# SOME APPLICATIONS OF THE MELLIN TRANSFORM IN STATISTICS

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**1. Summary.** It is well known that the Fourier transform is a powerful analytical tool in studying the distribution of sums of independent random variables. In this paper it is pointed out that the Mellin transform is a natural analytical tool to use in studying the distribution of products and quotients of independent random variables. Formulae are given for determining the probability density functions of the product and the quotient $\frac{\xi}{\eta}$, where $\xi$ and $\eta$ are independent positive random variables with p.d.f.'s $f(x)$ and $g(y)$, in terms of the Mellin transforms $F(s) = \int_0^\infty f(x)\, x^{s-1}\, dx$ and $G(s) = \int_0^\infty g(y) y^{s-1}\, dy$. An extension of the transform technique to random variables which are not everywhere positive is given. A number of examples including Student's $t$-distribution and Snedecor's $F$-distribution are worked out by the technique of this paper.

**2. Introduction.** It is well known [2], [3] that the Fourier transform is a useful analytical tool for studying the distribution of the sums of independent random variables. It is our purpose in this paper to study another transform which is useful in studying the distribution of the product of independent random variables. While it is perfectly true that one can reduce the study of the distribution of the random variable $\xi = \xi_1 \cdot \xi_2 \cdots \xi_n$, the product of $n$ independent random variables $\xi_1, \xi_2, \cdots, \xi_n$, to the study of the distribution of the random variable $\eta = \log \xi = \log \xi_1 + \log \xi_2 + \cdots + \log \xi_n$, the sum of $n$ independent random variables, it seems worth while to study the distribution problem directly. There are advantages inherent in the direct attack on the distribution problem which are lost to a considerable degree, if the problem is so transformed that the Fourier transform becomes applicable. In this paper we shall show that the direct application of the Mellin transform to the study of the distribution of products of independent random variables yields results of interest.

**3. Connection between Mellin transforms and products of independent random variables.** The key reason for the importance of Fourier transforms in studying the distribution of sums of independent random variables depends on the following result: if $\xi_1$ and $\xi_2$ are independent random variables with continuous[1] probability density functions, (henceforth abbreviated as p.d.f.), $f_1(x)$ and $f_2(x)$, respectively, then the p.d.f. $f(x)$ of the random variable $\xi = \xi_1 + \xi_2$ is expressible[1] as

$$(1) \qquad f(x) = \int_{-\infty}^{\infty} f_1(x - y) f_2(y)\, dy = \int_{-\infty}^{\infty} f_2(x - y) f_1(y)\, dy.$$

[1] In this paper we shall assume throughout that we are dealing with random variables with continuous p.d.f.'s. The argument can be extended with some changes to distribution functions which are perfectly general, but for simplicity this will not be done here.

But since these expressions are just the Fourier convolutions of $f_1(x)$ and $f_2(x)$, it is small wonder that the Fourier transform plays such a basic role in studying the distribution properties of sums of independent random variables.

Consider now the following result for products of independent random variables (4), (5): if $\xi_1$ is a random variable with continuous p.d.f. $f_1(x)$ and $\xi_2$, independent of $\xi_1$, is a positive random variable with continuous p.d.f. $f_2(x)$, then the p.d.f. $f(x)$ of the random variable $\xi = \xi_1\xi_2$ is expressible[2] as

$$f(x) = \int_0^\infty \frac{1}{y} f_1\left(\frac{x}{y}\right) f_2(y)\, dy. \tag{2}$$

But equation (2) is precisely in the form of a Mellin convolution of $f_1(x)$ and $f_2(x)$ and therefore it may be expected that the Mellin transform should be useful in studying the distribution of products of independent random variables.

It is useful to indicate briefly the properties of the Mellin transform. A detailed treatment of this transform will be found in [6] and we shall, therefore, stress only those portions of the theory of Mellin transforms which are of importance in the field of statistics. By definition, the Mellin transform $F(s)$, corresponding to a function $f(x)$ defined only[3] for $x \geq 0$, is

$$F(s) = \int_0^\infty f(x)x^{s-1}\, dx. \tag{3}$$

Under certain restrictions on $f(x)$ [6, p. 47], $F(s)$ considered as a function of the complex variable $s$ is a function of exponential type, analytic in a strip parallel to the imaginary axis. The width of the strip is governed by the order of magnitude of $f(x)$ in the neighborhood of the origin and for large values of $x$ and, in particular, the strip of analyticity becomes a half-plane if $f(x)$ decays exponentially as $x \to \infty$. There is a reciprocal formula enabling one to go from the transform $F(s)$ to the function $f(x)$. This transformation is:

$$f(x) = \frac{1}{2\pi i} \int_{c-i,\infty}^{c+i,\infty} x^{-s} F(s)\, ds \tag{4}$$

for all $x$ where $f(x)$ is continuous and where the path of integration is any line parallel to the imaginary axis and lying within the strip of analyticity of $F(s)$.

[2] More generally [4, p. 411], if $\xi_1$ and $\xi_2$ are independent random variables with continuous p.d.f.'s $f_1(x)$ and $f_2(x)$, then the p.d.f. of the random variable $\xi = \xi_1\xi_2$ is expressible as:

$$f(x) = \int_{-\infty}^\infty \frac{1}{|y|} f_1\left(\frac{x}{y}\right) f_2(y)\, dy = \int_{-\infty}^\infty \frac{1}{|y|} f_2\left(\frac{x}{y}\right) f_1(y)\, dy. \tag{2'}$$

In [4] analogous results are given for random variables with perfectly general distribution functions.

[3] The reason for this restriction is that there are technical difficulties in defining a Mellin transform directly for a function defined over $(-\infty, \infty)$. In [6], for instance, the Mellin transform theory is given for functions defined only for positive values of the argument. In statistical terminology this means that we are restricting ourselves for the moment to positive random variables. This is, of course, an unnatural restriction and we shall indicate later in the paper a simple device for treating such questions.

If, in particular, we are interested in applying Mellin transforms to p.d.f.'s of positive[4] random variables, the analysis can be carried out rigorously. Also, as in the case of the Fourier transform, one has the desirable property that there is a one-one correspondence between p.d.f.'s and their transforms.

A number of common p.d.f.'s of positive random variables have simple Mellin transforms. For example see Table 1.

In terms familiar to the mathematical statistician, the Mellin transform of a positive random variable $\xi$ with continuous p.d.f. $f(x)$ is $E(\xi^{s-1})$, where

$$F(s) = E(\xi^{s-1}) = \int_0^\infty x^{s-1} f(x)\, dx. \tag{5}$$

The following three basic properties hold: (*i*) The positive random variable $\eta = a\,\xi$, $a > 0$ has the Mellin transform $G(s) = a^{s-1} F(s)$. This is immediate since

$$G(s) = E(\eta^{s-1}) = E(a^{s-1}\,\xi^{s-1}) = a^{s-1} F(s). \tag{6}$$

(ii) The positive random variable $\eta = \xi^{\alpha}$ has the Mellin transform $G(s) = F(\alpha s - \alpha + 1)$. To prove this we note that

$$G(s) = E(\eta^{s-1}) = E(\xi^{\alpha s - \alpha}) = F(\alpha s - \alpha + 1). \tag{7}$$

In particular if $\alpha = -1$, i.e., $\eta = \frac{1}{\xi}$, then

$$G(s) = F(-s + 2).$$

This is a result which we shall have occasion to use later in the paper.

(iii) If $\xi_1$ and $\xi_2$ are independent positive random variables with Mellin transforms $F_1(s)$ and $F_2(s)$, respectively, then the Mellin transform of the product $\eta = \xi_1\xi_2$ is $G(s) = F_1(s)\, F_2(s)$. This is immediate since

$$\begin{aligned} G(s) = E(\eta^{s-1}) = E[(\xi_1\xi_2)^{s-1}] &= E(\xi_1^{s-1})\, E(\xi_2^{s-1}) \\ &= F_1(s)\, F_2(s). \end{aligned} \tag{8}$$

More generally if $\xi_1, \xi_2, \cdots, \xi_n$ are independent positive random variables with Mellin transforms $F_1(s), F_2(s), \cdots, F_n(s)$, then the Mellin transform of the random variable $\eta = \xi_1\xi_2 \cdots \xi_n$ is $G(s) = F_1(s)\, F_2(s) \cdots F_n(s)$. This relationship is fundamental and justifies the introduction of Mellin transforms in studying products of independent random variables.

From (8) it is clear that we can find the p.d.f. $g(y)$ of the random variable $\eta$ which is the product of two positive independent random variables $\xi_1$ and $\xi_2$ with continuous p.d.f.'s $f_1(x)$ and $f_2(x)$. In fact, by the Mellin inversion formula

$$g(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} G(s)\, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} F_1(s) F_2(s)\, ds, \tag{9}$$

[4] See footnote 3.

TABLE 1

| | p.d.f. | Mellin Transform | Region of Analyticity of Transform |
|---|---|---|---|
| (a) | $f(x) = 1, \quad 0 \leq x \leq 1$ <br> $= 0$, elsewhere | $F(s) = \frac{1}{s}$ | Half-plane, Re $(s) > 0$ |
| (b) | $f(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}, \; 0 < x < \infty$ <br> $\alpha > -1$ | $F(s) = \frac{\Gamma(\alpha+s)}{\Gamma(\alpha+1)}$ | Half-plane, Re $(s) > -\alpha$ |
| (c) | $f(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^\alpha(1-x)^\beta, \quad 0 < x < 1$ <br> $= 0$, elsewhere <br> $\alpha > -1, \beta > -1$ | $F(s) = \frac{\Gamma(\alpha+\beta+2)\Gamma(\alpha+s)}{\Gamma(\alpha+\beta+s+1)\Gamma(\alpha+1)}$ | Half-plane, Re $(s) > -\alpha$ |
| (d) | $f(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)} \frac{x^\alpha}{(1+x)^\beta}, \quad 0 < x < \infty$ <br> $\alpha > -1, \beta - \alpha > 1$ | $F(s) = \frac{\Gamma(\alpha+s)\Gamma(\beta-\alpha-s)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)}$ | Strip, $-\alpha <$ Re $(s) < \beta - \alpha$ |

where the path of integration is any line parallel to the imaginary axis and lying within the strip of analyticity of $G(s)$. As in the case of characteristic functions, it can be shown that there is a one-one correspondence between p.d.f.'s and their Mellin transforms. Therefore, it follows that the p.d.f. $g(y)$ computed in this way must be precisely equal to

$$(10) \qquad g(y) = \int_0^\infty \frac{1}{x} f_1\left(\frac{y}{x}\right) f_2(x)\, dx = \int_0^\infty \frac{1}{x} f_2\left(\frac{y}{x}\right) f_1(x)\, dx.$$

It is easy to verify this directly by showing that the Mellin transform of the right-hand side of (10) is $F_1(s)\, F_2(s)$ [6, p. 52], but this will not be done here. The essential point is that Equation (9), (which is sometimes easier to evaluate than Equation (10)), is a consequence of an algebraic formalism which is capable of revealing relationships which would otherwise remain hidden.

The p.d.f. $h(y)$ of $\eta = \frac{\xi_1}{\xi_2}$, the ratio of two positive random variables with continuous p.d.f.'s, can be reduced to finding the p.d.f. of the product of independent random variables $\xi_1$ and $\frac{1}{\xi_2}$. If $F_1(s)$ and $F_2(s)$ are the Mellin transform corresponding to $\xi_1$ and $\xi_2$, respectively, then by (ii) $F_2(-s + 2)$ is the Mellin transform of $\frac{1}{\xi_2}$ and, therefore, the Mellin transform $H(s)$ of $\eta = \frac{\xi_1}{\xi_2}$ is $F_1(s)\, F_2(-s + 2)$. Therefore, the p.d.f. $h(y)$ of $\eta$ is

$$(11) \quad h(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} H(s)\, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} F_1(s) F_2(-s + 2)\, ds.$$

This formula is useful in finding distributions such as Student's $t$ and Fisher's $z$.

**4. A modified Mellin transform procedure for finding the distribution of the product of independent random variables which are not everywhere positive.** Up to this point we have limited ourselves to the application of the Mellin transform to finding the distribution of the product or ratio of two positive independent random variables. While it is true that a number of interesting probability density functions are defined only for positive[5] values of the argument, it is certainly desirable that we be able to treat situations involving random variables capable of taking on both positive and negative values. A simple device for extending the Mellin transform treatment to the more general problem is to decompose the p.d.f.'s $f_1(x)$ and $f_2(x)$ of the independent random variables $\xi_1$ and $\xi_2$ into

$$f_1(x) = f_{11}(x) + f_{12}(x),$$
$$f_2(x) = f_{21}(x) + f_{22}(x),$$

[5] For example, distributions of type 3, the $\chi^2$ distribution, the distribution of the sample standard deviation and sample variance, the distribution of an even power of a random variable, etc. are all defined only for positive values of the argument.

where[6]

$$f_{11}(x) = 0, x < 0, \qquad f_{12}(x) = 0, x > 0,$$

$$f_{21}(x) = 0, x < 0, \qquad f_{22}(x) = 0, x > 0,$$

and then to operate on the pairs $[f_{11}(x), f_{21}(x)]$, $[f_{11}(x), f_{22}(x)]$, $[f_{12}(x), f_{21}(x)]$, and $[f_{12}(x), f_{22}(x)]$ separately. More specifically, the frequency distribution $h(y)$ corresponding to the random variable $\eta = \xi_1\xi_2$ is made up of the sum of four components $h_1(y)$, $h_2(y)$, $h_3(y)$, and $h_4(y)$. To compute $h_1(y)$ one can apply the Mellin transform directly to the evaluation of the expression

$$h_1(y) = \int_0^\infty \frac{1}{x} f_{11}\left(\frac{y}{x}\right) f_{21}(x)\, dx,$$

since both $f_{11}(x)$ and $f_{21}(x)$ are zero for negative values of $x$. The function $h_1(y)$ is zero for $y < 0$. To compute $h_2(y)$ we first evaluate

$$h_2^*(y) = \int_0^\infty \frac{1}{x} f_{11}\left(\frac{y}{x}\right) f_{22}(-x)\, dx.$$

Again $f_{11}(x)$ and $f_{22}(-x)$ are zero for negative values of $x$ and, therefore, the conventional Mellin transform can be applied in determining $h_2^*(y)$. It is clear that $h_2^*(y) = 0$ for $y < 0$ and, therefore, $h_2(y) = h_2^*(-y) = 0$ for $y > 0$. Similarly, one can find $h_3(y)$ and $h_4(y)$ where $h_3(y) = 0$ for $y > 0$ and $h_4(y) = 0$ for $y < 0$, and it is readily seen that[7]

$$h(y) = h_1(y) + h_2(y) + h_3(y) + h_4(y)$$

is the desired p.d.f. of $\eta = \xi_1\xi_2$.

**5. Examples of use of Mellin transforms in evaluating the product and quotient of independent random variables.** Example 1: *The distribution of* $\eta = \xi_1\xi_2$, *where* $\xi_1$ *and* $\xi_2$ *are independent random variables with p.d.f.'s* $f_1(x)$ *and* $f_2(x)$, *respectively, where*

$$f_1(x) = f_2(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < x < \infty.$$

In this case

$$f_1(x) = f_{11}(x) + f_{12}(x),$$

and

$$f_2(x) = f_{21}(x) + f_{22}(x),$$

[6] Of course, $f_{11}$, $f_{12}$, $f_{21}$, and $f_{22}$ are generally not p.d.f.'s since $\int_0^\infty f_{11}(x)\, dx$, $\int_{-\infty}^0 f_{12}(x)\, dx$, $\int_0^\infty f_{21}(x)\, dx$, $\int_{-\infty}^0 f_{22}(x)\, dx$ are no longer necessarily equal to one.

[7] As in footnote 6, $h_1$, $h_2$, $h_3$, and $h_4$ are, in general, not p.d.f.'s.

where

$$f_{11}(x) = 0, x < 0; f_{12}(x) = 0, x > 0;$$
$$f_{21}(x) = 0, x < 0; f_{22}(x) = 0, x > 0.$$

The random variable $\eta = \xi_1\xi_2$ has a p.d.f. $h(y) = h_1(y) + h_2(y) + h_3(y) + h_4(y)$ where

$h_1(y)$ is associated with $[f_{11}(x), f_{21}(x)]$,
$h_2(y)$ is associated with $[f_{11}(x), f_{22}(x)]$,
$h_3(y)$ is associated with $[f_{12}(x), f_{21}(x)]$,
and $h_4(y)$ is associated with $[f_{12}(x), f_{22}(x)]$.

It is sufficient to evaluate

$$h_1(y) = \int_0^\infty \frac{1}{x} f_{11}\left(\frac{y}{x}\right) f_{21}(x)\, dx.$$
$$= \int_0^\infty \frac{1}{x} f_{21}\left(\frac{y}{x}\right) f_{11}(x)\, dx.$$

In this case

$$F_{11}(s) = \int_0^\infty x^{s-1} f_{11}(x)\, dx = \int_0^\infty x^{s-1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{2^{\frac{1}{2}(s-3)}}{\sqrt{\pi}} \Gamma(s/2),$$

analytic for $\mathrm{Re}(s) > 0$
and

$$F_{21}(s) = \int_0^\infty x^{s-1} f_{21}(x)\, dx = \frac{2^{\frac{1}{2}(s-3)}}{\sqrt{\pi}} \Gamma(s/2).$$

Therefore,

$$H_1(s) = F_{11}(s) F_{21}(s) = \frac{2^{s-3}}{\pi} \Gamma^2(s/2)$$

$$h_1(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} H_1(s)\, ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} \frac{2^{s-3}}{\pi} \Gamma^2(s/2)\, ds, \qquad c > 0$$

$$= \frac{1}{2\pi} K_0(y), \qquad y > 0 \quad [6, \text{p. } 197]$$

where $K_0(y)$ is Bessel's function of the second kind with a purely imaginary argument of zero order. Similarly

$$h_2(y) = \frac{1}{2\pi} K_0(y), \qquad y < 0$$

$$h_3(y) = \frac{1}{2\pi} K_0(y), \qquad y < 0$$

$$h_4(y) = \frac{1}{2\pi} K_0(y), \qquad y > 0.$$

Therefore, $h(y) = h_1(y) + h_2(y) + h_3(y) + h_4(y)$

$$= \frac{1}{\pi} K_0(y), \qquad -\infty < y < \infty,$$

and this is the desired p.d.f. This result has been found by other methods and is given in [1, p. 1].

Example 2: *The distribution of* $\eta = \frac{\xi_1}{\xi_2}$ *where* $\xi_1$ *and* $\xi_2$ *are independent random variables with p.d.f.'s* $f_1(x)$ *and* $f_2(x)$*, respectively, where*

$$f_1(x) = f_2(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < y < \infty.$$

As in Example 1, one splits the determination of $h(y)$, the p.d.f. of $\eta$, into four parts: $h_1(y)$, $h_2(y)$, $h_3(y)$, $h_4(y)$. In the notation of Example 1 it is easy to show that $H_{11}(s)$ the Mellin transform of $h_1(y)$ is

$$F_{11}(s)F_{21}(-s+2) = \frac{2^{\frac{1}{2}(s-3)}}{\sqrt{\pi}} \Gamma(s/2) \frac{2^{\frac{1}{2}(s-3)}}{\sqrt{\pi}} \Gamma(-s/2+1) = \frac{1}{4} \frac{1}{\sin \frac{s\pi}{2}};$$

$$h_1(y) = \frac{1}{2\pi i} \int_{c-i,\infty}^{c+i,\infty} y^{-s} H(s)\, ds, \qquad 0 < c < 2,$$

$$= \frac{1}{2\pi i} \int_{c-i,\infty}^{c+i,\infty} \frac{1}{4} \frac{y^{-s}\, ds}{\sin \frac{s\pi}{2}}$$

$$= \frac{1}{2\pi} \frac{1}{1+y^2}, \qquad y \geq 0.$$

Similarly

$$h_2(y) = \frac{1}{2\pi} \frac{1}{1+y^2}, \qquad y \leq 0,$$

$$h_3(y) = \frac{1}{2\pi} \frac{1}{1+y^2}, \qquad y \leq 0,$$

$$h_4(y) = \frac{1}{2\pi} \frac{1}{1+y^2}, \qquad y \geq 0.$$

Therefore, $h(y) = h_1(y) + h_2(y) + h_3(y) + h_4(y)$

$$= \frac{1}{\pi} \frac{1}{1+y^2}, \qquad -\infty < y < \infty.$$

This result has been found by other methods and given in [4, p. 411].

Example 3: *F-Distribution.* Let $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n$ be $(m+n)$ independent random variables, each normally distributed with mean zero and standard deviation $\sigma$. Let

$$\xi = \sum_{i=1}^{m} \xi_i^2, \qquad \eta = \sum_{j=1}^{n} \eta_j^2.$$

We want to find the p.d.f. $h(z)$ of $\zeta$ where $\zeta = \xi/\eta$. The p.d.f.'s $f(x)$ and $g(y)$ of $\xi$ and $\eta$, respectively, are:

$$f(x) = \frac{x^{m/2-1}\, e^{-x/2\sigma^2}}{2^{m/2}\sigma^m\Gamma(m/2)}, \qquad x > 0,$$

and

$$g(y) = \frac{y^{n/2-1}\, e^{-y/2\sigma^2}}{2^{n/2}\,\sigma^n\,\Gamma(n/2)}, \qquad y > 0.$$

In this case

$$F(s) = \frac{2^{s-1}\sigma^{2s-2}\Gamma\left(s + \frac{m}{2} - 1\right)}{\Gamma(m/2)}, \qquad \text{analytic for Re } (s) > 1 - \frac{m}{2},$$

and

$$G(s) = \frac{2^{s-1}\sigma^{2s-2}\Gamma\left(s + \frac{n}{2} - 1\right)}{\Gamma(n/2)}, \qquad \text{analytic for Re } (s) > 1 - \frac{n}{2}.$$

The p.d.f. $h(z)$ has Mellin transform

$$H(s) = F(s)\, G(-s + 2)$$
$$= \frac{\Gamma\left(s + \frac{m}{2} - 1\right)\Gamma\left(-s + \frac{n}{2} + 1\right)}{\Gamma(m/2)\Gamma(n/2)}.$$

Therefore,

$$h(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s} H(s)\, dz, \qquad -\frac{m}{2} + 1 < c < \frac{n}{2} + 1,$$
$$= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma(m/2)\Gamma(n/2)}\, \frac{z^{m/2-1}}{(z+1)^{\frac{1}{2}(m+n)}}, \qquad z > 0.$$

A convenient way of carrying out the inversion is to use formula ($d$) in Table 1.

In a similar way one can find Student's distribution, i.e., the distribution of $\zeta = \xi_0/\eta$, where $\eta = \sqrt{\sum_1^n \xi_\nu^2/n}$, and where $\xi_0, \xi_1, \cdots, \xi_n$ are $n + 1$ independent random variables each having the distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2}, \qquad -\infty < x < \infty.$$

It should be mentioned in conclusion that the Mellin transform is a natural tool to use in situations involving the products and quotients of independent uniformly distributed random variables, or in finding products and/or quotients and/or Beta-distribution. In such cases formulae (b), (c) and (d) in Table 1 are useful.

## REFERENCES

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