LIMITS FOR QUANTILES

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In finding confidence limits for quantiles it is usual to determine two order statistics Z_i and Z_j which with a given probability contain the unknown quantile