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THE FOURTH DEGREE EXPONENTIAL DISTRIBUTION FUNCTION[1]

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We shall derive a recursion formula for the moments of the fourth degree exponential distribution function, state its more characteristic features, and show how the graduation of observed distributions may be accomplished by the method of moments and the method of maximum likelihood. The purpose of the note is to make possible a wider use of this function.

R. A. Fisher [1] introduced the fourth degree exponential function

$$y_t = k \exp\{-(\beta_4 t^4 + \beta_3 t^3 + \beta_2 t^2 + \beta_1 t)\}, \tag{1}$$

where $r_1 \le t \le r_2$, $t = (x - m)/\sigma$, m indicates the population mean, σ the population standard deviation, and where the β's are functions of

$$\alpha_n = \int_{r_1}^{r_2} t^n y_t \, dt.$$

A. L. O'Toole in two stimulating papers [2], [3], has studied (1); however his methods and results are unnecessarily complicated. O'Toole requires eight moments to determine parameters similar to the β's. Both Fisher and O'Toole considered the restricted class of (1) with range $(-\infty, \infty)$.

Let

$$u = t^n \exp\{-(\beta_4 t^4 + \beta_3 t^3 + \beta_2 t^2)\}, \; dv = e^{-\beta_1 t} \, dt \tag{2}$$

in

$$\alpha_n = \int_{r_1}^{r_2} t^n y_t \, dt, \quad \text{obtaining} \tag{3}$$

$$4\beta_4 \alpha_{n+3} + 3\beta_3 \alpha_{n+2} + 2\beta_2 \alpha_{n+1} + \beta_1 \alpha_n = n\alpha_{n-1}, \qquad n = 1, 2, 3, \cdots, \tag{4}$$

[1] Presented to the American Mathematical Society and the Institute of Mathematical Statistics, September 4, 1947.

and for $n = 0$, the right side of (4) is defined as zero. The result (4) is valid under the assumption

$$uv\Big]_{r_1}^{r_2} = 0. \tag{5}$$

Given the first six moments, β_4, β_3, β_2, β_1 are readily determined. It will be found that if $\beta_4 > 0$, $\beta_3 \neq 0$, then $r_1 = -\infty$, $r_2 = \infty$; while if $\beta_4 < 0$, and $\beta_3 \neq 0$, r_1 and r_2 will be finite. If we set $n = 0, 1, 2, 3$, in (4), the solutions are

$$\begin{aligned}
\beta_4 &= \{\alpha_3(\alpha_5 - 4\alpha_3) - (\alpha_4 - 3)(\alpha_4 - 1)\} \div 4D;\\
\beta_3 &= \{-\alpha_3(\alpha_6 - 3\alpha_4 - \alpha_3^2) + (\alpha_5 - \alpha_3)(\alpha_4 - 3)\} \div 3D;\\
\beta_2 &= \{(\alpha_3 - \alpha_5)(\alpha_5 - 4\alpha_3) + (\alpha_4 - 1)(\alpha_6 - \alpha_3^2 - 3\alpha_4)\} \div 2D;\\
\beta_1 &= \{\alpha_3(\alpha_6 - \alpha_3\alpha_5 - 3\alpha_4 + 3\alpha_3^2) - (\alpha_4 - 3)(\alpha_5 - \alpha_3\alpha_4)\} \div D,
\end{aligned} \tag{6}$$

where

$$D = (\alpha_6 - \alpha_4^2 - \alpha_3^2)(\alpha_4 - \alpha_3^2 - 1) - (\alpha_5 - \alpha_3 - \alpha_3\alpha_4)^2 \geqq 0.$$

To prove $D \geqq 0$ we adopt the method of J. E. Wilkins Jr. [4]. In only a trivial case is $D = 0$. Let

$$G(a, b, c, d) = \int_{r_1}^{r_2} (a + bt + ct^2 + dt^3)^2 y_t \, dt \geqq 0,$$

where y_t is any probability function with range $r_1 \leqq t \leqq r_2$. Since $G(a, b, c, d)$ is a semi-definite quadratic form, its discriminant will be non-negative. But its discriminant is easily seen to be equal to D, thus

$$D = \begin{vmatrix} \alpha_3 & 1 & 0 & 1 \\ \alpha_4 & \alpha_3 & 1 & 0 \\ \alpha_5 & \alpha_4 & \alpha_3 & 1 \\ \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 \end{vmatrix} \geqq 0. \tag{7}$$

We summarize without proofs the essential features of the fourth degree exponential. Near the normal point, $\alpha_4 = 3$, $\alpha_3 = 0$, the fourth degree exponential function, the Pearson system, and the Gram-Charlier Type A are essentially alike. Type C [5] while similar is not the same. Note that β_4 may be negative and in such a case r_1 and r_2 are the two real zeros of the derivative of (1). The exponential may be bimodal as well as unimodal and the normal curve is the special case $\beta_4 = \beta_3 = \beta_1 = 0$. Various special cases where a particular β is zero are readily handled by either (4) or (6). The graduation of both unimodal and bimodal observed distributions will be published elsewhere.

Let

$$y_t = k \exp - \sum_{j=1}^{r} \beta_j t^j, \qquad r_1 \leq t \leq r_2, \tag{8}$$

where

(9) $$\frac{1}{k} = \int_{r_1}^{r_2} \exp - \sum_{j=1}^{r} \beta_j t^j \, dt.$$

The likelihood, L, in a sample of N is given by

(10) $$L = k^N \exp\left\{-\left\{\beta_r \sum_{i=1}^{N} t_i^r + \beta_{r-1} \sum_{i=1}^{N} t_i^{r-1} + \cdots + \beta_1 \sum_{i=1}^{N} t_i\right\}\right\}$$

where $t_i = (x_i - m)/\sigma$. Then

(11) $$\frac{\partial \log L}{L \, \partial\beta_j} = \frac{N}{k}\frac{\partial k}{\partial \beta_j} - \sum t_i^j, \quad \text{and}$$

(12) $$\frac{1}{k}\frac{\partial k}{\partial \beta_j} = k\left\{\int_{r_1}^{r_2} t^j \exp\left\{-\sum_{j=1}^{r} \beta_j t^j \, dt\right\} - \frac{\partial r_2}{\partial \beta_j} \exp\left\{-\sum_{j=1}^{r} \beta_j r_2^j\right\} + \frac{\partial r_1}{\partial \beta_j} \exp\left\{-\sum_{j=1}^{r} \beta_j r_1^j\right\}\right\}.$$

If we assume either r_1 and r_2 constant, or $\exp\left\{-\sum_{j=1}^{r} \beta_j r_2^j\right\}$ and $\exp\left\{-\sum_{j=1}^{r} \beta_j r_1^j\right\}$ negligible, then (12) becomes

$$k \int_{r_1}^{r_2} t^j \exp\left\{-\sum_{j=1}^{r} \beta_j t^j\right\} dt \quad \text{and} \quad \frac{\partial \log L}{L \, \partial \beta_j} = 0 \qquad \text{implies}$$

(13) $$\frac{\int_{r_1}^{r_2} t^j \exp\left\{-\sum_{j=1}^{r} \beta_j t^j\right\} dt}{\int_{r_1}^{r_2} \exp\left\{-\sum_{j=1}^{r} \beta_j t^j\right\} dt} = \frac{\sum t_i^j}{N} = a_j, \qquad j = 1, 2, \cdots r,$$

where a_j is the sample estimate of α_j. For, if in $\sum t_i^j/N$ we let $j = 1, 2$, we find by (13) that $\bar{x} = m$, and $\sigma^2 = \sum(x_i - \bar{x})^2/N$. The solution of (13) provides estimates of β_4, β_3, β_2, and β_1, if we set $r = 4$. Naturally more time is required for the solution of (13) as compared with the method of moments, but the maximum likelihood estimates are asymptotically efficient. The system (13) must be solved by successive approximations. To determine the moments solution all we do is to replace α_j by a_j in equations (6). This affords a point of departure from which the maximum likelihood equations may be solved. The two methods are not the same.

The fourth degree exponential is readily generalized to a fourth (or rth) degree multivariate function including the normal multivariate function as a special case.

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AN APPROXIMATION TO THE BINOMIAL SUMMATION

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We consider the binomial expansion $(q + p)^n$, where $q = 1 - p$ and n is a positive integer. For given values of n, p, r, and s, where $np < r < s \leq n$, we are often interested in the probability $P(r \leq x \leq s)$ that the number of successes x will satisfy $r \leq x \leq s$.

When n does not exceed 50, we can use tables of the Incomplete Beta Function, or other convenient and accurate tables. For "large" values of n, we can use normal tables. When p is "small", we can use Poisson tables. However, it is often true that p is fairly small, and yet not small enough to give really accurate results when Poisson tables are employed in the usual way, while n is too large for use of the tables of the Incomplete Beta Function and yet too small for accurate use of normal tables.

It frequently happens that an upper bound for $P(r \leq x \leq s)$ would serve our purpose. We propose to show how to find this from Poisson tables with greater accuracy than could be obtained by using these tables in the ordinary way.

We shall denote the general term of the binomial expansion by $B_i = \binom{n}{i} p^i q^{n-i}$ and the general term of the corresponding Poisson distribution with the same value of p by $P_i = (pn)^i e^{-pn}/i!$. We shall also consider a second Poisson distribution whose general term is given by $P'_i = (p'n)^i e^{-p'n}/i!$, where $p' \neq p$ will be determined later.

We shall use the following notations:

$$U_i = B_{i+1}/B_i = (n - i)(p)/(i + 1)(1 - p); \tag{1}$$

$$V_i = P_{i+1}/P_i = pn/(i + 1); \tag{2}$$

$$V'_i = P'_{i+1}/P'_i = p'n/(i + 1); \tag{3}$$

$$U_i - V_i = p(np - i)/(i + 1)(1 - p). \tag{4}$$

From (4) we obtain at once the following:

Lemma I. *$U_i > V_i$ or $U_i < V_i$ according as $i < np$ or $i > np$.*

Thus, the size of the general term of the binomial expansion falls off more steeply to the right of $i = np$ than does that of the general Poisson term.