TESTING COMPOUND SYMMETRY IN A NORMAL MULTIVARIATE DISTRIBUTION

By David F. Votaw, Jr.

Yale University

Summary. In this paper test criteria are developed for testing hypotheses of "compound symmetry" in a normal multivariate population of t variates ($t \geq 3$) on basis of samples. A feature common to the twelve hypotheses considered is that the set of t variates is partitioned into mutually exclusive subsets of variates. In regard to the partitioning, the twelve hypotheses can be divided into two contrasting but very similar types, and the six in one type can be paired off in a natural way with the six in the other type. Three of the hypotheses within a given type are associated with the case of a single sample and moreover are simple modifications of one another; the remaining three are direct extensions of the first three, respectively, to the case of k samples ($k \geq 2$). The gist of any of the hypotheses is indicated in the following statement of one, denoted by $H_1(mvc)$: *within each subset of variates the means are equal, the variances are equal and the covariances are equal and between any two distinct subsets the covariances are equal.*

The twelve sample criteria for testing the hypotheses are developed by the Neyman-Pearson likelihood-ratio method. The following results are obtained for each criterion (assuming that the respective null hypotheses are true) for any admissible partition of the t variates into subsets and for any sample size, N, for which the criterion's distribution exists: (i) the exact moments; (ii) an identification of the exact distribution as the distribution of a product of independent beta variates; (iii) the approximate distribution for large N. Exact distributions of the single-sample criteria are given explicitly for special values of t and special partitionings.

Certain psychometric and medical research problems in which hypotheses of compound symmetry are relevant are discussed in section 1. Sections 2–6 give statements of the hypotheses and an illustration, for $H_1(mvc)$, of the technique of obtaining the moments and identifying the distributions. Results for the other criteria are given in sections 7–8. Illustrative examples showing applications of the results are given in section 9.

1. Introduction. In studying psychological examinations, or other measuring devices, one may wish to test whether several forms of an examination may be used interchangeably. Consider the case of three forms, and assume that scores of individuals on the three forms have a normal 3-variate distribution. The hypothesis of interchangeability is equivalent to the hypothesis that in the normal distribution the means are equal, the variances are equal, and the covariances are equal. When this hypothesis is true, the normal distribution is in-

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variant over all permutations of the variates and is said to have *complete symmetry*. It is frequently more important, however, not only to test that the forms have completely symmetric relations with each other but also that they are interchangeable with regard to their relation to some outside criterion measure (e.g., the criterion might be skill in a given task). Assuming that the scores of individuals on the three forms and the criterion have a normal 4-variate distribution, the hypothesis of interchangeability is equivalent to the hypothesis of equality of means, equality of variances, and equality of covariances among the three forms and equality of covariances between forms and criterion. When this hypothesis is true, the 4-variate normal distribution is invariant over all permutations of the three variates (associated with forms) among themselves, and so the variance-covariance matrix has the following form:

$$\left\| \begin{array}{c|ccc} A & C & C & C \\ \hline C & B & D & D \\ C & D & B & D \\ C & D & D & B \end{array} \right\|,$$

where the quantity A represents the variance of the criterion measure. A normal distribution for which this hypothesis is true is said to have *compound symmetry* (of type I). A more general case of compound symmetry (of type I) arises when there are several examinations (no two of which need have the same number of forms) and several outside criteria.

The hypothesis of complete symmetry may arise in certain medical-research problems. For example, suppose a measurement (e.g., $\% CO_2$ in blood) is made at each of three times (say T_1, T_2, T_3) on an experimental animal and assume that the three quantities have a normal 3-variate distribution; one may then be interested in testing the hypothesis of complete symmetry on basis of measurements (considered as a random sample) made on several experimental animals. More generally, let there be two characteristics, say U and W (e.g., $\% CO_2$ in blood and $\% O_2$ in blood), which are both measured at each of two times, T_1, T_2. Let it be assumed that the four quantities—which we represent as UT_1, UT_2, WT_1, WT_2—have a normal 4-variate distribution. One may then be interested in testing the hypothesis that the means of the first two variates are equal, the means of the second two are equal, and the variance-covariance matrix has the form:

$$\left\| \begin{array}{cc|cc} E & F & K & L \\ F & E & L & K \\ \hline K & L & G & J \\ L & K & J & G \end{array} \right\|.$$

When this hypothesis is true, the 4-variate distribution is said to have *compound symmetry* (of type II). A more general case of compound symmetry (of type II) arises when there are h characteristics and n times ($h, n = 2, 3, \cdots$).

Either of the two types of compound symmetry is a direct extension of complete symmetry. Wilks [5] has thoroughly treated the sampling theory of certain criteria for testing various hypotheses of complete symmetry regarding a normal multivariate distribution.

The problems dealt with in this paper are: (i) to give sample criteria for testing hypotheses of compound symmetry regarding a normal multivariate distribution, and (ii) to give the moments and identify the distribution of each sample criterion when the corresponding hypothesis is true.

The hypotheses are stated in section 2. Certain properties of compound symmetric normal distributions are given in section 3. Sections 4, 5, and 6 together give the method of deriving each sample criterion and the methods of obtaining the criterion's moments and identifying its distribution; the methods are illustrated for one of the hypotheses. Sections 7–8 give the other criteria and their moments together with approximate distributions of the criteria for large sample sizes. Exact distributions of some of the criteria are given in section 7g for certain special compound symmetries. Section 9 contains two illustrative examples.

2. Statements of hypotheses. Let Π be a normal t-variate population and X_i $(i = 1, \cdots, t)$ $(t \geq 3)$ be the i-th variate in Π. Let the set of variates X_1, $X_2, \cdots, X_t$ be partitioned into q mutually exclusive subsets of which, say, b subsets contain exactly one variate each and the remaining $q - b = h$ subsets (where $h \geq 1$) contain $n_1, n_2, \cdots, n_h$, variates, respectively, where $n_a \geq 2$ $(a = 1, \cdots, h; b + \sum_{a=1}^{h} n_a = t)$. No generality is lost in assuming that the t variates are ordered so that the first b belong to the b subsets containing one variate each, the next n_1 variates belong to the $(b + 1)$-th subset, $\cdots$, the last n_h variates to the q-th subset, where $n_1 \leq n_2 \leq \cdots \leq n_h$. Let $(1^b, n_1, n_2, \cdots, n_h)$ represent such a partition of the variates $X_1, \cdots, X_t$ into subsets; when $b = 0$ the term 1^b will be omitted. The notation can be simplified when $n_1, n_2, \cdots$, n_h are not all unequal; e.g., $(1^b, 2, 2)$ can be written as $(1^b, 2^2)$.

In the statement of each of the following six hypotheses it is assumed that there is a preassigned partition $(1^b, n_1, n_2, \cdots, n_h)$ of the t variates into q subsets $(q = b + h)$.

(1) $H_1(mvc)$: The hypothesis that within each subset of variates the means are equal, the variances are equal, and the covariances are equal and that between any two distinct subsets of variates the covariances are equal.

(2) $H_1(vc)$: The hypothesis that within each subset of variates the variances are equal and the covariances are equal and that between any two distinct subsets of variates the covariances are equal.

(3) $H_1(m)$: The hypothesis that within each subset of variates the means are equal, given that $H_1(vc)$ is true.

(4) $H_k(MVC \mid mvc)$: the hypothesis that k normal t-variate distributions are the same given that they all satisfy $H_1(mvc)$ for a particular division of the variates into subsets $(k \geq 2)$.

(5) $H_k(VC \mid mvc)$: The hypothesis that k normal t-variate distributions have the same variance-covariance matrix, given that they all satisfy $H_1(mvc)$ for a particular division of the variates into subsets ($k \geq 2$).

(6) $H_k(M \mid mVC)$: The hypothesis that k normal t-variate distributions are the same, given that they all satisfy $H_1(mvc)$ for a particular division of the variates into subsets and that they all have the same variance-covariance matrix ($k \geq 2$).

Any of the hypotheses stated above can be expressed in terms of an invariance condition on the normal t-variate distribution (or distributions); e.g., $H_1(mvc)$ is equivalent to the hypothesis that the distribution is invariant over all permutations of the variates within subsets. The pattern of symmetry present in the variance-covariance matrix of the distribution when any of the above six hypotheses is true is given in section 3 (see (3.2)).

Six additional hypotheses, $\bar{H}_1(mvc)$, $\bar{H}_1(vc)$, $\cdots$, $\bar{H}_k(M \mid mVC)$, which are modifications of $H_1(mvc)$, $H_1(vc)$, $\cdots$, $H_k(M \mid mVC)$, respectively, will also be considered. In regard to any of these six $\bar{H}$ hypotheses, it is assumed that there is a partition $(n^h)(n = 2, 3, \cdots)$ of the t variates ($t = nh$) and that in each subset the variates are in a given order; thus each subset has n variates and between any two distinct subsets of variates there are n^2 covariances, which form an $n \times n$ "block" in the variance-covariance matrix of the distribution (see (3.4)). The hypotheses may now be stated as follows:

$\bar{H}_1(mvc)$: The hypothesis that within each subset of variates the means are equal, the variances are equal, and the covariances are equal and that between any two distinct subsets of variates the diagonal covariances are equal and the off-diagonal covariances are equal.

$\bar{H}_1(vc)$: The hypothesis that within each subset of variates the variances are equal and the covariances are equal and that between any two distinct subsets of variates the diagonal covariances are equal and the off-diagonal covariances are equal.

The statement of any of the hypotheses $\bar{H}_1(m)$, $\bar{H}_k(MVC \mid mvc)$, $\bar{H}_k(VC \mid mvc)$, and $\bar{H}_k(M \mid mVC)$ is obtained from the statement of the corresponding H hypothesis by simply substituting $\bar{H}$ for H. The pattern of symmetry present in the variance-covariance matrix of the distribution when any of the six $\bar{H}$ hypotheses is true is given in section 3 (see (3.4)), from which the appropriate invariance condition on the normal distribution can be obtained.

A test of any of the hypotheses $H_1(mvc)$, $\bar{H}_1(mvc)$, $H_1(vc)$, $\bar{H}_1(vc)$, $H_1(m)$, $\bar{H}_1(m)$ is based on a random sample from a normal multivariate distribution; a test of any of the remaining hypotheses is based on k random samples from k normal multivariate distributions, respectively, ($k \geq 2$).

A normal distribution for which an H or $\bar{H}$ hypothesis is true will be called *compound symmetric*. In the special case where the compound symmetry holds for a partition (t) of the t variates, any H hypothesis and the $\bar{H}$ hypothesis corresponding to it are identical; in this case the normal distribution will be called *completely symmetric*. Problems (i) and (ii) (see section 1) have been solved completely by Wilks [5] for $H_1(mvc)$, $H_1(vc)$, and $H_1(m)$ for the case of complete symmetry.

3. Block symmetric matrices and vectors. Let m_i be the mean value of X_i and $\| \rho_{ij}\sigma_i\sigma_j \|$ be the variance-covariance matrix of $X_1, \cdots, X_t$ $(i, j = 1, \cdots, t)$ (ρ_{ij} is the coefficient of correlation between X_i and X_j). The joint probability density function[1] of $X_1, X_2, \cdots, X_t$ is

$$(3.1) \quad f(X_1, X_2, \cdots, X_t) = |G_{ij}|^{1/2}\pi^{-t/2} \exp\left[-\sum_{i,j} G_{ij}(X_i - m_i)(X_j - m_j)\right],$$

where $\| G_{ij} \|$ is positive definite and its inverse $\| G^{ij} \| = \| 2\,\rho_{ij}\sigma_i\sigma_j \|$.

When any of the H hypotheses is true (see section 2), we represent $\| G^{ij} \|$ by $\| A^{ij} \|$ (also $\| G_{ij} \|$ by $\| A_{ij} \|$) which can be written as (3.2) (see page 452), where $A^{ss'} = A^{s's}$ $(s, s' = 1, \cdots, b)$ and $D^{aa'} = D^{a'a}(a, a' = 1, \cdots, h; a \neq a')$. The A's and B's with single superscripts and the C's and D's have been introduced to indicate the *block pattern* clearly. In general $C^{sa} = C^{as}$ only if $a = s(s = 1, \cdots, b;\ a = 1, \cdots, h)$. $\| A_{ij} \|$ and $\| A^{ij} \|$ have the same *block pattern* of symmetry.

The blocks in (3.2) are formed by making a partition $(1^b, n_1, n_2, \cdots, n_h)$ of the t rows and t columns of $\| A^{ij} \|$. A matrix having the block pattern of symmetry of (3.2) will be called *block symmetric of type I*. Clearly a block symmetric matrix of type I is invariant over all permutations of its rows and columns within the subsets determined by $(1^b, n_1, \cdots, n_h)$, if the row interchanges and column interchanges are the same. Also, a t-component vector will be called *block symmetric* if the order of values of the components is invariant over all permutations of the components within groups determined by $(1^b, n_1, \cdots, n_h)$.

The determinant of the block symmetric matrix $\| A_{ij} \|$ is

$$(3.3) \qquad |A_{ij}| = K \prod_1^h (A_a - B_a)^{n_a - 1},$$

where

$$K = \left|\begin{array}{cccc|cccc} & & & & C'_{11} & C'_{12} & \cdots & C'_{1h} \\ & & & & \cdot & \cdot & \cdot & \cdot \\ & A_{ss'} & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & C'_{b1} & C'_{b2} & \cdots & C'_{bh} \\ \hline C'_{11} & C'_{21} & \cdots & C'_{b1} & A'_1 & D'_{12} & \cdots & D'_{1h} \\ C'_{12} & C'_{22} & \cdots & C'_{b2} & D'_{21} & A'_2 & \cdots & D'_{2h} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C'_{1h} & C'_{2h} & \cdots & C'_{bh} & D'_{h1} & D'_{h2} & \cdots & A'_h \end{array}\right|,$$

[1] In general a chance quantity and the variable of its distribution function will be denoted by the same symbol.

$$
(3.2) \qquad \| A^{ij} \| = \left\| \begin{array}{cccc|cccc|cccc|c|cccc}
A^{11} & A^{12} & \cdots & A^{1b} & C^{11} & C^{11} & \cdots & C^{11} & C^{12} & C^{12} & \cdots & C^{12} & & C^{1h} & C^{1h} & \cdots & C^{1h} \\
A^{21} & A^{22} & \cdots & A^{2b} & C^{21} & C^{21} & \cdots & C^{21} & C^{22} & C^{22} & \cdots & C^{22} & & C^{2h} & C^{2h} & \cdots & C^{2h} \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots & \cdots & \vdots & & \ddots & \vdots \\
A^{b1} & A^{b2} & \cdots & A^{bb} & C^{b1} & C^{b1} & \cdots & C^{b1} & C^{b2} & C^{b2} & \cdots & C^{b2} & & C^{bh} & C^{bh} & \cdots & C^{bh} \\
\hline
C^{11} & C^{21} & \cdots & C^{b1} & A^{1} & B^{1} & \cdots & B^{1} & D^{12} & D^{12} & \cdots & D^{12} & & D^{1h} & D^{1h} & \cdots & D^{1h} \\
C^{11} & C^{21} & \cdots & C^{b1} & B^{1} & A^{1} & \cdots & B^{1} & D^{12} & D^{12} & \cdots & D^{12} & & D^{1h} & D^{1h} & \cdots & D^{1h} \\
\vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots & \cdots & \vdots & & \ddots & \vdots \\
C^{11} & C^{21} & \cdots & C^{b1} & B^{1} & B^{1} & \cdots & A^{1} & D^{12} & D^{12} & \cdots & D^{12} & & D^{1h} & D^{1h} & \cdots & D^{1h} \\
\hline
C^{12} & C^{22} & \cdots & C^{b2} & D^{21} & D^{21} & \cdots & D^{21} & A^{2} & B^{2} & \cdots & B^{2} & & D^{2h} & D^{2h} & \cdots & D^{2h} \\
C^{12} & C^{22} & \cdots & C^{b2} & D^{21} & D^{21} & \cdots & D^{21} & B^{2} & A^{2} & \cdots & B^{2} & & D^{2h} & D^{2h} & \cdots & D^{2h} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots & \cdots & \vdots & & \ddots & \vdots \\
C^{12} & C^{22} & \cdots & C^{b2} & D^{21} & D^{21} & \cdots & D^{21} & B^{2} & B^{2} & \cdots & A^{2} & & D^{2h} & D^{2h} & \cdots & D^{2h} \\
\hline
 & & \vdots & & & \vdots & & & & & \vdots & & \ddots & & & \vdots & \\
\hline
C^{1h} & C^{2h} & \cdots & C^{bh} & D^{h1} & D^{h1} & \cdots & D^{h1} & D^{h2} & D^{h2} & \cdots & D^{h2} & \cdots & A^{h} & B^{h} & \cdots & B^{h} \\
C^{1h} & C^{2h} & \cdots & C^{bh} & D^{h1} & D^{h1} & \cdots & D^{h1} & D^{h2} & D^{h2} & \cdots & D^{h2} & \cdots & B^{h} & A^{h} & \cdots & B^{h} \\
\vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots & & \vdots & & \ddots & \vdots \\
C^{1h} & C^{2h} & \cdots & C^{bh} & D^{h1} & D^{h1} & \cdots & D^{h1} & D^{h2} & D^{h2} & \cdots & D^{h2} & \cdots & B^{h} & B^{h} & \cdots & A^{h}
\end{array} \right\|,
$$

where $C'_{sa} = C_{sa}\sqrt{n_a}$, $A'_a = A_a + (n_a - 1)B_a$, and $D'_{aa'} = D_{aa'}\sqrt{n_a n_{a'}}$ $(s = 1, \cdots, b; a, a' = 1, \cdots, h; a \neq a')$; $A_{ss'}$, C_{sa}, A_a, B_a, and $D_{aa'}$ are the cofactors of $A^{ss'}$, C^{sa}, A^a, B^a, and $D^{aa'}$, respectively in (3.2). The ellipsoid, defined by $A_{ij}(X_i - m_i)(X_j - m_j) = r_0$ (r_0 fixed and > 0), has $(n_a - 1)$ axes of equal length $(a = 1, \cdots, h)$; and each of the remaining q axes is inclined to the coordinate axes so that its direction cosines have the same block symmetry as the set of diagonal elements in (3.2).

When any of the $\bar{H}$ hypotheses is true, we represent $\| G^{ij} \|$ by $\| \bar{A}^{ij} \|$ (also $\| G_{ij} \|$ by $\| \bar{A}_{ij} \|$) which can be written as

$$\| \bar{A}^{ij} \| = \tag{3.4}$$

$$\left\| \begin{array}{cccc|cccc|c|cccc}
\bar{A}^1 & \bar{B}^1 & \cdots & \bar{B}^1 & \bar{C}^{12} & \bar{D}^{12} & \cdots & \bar{D}^{12} & & \bar{C}^{1h} & \bar{D}^{1h} & \cdots & \bar{D}^{1h} \\
\bar{B}^1 & \bar{A}^1 & \cdots & \bar{B}^1 & \bar{D}^{12} & \bar{C}^{12} & \cdots & \bar{D}^{12} & & \bar{D}^{1h} & \bar{C}^{1h} & \cdots & \bar{D}^{1h} \\
\cdot & \cdot\ \cdot & & \cdot & \cdot & \cdot\ \cdot & & \cdot & \cdots & \cdot & \cdot\ \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & & \cdot & \cdot & \cdot\ \cdot & \cdot \\
\bar{B}^1 & \bar{B}^1 & \cdots & \bar{A}^1 & \bar{D}^{12} & \bar{D}^{12} & \cdots & \bar{C}^{12} & & \bar{D}^{1h} & \bar{D}^{1h} & \cdots & \bar{C}^{1h} \\
\hline
\bar{C}^{21} & \bar{D}^{21} & \cdots & \bar{D}^{21} & \bar{A}^2 & \bar{B}^2 & \cdots & \bar{B}^2 & & \bar{C}^{2h} & \bar{D}^{2h} & \cdots & \bar{D}^{2h} \\
\bar{D}^{21} & \bar{C}^{21} & \cdots & \bar{D}^{21} & \bar{B}^2 & \bar{A}^2 & \cdots & \bar{B}^2 & & \bar{D}^{2h} & \bar{C}^{2h} & \cdots & \bar{D}^{2h} \\
\cdot & \cdot\ \cdot & & \cdot & \cdot & \cdot\ \cdot & & \cdot & \cdots & \cdot & \cdot\ \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & & \cdot & \cdot & \cdot\ \cdot & \cdot \\
\bar{D}^{21} & \bar{D}^{21} & \cdots & \bar{C}^{21} & \bar{B}^2 & \bar{B}^2 & \cdots & \bar{A}^2 & & \bar{D}^{2h} & \bar{D}^{2h} & \cdots & \bar{C}^{2h} \\
\hline
 & \vdots & & & & \vdots & & & \ddots & & \vdots & & \\
\hline
\bar{C}^{h1} & \bar{D}^{h1} & \cdots & \bar{D}^{h1} & \bar{C}^{2h} & \bar{D}^{2h} & \cdots & \bar{D}^{2h} & & \bar{A}^h & \bar{B}^h & \cdots & \bar{B}^h \\
\bar{D}^{h1} & \bar{C}^{h1} & \cdots & \bar{D}^{h1} & \bar{D}^{2h} & \bar{C}^{2h} & \cdots & \bar{D}^{2h} & & \bar{B}^h & \bar{A}^h & \cdots & \bar{B}^h \\
\cdot & \cdot\ \cdot & & \cdot & \cdot & \cdot\ \cdot & & \cdot & \cdots & \cdot & \cdot\ \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & & \cdot & \cdot & \cdot\ \cdot & \cdot \\
\bar{D}^{h1} & \bar{D}^{h1} & \cdots & \bar{C}^{h1} & \bar{D}^{2h} & \bar{D}^{2h} & \cdots & \bar{C}^{2h} & & \bar{B}^h & \bar{B}^h & \cdots & \bar{A}^h
\end{array} \right\|,$$

where the blocks are formed by a partition (n^h) of the t rows and t columns; thus each block is an $n \times n$ array. $\| \bar{A}^{ij} \|$ and $\| \bar{A}_{ij} \|$ have the same block pattern of symmetry.

A matrix having the block pattern of symmetry of (3.4) will be called *block symmetric of type II*. The determinant of $\| \bar{A}_{ij} \|$ is

$$| \bar{A}_{ij} | = \bar{K}^{n-1}\bar{Q}, \tag{3.5}$$

where

$$\bar{K} = \begin{vmatrix} \bar{A}_1 - \bar{B}_1 & \bar{C}_{12} - \bar{D}_{12} & \cdots & \bar{C}_{1h} - \bar{D}_{1h} \\ \bar{C}_{21} - \bar{D}_{21} & \bar{A}_2 - \bar{B}_2 & \cdots & \bar{C}_{2h} - \bar{D}_{2h} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \bar{C}_{h1} - \bar{D}_{h1} & \bar{C}_{h2} - \bar{D}_{h2} & \cdots & \bar{A}_h - \bar{B}_h \end{vmatrix},$$

$$\bar{Q} = \begin{vmatrix} \bar{A}'_1 & \bar{C}'_{12} & \cdots & \bar{C}'_{1h} \\ \bar{C}'_{21} & \bar{A}'_2 & \cdots & \bar{C}'_{2h} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \bar{C}'_{h1} & \bar{C}'_{h2} & \cdots & \bar{A}'_h \end{vmatrix},$$

where $\bar{A}'_a = \bar{A}_a + (n - 1)\bar{B}_a$ and $\bar{C}'_{aa'} = \bar{C}_{aa'} + (n - 1)\bar{D}_{aa'}$ $(a, a' = 1, 2, \cdots, h; a \neq a')$; $\bar{A}_a$, $\bar{B}_a$, $\bar{C}_{aa'}$, and $\bar{D}_{aa'}$ are the cofactors of $\bar{A}^a$, $\bar{B}^a$, $\bar{C}^{aa'}$, $\bar{D}'^{aa}$, respectively, in (3.4).

4. Method of obtaining the sample criteria. The probability distribution, P, of a simple, random sample, say $O_N(X_{1\alpha}, X_{2\alpha}, \cdots, X_{t\alpha})(\alpha = 1, 2, \cdots, N)$, from Π is

$$(4.1) \qquad P = \pi^{-Nt/2} | G_{ij} |^{N/2} \exp \left[-\sum_{i,j,\alpha} G_{ij}(X_{i\alpha} - m_i)(X_{j\alpha} - m_j)\right].$$

For O_N fixed, P is the likelihood function of the parameters $m_1, m_2, \cdots, m_t$, and G_{ij} $(i, j = 1, 2, \cdots, t)$. To obtain sample criteria for testing the H and $\bar{H}$ hypotheses we shall employ the Neyman-Pearson likelihood-ratio method. The details of applying this method will be given for only one of the hypotheses, since the technique of application is the same for all the hypotheses under consideration.

In applying the likelihood-ratio method we maximize P under two different sets of conditions and form the ratio of the two maxima. To derive a criterion for, say, $H_1(mvc)$, we first maximize P over the set, Ω, of admissible values of the parameters in (4.1); secondly, we maximize P over the set, ω, of admissible values of the parameters in (4.1) that satisfy $H_1(mvc)$. Let P_Ω and P_ω be these maxima, respectively. The likelihood-ratio criterion for $H_1(mvc)$ is $\lambda_1(mvc) = P_\omega/P_\Omega$; thus $0 \leq \lambda_1(mvc) \leq 1$. The sample criterion, $L_1(mvc)$, for $H_1(mvc)$ will be chosen as a single-valued function of $\lambda_1(mvc)$.

4a. Derivation of the criterion $L_1(mvc)$. The parameter spaces, Ω, and, ω, can be specified as follows:

$$\Omega \begin{cases} (1) & \| G_{ij} \| \text{ positive definite;} \\ (2) & -\infty < m_i < +\infty \, (i = 1, 2, \cdots, t); \end{cases}$$

$$\omega \begin{cases} (1) & \| A_{ij} \| \text{ positive definite and block symmetric (of type I);} \\ (2) & -\infty < m_i < +\infty, \, (m_1, m_2, \cdots, m_t) \text{ block symmetric.} \end{cases}$$

The block symmetries in $\omega(1)$ and $\omega(2)$ are for the same partition $(1^b, n_1, \cdots, n_h)$ of the t variates (see sections 2 and 3).

Maximizing P is equivalent to maximizing

$$
(4.2)\qquad L = \ln P = -(Nt/2)\ln\pi + (N/2)\ln|G_{ij}| - \sum_{i,j,\alpha} G_{ij}(X_{i\alpha} - m_i)(X_{j\alpha} - m_j).
$$

Solving the simultaneous equations $\partial L/\partial m_i = 0 (i = 1, \cdots, t)$ and $\partial L/\partial G_{ij} = 0 (i, j = 1, \cdots, t; i \le j)$ for m_i and G^{ij}, we have

$$
(4.3)\qquad \begin{aligned} \hat{m}_i &= (1/N)\sum_{\alpha=1}^{N} X_{i\alpha} = \bar{X}_i, \\ (N/2)\hat{G}^{ij} &= \sum_{\alpha=1}^{N}(X_{i\alpha} - \bar{X}_i)(X_{j\alpha} - \bar{X}_j) = v_{ij}; \end{aligned}
$$

substituting these values of the parameters into (4.1) we find that

$$
(4.4)\qquad P_\Omega = \pi^{-Nt/2}(2/N)^{N/2}|v_{ij}|^{-N/2}\exp[-Nt/2].
$$

In (4.3) and (4.5) each expression at the extreme right is defined by the corresponding expressions at the left.

In $\omega(2)$ there are $b + h$ groups of means, the means within a group being all equal; let m'_s be the s-th mean and m'_{r_a} be the common value of the means in the $(b + a)$-th group. Solving the simultaneous equations $\partial L/\partial m'_s = 0$, $\partial L/\partial m'_{r_a} = 0$, $\partial L/\partial A_{ss'} = 0$, $\partial L/\partial C_{sa} = 0$, $\partial L/\partial A_a = 0$, $\partial L/\partial B_a = 0$, $\partial L/\partial D_{aa'} = 0 (s, s' = 1, \cdots, b; a, a' = 1, \cdots, h; a \ne a')$, we find that

$$
(4.5)\qquad \begin{aligned} \hat{m}'_s &= \bar{X}_s, \\ \hat{m}'_{r_a} &= (1/Nn_a)\sum_{\alpha, i_a} X_{i_a\alpha} = \bar{X}'_{r_a}, \end{aligned}
$$

$$
\begin{aligned}
(N/2)\hat{A}^{ss'} &= \sum_{\alpha=1}^{N}(X_{s\alpha} - \bar{X}_s)(X_{s'\alpha} - \bar{X}_{s'}) = v_{ss'}, \\
(N/2)\hat{C}^{sa} &= (1/n_a)\sum_{\alpha, i_a}(X_{s\alpha} - \bar{X}_s)(X_{i_a} - \bar{X}'_{ra}) = u'_{sa}, \\
(N/2)\hat{A}^{a} &= (1/n_a)\sum_{\alpha, i_a}(X_{i_a\alpha} - \bar{X}_{r_a})^2 = v'_a, \\
(N/2)\hat{B}^{a} &= [1/n_a(n_a - 1)]\sum_{\alpha, i_a, j_a}(X_{i_a\alpha} - \bar{X}'_{r_a})(X_{j_a\alpha} - \bar{X}'_{r_a}) = w'_a, \\
(N/2)\hat{D}^{aa'} &= (1/n_a n_{a'})\sum_{\alpha, i_a, i_{a'}}(X_{i_a\alpha} - \bar{X}'_{r_a})(X_{i_{a'}\alpha} - \bar{X}'_{r_a}) = z'_{aa'},
\end{aligned}
$$

where $i_a, j_a = b + \bar{n}_a + 1, \cdots, b + \bar{n}_{a+1}$; $i_a \ne j_a$; $\bar{n}_a = n_1 + \cdots + n_{a-1}$; $\bar{n}_1 = 0$; $a, a' = 1, \cdots, h$; $a \ne a'$.

When $H_1(mvc)$ is true, the maximum likelihood estimates of m_i, σ_i, and $\rho_{ij}(i, j = 1, \cdots, t)$ would be obtained by means of (4.5) and the definition of $\|A^{ij}\|$ given just after (3.1).

Substituting the expressions in (4.5) into (4.1) we find that

$$
(4.6)\qquad P_\omega = \pi^{-Nt/2}|v'_{ij}|^{-N/2}(2/N)^{N/2}\exp[-Nt/2],
$$

where

$$
(4.7) \qquad \| v'_{ij} \| = \left\| \begin{array}{cccc:cccc:cccc:c:cccc}
v_{11} & v_{12} & \cdots & v_{1b} & u'_{11} & u'_{11} & \cdots & u'_{11} & u'_{12} & u'_{12} & \cdots & u'_{12} & & u'_{1h} & u'_{1h} & \cdots & u'_{1h} \\
v_{21} & v_{22} & \cdots & v_{2b} & u'_{21} & u'_{21} & \cdots & u'_{21} & u'_{22} & u'_{22} & \cdots & u'_{22} & & u'_{2h} & u'_{2h} & \cdots & u'_{2h} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
v_{b1} & v_{b2} & \cdots & v_{bb} & u'_{b1} & u'_{b1} & \cdots & u'_{b1} & u'_{b2} & u'_{b2} & \cdots & u'_{b2} & & u'_{bh} & u'_{bh} & \cdots & u'_{bh} \\
\hdashline
u'_{11} & u'_{21} & \cdots & u'_{b1} & v'_{1} & w'_{1} & \cdots & w'_{1} & z'_{12} & z'_{12} & \cdots & z'_{12} & & z'_{1h} & z'_{1h} & \cdots & z'_{1h} \\
u'_{11} & u'_{21} & \cdots & u'_{b1} & w'_{1} & v'_{1} & \cdots & w'_{1} & z'_{12} & z'_{12} & \cdots & z'_{12} & & z'_{1h} & z'_{1h} & \cdots & z'_{1h} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
u'_{11} & u'_{21} & \cdots & u'_{b1} & w'_{1} & w'_{1} & \cdots & v'_{1} & z'_{12} & z'_{12} & \cdots & z'_{12} & & z'_{1h} & z'_{1h} & \cdots & z'_{1h} \\
\hdashline
u'_{12} & u'_{22} & \cdots & u'_{b2} & z'_{21} & z'_{21} & \cdots & z'_{21} & v'_{2} & w'_{2} & \cdots & w'_{2} & & z'_{2h} & z'_{2h} & \cdots & z'_{2h} \\
u'_{12} & u'_{22} & \cdots & u'_{b2} & z'_{21} & z'_{21} & \cdots & z'_{21} & w'_{2} & v'_{2} & \cdots & w'_{2} & & z'_{2h} & z'_{2h} & \cdots & z'_{2h} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
u'_{12} & u'_{22} & \cdots & u'_{b2} & z'_{21} & z'_{21} & \cdots & z'_{21} & w'_{2} & w'_{2} & \cdots & v'_{2} & & z'_{2h} & z'_{2h} & \cdots & z'_{2h} \\
\hdashline
 & & \vdots & & & & \vdots & & & & \vdots & & \ddots & & & \vdots & \\
\hdashline
u'_{1h} & u'_{2h} & \cdots & u'_{bh} & z'_{h1} & z'_{h1} & \cdots & z'_{h1} & z'_{h2} & z'_{h2} & \cdots & z'_{h2} & & v'_{h} & w'_{h} & \cdots & w'_{h} \\
u'_{1h} & u'_{2h} & \cdots & u'_{bh} & z'_{h1} & z'_{h1} & \cdots & z'_{h1} & z'_{h2} & z'_{h2} & \cdots & z'_{h2} & & w'_{h} & v'_{h} & \cdots & w'_{h} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
u'_{1h} & u'_{2h} & \cdots & u'_{bh} & z'_{h1} & z'_{h1} & \cdots & z'_{h1} & z'_{h2} & z'_{h2} & \cdots & z'_{h2} & & w'_{h} & w'_{h} & \cdots & v'_{h}
\end{array} \right\|
$$

From (4.4) and (4.6) it follows that the likelihood-ratio criterion for $H_1(mvc)$ is:

$$\lambda_1(mvc) = [\,|\,v_{ij}\,|\,/\,|\,v'_{ij}\,|\,]^{(N/2)}, \qquad (i, j = 1, \cdots, t). \tag{4.8}$$

Finally, as the sample criterion for $H_1(mvc)$ we choose

$$L_1(mvc) = [\lambda_1(mvc)]^{(2/N)} = [\,|\,v_{ij}\,|\,/\,|\,v'_{ij}\,|\,]. \tag{4.9}$$

4*b*. *Preliminary calculations for evaluation of moments of* $L_1(mvc)$. The determinant $|\,v'_{ij}\,|$ in (4.9) is block symmetric. From (3.2), (3.3), and (4.9) it follows that:

$$L_1(mvc) = |\,v_{ij}\,| \left[\prod_{a=1}^{h} (v'_a - w'_a)^{-(n_a-1)}\right] |\,v''_{rr'}\,|^{-1}, \tag{4.10}$$

where

$$v''_{ss'} = v_{ss'};$$
$$v''_{sr_a} = u'_{sa}\sqrt{n_a};$$
$$v''_{r_ar_a} = v'_a + (n_a - 1)w'_a;$$
$$v''_{r_ar_{a'}} = \sqrt{n_a n_{a'}}\, z'_{aa'},$$

$(s, s' = 1, \cdots, b;\ r, r' = 1, \cdots, b + h;\ r_a = b + a;\ a = 1, \cdots, h)$.

Let $Y_{i\alpha} = X_{i\alpha} - m_i$ and $\bar{Y}_i = (1/N)\sum_{\alpha=1}^{N} Y_{i\alpha}$, $(i = 1, \cdots, t)$. Clearly $v_{ij} = \sum_{\alpha=1}^{N} (Y_{i\alpha} - \bar{Y}_i)(Y_{j\alpha} - \bar{Y}_j)$. When $H_1(mvc)$ is true, u'_{sa}, v'_a, w'_a, and $z'_{aa'}$, in $L_1(mvc)$, can be expressed exactly as they are expressed in (4.5) with Y substituted for X, and $(v'_a - w'_a)$ and $v''_{rr'}$ in (4.10) can be expressed as follows:

$$\begin{aligned} v'_a - w'_a = {} & (1/n_a)\Big\{\sum_{i_a} v_{i_ai_a} - [1/(n_a - 1)] \sum_{i_a \neq j_a} v_{i_aj_a}\Big\} \\ & + (N/n_a)\sum_{i_a} \bar{Y}^2_{i_a} - [N/n_a(n_a - 1)] \sum_{i_a \neq j_a} \bar{Y}_{i_a}\bar{Y}_{j_a}; \end{aligned}$$

$$\begin{aligned} v''_{ss'} &= v_{ss'}; \\ v''_{sr_a} &= (1/\sqrt{n_a}) \sum_{i_a} v_{si_a}; \\ v''_{r_ar_a} &= (1/n_a) \sum_{i_a, j_a} v_{i_aj_a}; \\ v''_{r_ar_{a'}} &= (1/\sqrt{n_a n_{a'}}) \sum_{i_a, j_{a'}} v_{i_aj_a}. \end{aligned} \tag{4.11}$$

From (4.10) and (4.11) it follows that when $H_1(mvc)$ is true, each element of the determinants on which $L_1(mvc)$ depends consists of: (a) a quadratic form in $\bar{Y}_i$ and a linear function of the v_{ij}; or (b) merely a linear function of the

$$v_{ij} \qquad (i, j = 1, \cdots, t).$$

The joint probability density function of the v_{ij} and $\bar{Y}_i$ is

$$f(v_{ij})g(\bar{Y}_1, \cdots, \bar{Y}_t), \tag{4.12}$$

where

$$f(v_{ij}) = \frac{|G_{ij}|^{(N-1)/2} |v_{ij}|^{(N-t-2)/2} \exp[-\sum_{i,j} G_{ij}v_{ij}]}{\pi^{t(t-1)/4}\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)\cdots\Gamma\left(\frac{N-t}{2}\right)},$$

($\|G_{ij}\|$ positive definite; $N > t$), which is the Wishart distribution [9, p. 120], and

$$g(\bar{Y}_1, \cdots, \bar{Y}_t) = |G_{ij}|^{1/2} N^{t/2} \pi^{-t/2} \exp[-N \sum_{i,j} G_{ij}\bar{Y}_i\bar{Y}_j] = g(\bar{Y}), \quad \text{say},$$

which is a normal t-variate distribution. The d-th moment ($d = 0, 1, \cdots$) of $L_1(mvc)$, when $H_1(mvc)$ is true, is

$$E[L_1(mvc)]^d = \int_R f(v_{ij})g(\bar{Y}) |v_{ij}|^d |v''_{rr'}|^{-d} \cdot \prod_{a=1}^{h} (v'_a - w'_a)^{-d(n_a-1)} \left(\prod_1^t d\bar{Y}_1\right) \prod_{i\geq j} dv_{ij}, \tag{4.13}$$

where the domain, R, of integration is $-\infty < \bar{Y}_i < +\infty$ $\|v_{ij}\|$ positive semi-definite ($i, j = 1, \cdots, t$). The integral in (4.13) is evaluated in section 6 (by means of Wilks' moment-generating operators) for the case where $H_1(mvc)$ is true.

5. Remarks on Wilks' Moment-Generating Operators. Wilks' operators are applicable to a far wider class of problems than those treated in this paper. The following discussion is confined to a special use of the operators.

From (4.12) it follows that

$$\int_{R'} \frac{|v_{ij}|^{(N-t-2)/2} \exp[-\sum_{i,j} G_{ij}v_{ij}] \prod_{i\geq j} dv_{ij}}{\pi^{t(t-1)/4} \prod_{i=1}^{t} \Gamma[(N-i)/2]} = |G_{ij}|^{-(N-1)/2}, \tag{5.1}$$

where R' is the region in the space of v_{ij} for which $\|v_{ij}\|$ is positive definite, and $\|G_{ij}\|$ is positive semi-definite. (Of course, the probability that $\|v_{ij}\|$ is not positive definite is 0.) Let $G'_{ij} = G_{ij} + \beta_{ij}(i, j = 1, \cdots, t)$; if all the β_{ij} are sufficiently small, $\|G'_{ij}\|$ is positive definite, and we have

$$|G_{ij}|^{(N-1)/2} \int_{R'} \frac{|v_{ij}|^{N-t-2/2} \exp[-\sum_{i,j} G_{ij}v_{ij}] \prod_{i\geq j} dv_{ij}}{\pi^{t(t-1)/4} \prod_{i=1}^{t} \Gamma[(N-i)/2]} = |G_{ij}|^{(N-1)/2} |G'_{ij}|^{-(N-1)/2}, \tag{5.2}$$

which is $E(g)$, where $g = \exp[-\sum_{i,j} \beta_{ij}v_{ij}]$.

Let I^1_{ij} be an operator (whose operand is a function of all the β_{ij}) which represents the following set of operations: (a) replacement of each β_{ij} in the operand by $B_{ij} + \xi_i\xi_j$; (b) integration (of the result of (a)) with respect to $\xi_i (i = 1, \cdots, t)$ from $-\infty$ to $+\infty$; (c) multiplication of the result of (a) and (b) by $\pi^{-t/2}$. From (3.1) it follows that

$$(5.3) \qquad I_{ij}^1(g) = I_{ij}^1 (\exp [-\sum_{i,j} \beta_{ij} v_{ij}]) = g \mid v_{ij} \mid^{-1/2}, \qquad (\| v_{ij} \| \text{ pos. def.});$$

and if all the β_{ij} are set equal to 0 after performing the I-operations, then $g = 1$ and (5.3) yields $\mid v_{ij} \mid^{-1/2}$. Let I_{ij}^λ be λ repetitions ($\lambda = 1, 2, \cdots$) of I_{ij}^1. Clearly,

$$(5.4) \qquad E[I_{ij}^\lambda(g)] \mid_{\beta_{ij}=0} = E[g \mid v_{ij} \mid^{-\lambda/2}] \mid_{\beta_{ij}=0} = E[\mid v_{ij} \mid^{-\lambda/2}].$$

Under all conditions of their use in this paper the I operations are interchangeable with the E operation [8; p. 316]; thus,

$$E[I_{ij}^\lambda g] = I_{ij}^\lambda[E(g)].$$

From (5.2), (5.4), and [8, pp. 318–320] we have

$$(5.5) \qquad \begin{aligned} E[\mid v_{ij} \mid^{-\lambda/2}] &= \mid G_{ij} \mid^{(N-1)/2} \{I_{ij}^\lambda \mid G'_{ij} \mid^{-(N-1)/2}\} \mid_{\beta_{ij}=0} \\ &= \mid G_{ij} \mid^{\lambda/2} \prod_{i=1}^{t} \psi[N - i, -\lambda], \end{aligned}$$

where $N \geq t + \lambda + 1$ and $\psi(R, S) = \left[\Gamma\left(\frac{R+S}{2}\right)\right] \Big/ \left[\Gamma\left(\frac{R}{2}\right)\right]$.

The operator I_{ij} may be used, as indicated above, to find negative half-integer moments of $\mid v_{ij} \mid$. To obtain positive half-integer moments of $\mid v_{ij} \mid$ we may use an inverse operator $I_{ij}^{-\lambda}$ [8, pp. 321–323] ($\lambda = 1, 2, \cdots$) which has been defined in such a way that

$$(5.6) \qquad \begin{aligned} \mid G_{ij} \mid^{(N-1)/2} \mid \{I_{ij}^{-\lambda} \mid G'_{ij} \mid^{-(N-1)/2}\} \mid_{\beta_{ij}=0} &= E[\mid v_{ij} \mid^{\lambda/2}] \\ &= \mid G_{ij} \mid^{-\lambda/2} \left(\prod_{i=1}^{t} \psi[N - i, \lambda]\right). \end{aligned}$$

The equality between the second and third expressions in (5.6) can be obtained from (5.1) by replacing N by $N + \lambda$ (see [7]).

In (5.5) and (5.6) the β's are not necessary; however, in (4.13) and in similar expressions for the moments of the other criteria there are several determinants; each determinant requires a distinct I-operator, and it is of great convenience to introduce a distinct set of β's for each I operator. The β's associated with a given operator may initially appear in more than one of the determinants in the operand. The order in which several I-operators are used is illustrated in the following case for two:

$$(5.7) \qquad [I_{ij}^\lambda \mid G'_{ij} \mid^{-k'} (I_{ij}^{-\rho} \mid G''_{ij} \mid^{-k''}) \mid_{\beta''_{ij}=0}] \mid_{\beta'_{ij}=0}.$$

where $\lambda, \rho > 0$ and the values of k' and k'' are such that the value of the expression is well defined. The notation in (5.7) means that $I_{ij}^{-\rho}$ is applied to $\mid G''_{ij} \mid^{-k''}$, the β's associated with $I_{ij}^{-\rho}$ are set equal to zero, then I_{ij}^λ is applied to the product of $| G'_{ij} |^{-k'}$ and the results of the previous operations, and then the β's associated with I^{λ}_{ij} are set equal to zero. The interchangeability of the order of I operations is discussed in [8, p. 324].

6. The moments and distribution of L_1 (*mvc*) when H_1 (*mvc*) is true. To evaluate (4.13) we let

$$g = \exp\left[-\sum_{i,j}\beta_{ij}v_{ij} - \sum_a \beta'_a(v'_a - w'_a) - \sum_{r,r'}\beta''_{rr'}v''_{rr'}\right]. \tag{6.1}$$

From (4.11) and (4.12) we have

$$E(g) = | A_{ij} |^{N/2} \, | A'_{ij} |^{-(N-1)/2} \, | A''_{ij} |^{-1/2}, \tag{6.2}$$

where

$$\begin{aligned}
A'_{ss'} &= A_{ss'} + \beta_{ss'} + \beta''_{ss'}, \\
A'_{si_a} &= C_{sa} + \beta_{si_a} + \beta''_{sr_a}/\sqrt{n_a}, \\
A'_{i_ai_a} &= A_a + \beta_{i_ai_a} + \beta'_a/n_a + \beta''_{r_ar_a}/n_a, \\
A'_{i_aj_a} &= B_a + \beta_{i_aj_a} - \beta'_a/(n_a - 1)n_a + \beta''_{r_ar_a}/n_a, \qquad (i_a \neq j_a), \\
A'_{i_aj_{a'}} &= D_{aa'} + \beta_{i_aj_{a'}} + \beta''_{r_ar_{a'}}/\sqrt{n_an_{a'}}, \qquad (a \neq a'), \\
A''_{ss'} &= A_{ss'}, \\
A''_{si_a} &= C_{sa}, \\
A''_{i_ai_a} &= A_a + \beta'_a/n_a, \\
A''_{i_aj_a} &= B_a - \beta'_a/n_a(n_a - 1), \qquad (i_a \neq j_a), \\
A''_{i_aj_{a'}} &= D_{aa'}, \qquad (a \neq a').
\end{aligned}$$

When $H_1(mvc)$ is true, we have

$$\begin{aligned}
E[L_1(mvc)]^d &= | A_{ij} |^{N/2} \left\{ \prod_{a=1}^{h} I_a^{2d(n_a-1)} \, | A''_{ij} | \, [I^{2d}_{rr'}(I^{-2d}_{ij} \, | A'_{ij} |^{-(N-1/2)})_{\beta_{ij}=0}]_{\beta''_{rr'}=0} \right\}_{\beta'_a=0} \\
&= \left\{ \prod_{i=1}^{t} \psi(N - i, 2d) \right\} \left\{ \prod_{r=1}^{q} \psi(N + 2d - r, -2d) \right\} \\
&\quad \times \left\{ \prod_{a=1}^{h} \psi[(N + 2d)(n_a - 1), -2d(n_a - 1)] \right\} \left\{ \prod_{a=1}^{h} (n_a - 1)^{d(n_a-1)} \right\}, \\
&\qquad (d = 0, 1, 2, \cdots ; N > t),
\end{aligned} \tag{6.3}$$

where $q = b + h$ and $\psi(R, S)$ is defined in (5.5). In (6.3) the assumption that $H_1(mvc)$ is true implies that after we apply I^{-2d}_{ij} and set the β_{ij} equal to 0 all remaining determinants are block symmetric; we may then use (3.3) before applying $I_{rr'}^{2d}$ and $I_a^{2d(n_a-1)}$, $(a = 1, \cdots, h)$. The expression in (6.3) may be written as follows:

$$E[L_1(mvc)]^d = \left\{ \prod_{i=1+q}^{t} \frac{\Gamma\left(\frac{N-i}{2} + d\right)}{\Gamma\left(\frac{N-i}{2}\right)} \right\} \left\{ \prod_{a=1}^{h} \frac{\Gamma\left(\frac{N(n_a-1)}{2}\right)}{\Gamma\left(\frac{N(n_a-1)}{2} + d(n_a-1)\right)} \right\} \left(\prod_{a=1}^{h} (n_a - 1)^{d(n_a-1)} \right) \tag{6.4}$$

$$= \prod_{a=1}^{h} \prod_{s_a=1}^{n_a-1} \left\{ \frac{\left(\frac{N - q - s_a - \bar{n}_a + a - 1}{2}\right)_d}{\left(\frac{N}{2} + \frac{(s_a - 1)}{(n_a - 1)}\right)_d} \right\},$$

where $\bar{n}_a$ is defined in (4.5) and $(T)_d = \Gamma(T + d)/\Gamma(T)$.

We now consider the problem of identifying from (6.4) the distribution of $L_1(mvc)$ (when $H_1(mvc)$ is true). Let θ be a beta variate, i.e., a variate whose c.d.f., $F(\theta)$, is

$$F(\theta) = I_\theta(P, Q), \qquad (0 \le \theta \le 1; P, Q > 0), \tag{6.5}$$

which is the Incomplete Beta Function ratio. $I_\theta(P, Q)$ is tabulated in [1] and [3]. The d-th moment of θ is:

$$E(\theta)^d = \frac{\Gamma(P + d)}{\Gamma(P)} \frac{\Gamma(P + Q)}{\Gamma(P + Q + d)} = (P)_d/(P + Q)_d, \tag{6.6}$$

$(d = 0, 1, \cdots)$. Let

$$\tau = \prod_{j=1}^{c} \theta_j \qquad (c = 1, 2, \cdots), \tag{6.7}$$

where the $\theta_j (j = 1, \cdots, c)$ are mutually independent and each θ_j is a beta variate, having parameters p_j, q_j, say. The d-th moment of τ is

$$E(\tau)^d = \prod_{j=1}^{c} (p_j)_d/(p_j + q_j)_d, \qquad (d = 0, 1, \cdots). \tag{6.8}$$

Given a variate, say μ $(0 \le \mu \le 1)$, whose d-th moment $(d = 0, 1, \cdots)$ is given by (6.8) we can infer by means of the solution of the Hausdorff problem of moments that μ and τ have the same exact probability distribution function (see Corollary 1.1 [2, p. 11]). It should be noted that (6.4) can be written as

$$E[L_1(mvc)]^d = \prod_{a=1}^{h} \prod_{s_a=1}^{r_a-1} [(p_{as_a})_d/(p_{as_a} + q_{as_a})_d], \tag{6.9}$$

where $$p_{as_a} = [(N - q - s_a - \bar{n}_a + a - 1)/2] > 0,$$
$$q_{as_a} = \left[\frac{(s_a - 1)}{(n_a - 1)} + \frac{q + s_a + \bar{n}_a - a + 1}{2}\right] > 0;$$

thus (6.4) is a special case of (6.8).

The exact probability (density) function, say $g(\tau)$, of τ has been obtained by Wilks [7, p. 475] and is:

$$\begin{aligned} g(\tau) = {} & K\tau^{p_c - 1}(1-\tau)^{\zeta_c - \eta_c - 1}\int_0^1 \cdots \int_0^1 v_1^{q_1-1} v_2^{q_2-1} \cdots v_{c-1}^{q_{c-1}-1} \\ & \times (1 - v_1)^{\zeta_{c-1}-\eta_{c-1}-1}(1-v_2)^{\zeta_{c-2}-\eta_{c-2}-1} \cdots (1-v_{c-1})^{\zeta_1 - \eta_1 - 1} \\ & \times [1 - v_1(1-\tau)]^{p_1 - p_2 - q_2}[1 - \{v_1 + v_2(1-v_1)\}(1-\tau)]^{p_2 - p_3 - q_3} \cdots \\ & \times [1 - \{v_1 + v_2(1-v_1) + \cdots + v_{c-1}(1-v_1)(1-v_2)\cdots(1-v_{c-2})\} \\ & \qquad (1-\tau)^{p_{c-1}-p_c-q_c}] \\ & \times \prod_{j=1}^{c-1} dv_j, \end{aligned} \tag{6.10}$$

where $K = \prod_{j=1}^{c}\left[\frac{\Gamma(p_j + q_j)}{\Gamma(p_j)\Gamma(q_j)}\right]$, $\zeta_j = \sum_{j'=0}^{j-1}(p_{c-j'} + q_{c-j'})$,

$\eta_j = \sum_{j'=0}^{j-1} p_{c-j'}$. An approximation of the distribution of a product of independent beta variates by the distribution of a single beta variate is given in [4].

The results of this section may be summarized as follows: *If $H_1(mvc)$ is true, the d-th moment ($d = 0, 1, \cdots$) of the exact distribution of $L_1(mvc)$ is given by* (6.4). Also, if $H_1(mvc)$ is true, the exact distribution of $L_1(mvc)$ is given by (6.10), where the p_j, q_j, and c can be specified by means of (6.4). The cumulative distribution of $L_1(mvc)$ is given for certain special cases in section 7g.

7. Single Sample Criteria. The solutions of problems (i) and (ii) (see section 1) for $H_1(mvc)$ are contained in (4.9) and the summary at the end of section 6. In the present section solutions of problems (i) and (ii) are given for each of the remaining two H_1 hypotheses and the three $\bar{H}_1$ hypotheses (all of which are stated in section 2). For any of the hypotheses the sample criterion is chosen as a single-valued function of the likelihood-ratio criterion for the hypothesis. The methods of determining the moments and identifying the distribution of each sample criterion (when the corresponding null hypothesis is true) are entirely similar to those used in sections 4, 5, and 6 in regard to $H_1(mvc)$. Section 7g gives the exact distributions of the single-sample criteria for certain special compound symmetries.

Each criterion discussed in this section is based on a sample

$$O_N(X_{1\alpha}, X_{2\alpha}, \cdots, X_{t\alpha})(\alpha = 1, \cdots, N; N > t)$$

of size N from a normal t-variate distribution ($t = 3, 4, \cdots$). As in the case of $H_1(mvc)$, it is presupposed for testing $H_1(vc)$ or $H_1(m)$ that there is a certain partition $(1^b, n_1, n_2, \cdots, n_h)$ of the t-variates; for testing $\bar{H}_1(mvc)$, $\bar{H}_1(vc)$, or $\bar{H}_1(m)$ it is presupposed that there is a certain partition (n^h) of the t variates (see sections 2 and 3).

7a. The test $L_1(vc)$ for the hypothesis $H_1(vc)$. For the sample criterion for $H_1(vc)$ we choose

$$L_1(vc) = [\lambda_1(vc)]^{2/N} = |v_{ij}| / |\bar{v}_{ij}|, \qquad (i, j = 1, \cdots, t) \tag{7.1}$$

where $\lambda_1(vc)$ is the likelihood-ratio criterion for $H_1(vc)$, v_{ij} is defined in (4.3), and

$$\begin{aligned}
\bar{v}_{ss'} &= v_{ss'}, \\
\bar{v}_{si_a} &= (1/n_a) \sum_{j_a} v_{sj_a}, \\
\bar{v}_{i_ai_a} &= (1/n_a) \sum_{j_a} v_{j_aj_a}, \\
\bar{v}_{i_aj_a} &= [1/n_a(n_a - 1)] \sum_{i'_a \neq j'_a} v_{i'_aj'_a}, \\
\bar{v}_{i_aj_{a'}} &= (1/n_a n_{a'}) \sum_{i'_a, j'_{a'}} v_{i'_aj'_{a'}},
\end{aligned}$$

($s, s' = 1, \cdots, b$; $a, a' = 1, \cdots, h$; $a \neq a'$; $i_a, i'_a, j_a, j'_a = b + \bar{n}_a + 1, \cdots, b + \bar{n}_{a+1}$; $\bar{n}_a = n_1 + \cdots + n_{a-1}$; $\bar{n}_1 = 0$). Since $\| \bar{v}_{ij} \|$ is a block symmetric matrix, there is an expression for $|\bar{v}_{ij}|$ that is entirely similar in form to the expression in (3.3) for $|A_{ij}|$ (see also (4.9) and (4.10)).

If $H_1(vc)$ is true,

$$\begin{aligned}
E[L_1(vc)]^d &= \left\{\prod_{i=1}^{t} \psi(N - i, 2d)\right\} \\
&\quad \left\{\prod_{a=1}^{h} \psi[(N - 1 + 2d)(n_a - 1), -2d(n_a - 1)]\right\} \\
&\quad \times \left\{\prod_{r=1}^{q} \psi[N - r + 2d, -2d]\right\} \left\{\prod_{a=1}^{h} (n_a - 1)^{d(n_a-1)}\right\} \\
&= \prod_{a=1}^{h} \prod_{s_a=1}^{n_a-1} \left\{ \frac{\left(\frac{N - q - s_a - n_a + a - 1}{2}\right)_d}{\left(\frac{N-1}{2} + \frac{(s_a - 1)}{(n_a - 1)}\right)_d} \right\}, \qquad (d = 0, 1, \cdots),
\end{aligned} \tag{7.2}$$

where $q = b + h$ and $\psi(R, S)$, $\bar{n}_a$ and $(T)_d$ are defined in (5.5), (4.5), and (6.4), respectively. From (7.2) and the argument given after (6.8) it follows that if $H_1(vc)$ is true, the exact distribution of $L_1(vc)$ is given by (6.10), where the p_j, q_j, and c can be specified by means of (7.2).

7b. *The test $L_1(m)$ for the hypothesis $H_1(m)$.* For the sample criterion for $H_1(m)$ we choose

$$(7.3) \qquad L_1(m) = [\lambda_1(m)]^{2/N} = \frac{|\tilde{v}_{ij}|}{|v'_{ij}|}, \qquad (i, j = 1, \cdots, t),$$

where $\lambda_1(m)$ is the likelihood-ratio criterion for $H_1(m)$ and v'_{ij} and $\tilde{v}_{ij}$ are defined in (4.7) and (7.1), respectively. In passing we note that

$$(7.4) \qquad [L_1(m)][L_1(vc)] = L_1(mvc).$$

If $H_1(m)$ is true,

$$(7.5) \qquad \begin{aligned} E[L_1(m)]^d &= \prod_{a=1}^{h} \{\psi[(N-1)(n_a-1), 2a(n_a-1)] \\ &\quad \times \psi[(n_a-1)(N+2a), -2a(n_a-1)]\} \\ &= \prod_{a=1}^{h}\prod_{s_a=1}^{r_a-1}\left\{\frac{\left(\frac{N-1}{2}+\frac{s_a-1}{n_a-1}\right)_d}{\left(\frac{N}{2}+\frac{s_a-1}{n_a-1}\right)_d}\right\}, \qquad (d = 0, 1, \cdots). \end{aligned}$$

If $H_1(m)$ is true, the exact distribution of $L_1(m)$ is given by (6.10), where the p_j, q_j and c can be specified by means of (7.5). It follows from (7.5) that the exact distribution of $L_1(m)$, when $H_1(m)$ is true, does not depend on b.

7c. *The test $\bar{L}_1(mvc)$ for the hypothesis $\bar{H}_1(mvc)$.* The sample criterion, $\bar{L}_1(mvc)$, for $\bar{H}_1(mvc)$ (see section 2) is

$$(7.6) \qquad \bar{L}_1(mvc) = [\bar{\lambda}_1(mvc)]^{2/N} = |v_{ij}| / |\bar{v}'_{ij}|, \qquad (i, j = 1, \cdots, t)$$

where $\bar{\lambda}_1(mvc)$ is the likelihood-ratio criterion for $\bar{H}_1(mvc)$, v_{ij} is defined in (4.3), and

$$\bar{v}'_{i_a i_a} = (1/n) \sum_{\alpha, j_a} (X_{j_a\alpha} - \bar{X}'_a)^2,$$

$$\bar{v}'_{i_a j_a} = [1/n(n-1)] \sum_{\substack{\alpha \\ i'_a \neq j'_a}} (X_{i'_a\alpha} - \bar{X}'_a)(X_{j'_a\alpha} - \bar{X}'_a), \qquad (i_a \neq j_a),$$

$$\bar{v}'_{i_a k_{a'}} = (1/n) \sum_{\alpha, j_a, k'_{a'}} (X_{j_a\alpha} - \bar{X}'_a)(X_{k'_{a'}\alpha} - \bar{X}'_{a'}),$$
$$(k'_{a'} = j_a + n(a' - a); a \neq a'),$$

$$\bar{v}'_{i_a h_{a'}} = [1/n(n-1)] \sum_{\alpha, j_a, h'_{a'}} (X_{j_a\alpha} - \bar{X}'_a)(X_{h'_{a'}\alpha} - \bar{X}'_{a'}),$$
$$(h'_{a'} \neq j_a + n(a - a'); a \neq a'),$$

$(a = 1, \cdots, h; i_a, j_a, h_a, k_a = (a-1)n + 1, \cdots, an; k_{a'} = i_a + n(a' - a); h_{a'} \neq i_a + n(a' - a); \alpha = 1, \cdots, N)$. $\|\bar{v}'_{ij}\|$ is a block symmetric matrix, of type II (see (3.4)), in which the blocks are formed by a partition (n^h) $(t = nh)$ of the rows and columns; there is an expression for $|\bar{v}'_{ij}|$ that is entirely similar in form to the expression in (3.5) for $|\bar{A}_{ij}|$.

If $\bar{H}_1(mvc)$ is true,

$$E[\bar{L}_1(mvc)]^d = (n-1)^{hd(n-1)} \left\{ \prod_{i=h+1}^{t} \psi(N-i, 2d) \right\} \times \left\{ \prod_{a=1}^{h} \psi[(N+2d)(n-1)+1-a, -2d(n-1)] \right\} = \prod_{a=1}^{h} \prod_{s=1}^{n-1} \left\{ \frac{\left(\frac{N-h-s-(n-1)(a-1)}{2} \right)_d}{\left(\frac{N}{2} + \frac{1-a}{2(n-1)} + \frac{s-1}{n-1} \right)_d} \right\}, \tag{7.7}$$

$$(d = 0, 1, \cdots).$$

If $\bar{H}_1(mvc)$ is true, the exact distribution of $\bar{L}_1(mvc)$ is given by (6.10), where the p_j, q_j and c can be specified by means of (7.7).

7d. The test $\bar{L}_1(vc)$ for the hypothesis $\bar{H}_1(vc)$. The sample criterion, $\bar{L}_1(vc)$ for $\bar{H}_1(vc)$ (see section 2) is

$$\bar{L}_1(vc) = [\bar{\lambda}_1(vc)]^{2/N} = |v_{ij}| / |\bar{v}_{ij}| \qquad (i, j = 1, \cdots, t), \tag{7.8}$$

where $\bar{\lambda}_1(vc)$ is the likelihood-ratio criterion for $\bar{H}_1(vc)$, v_{ij} is defined in (4.3), and

$$\bar{v}_{i_a i_a} = (1/n) \sum_{j_a} v_{j_a j_a},$$

$$\bar{v}_{i_a j_a} = [1/n(n-1)] \sum_{i'_a \neq j'_a} v_{i'_a j'_a}, \qquad (i_a \neq j_a),$$

$$\bar{v}_{i_a k_{a'}} = (1/n) \sum_{j_a, k'_{a'}} v_{j_a k'_{a'}}, \qquad (k'_{a'} = j_a + n(a'-a);\ a \neq a'),$$

$$\bar{v}_{i_a h_{a'}} = [1/n(n-1)] \sum_{j_a, h'_{a'}} v_{j_a h'_{a'}}, \qquad (h'_{a'} \neq j_a + n(a'-a);\ a \neq a'),$$

where the ranges of a, i_a, j_a, h_a, k_a are given in (7.6). There is an expression for $|\bar{v}_{ij}|$ which is entirely similar in form to the expression in (3.5) for $|\bar{A}_{ij}|$.

If $\bar{H}_1(vc)$ is true,

$$E[\bar{L}_1(vc)]^d = (n-1)^{hd(n-1)} \left[\prod_{i=h+1}^{t} \psi(N-i, 2d) \right] \times \left\{ \sum_{a=1}^{h} \psi[(N-1+2d)(n-1)+1-a, -2d\,(n-1)] \right\} = \prod_{a=1}^{h} \prod_{s=1}^{n-1} \left\{ \frac{\left(\frac{N-h-s-(n-1)(a-1)}{2} \right)_d}{\left(\frac{N-1}{2} + \frac{1-a}{2(n-1)} + \frac{s-1}{n-1} \right)_d} \right\}, \qquad (d = 0, 1, \cdots). \tag{7.9}$$

If $\bar{H}_1(vc)$ is true, the exact distribution of $\bar{L}_1(vc)$ is given by (6.10), where the p_j, q_j and c can be specified by means of (7.9).

7e. The test $\bar{L}_1(m)$ for the hypothesis $\bar{H}_1(m)$. The sample criterion $\bar{L}_1(m)$, for $\bar{H}_1(m)$ (see section 2) is

$$(7.10) \qquad \bar{L}_1(m) = [\bar{\lambda}_1(m)]^{2/N} = \frac{\bar{L}_1(mvc)}{\bar{L}_1(vc)} = \frac{|\bar{v}_{ij}|}{|\bar{v}'_{ij}|},$$

where $\bar{\lambda}_1(m)$ is the likelihood-ratio criterion for $\bar{H}_1(m)$ and $\|\bar{v}_{ij}\|$ and $\|\bar{v}'_{ij}\|$ are given in (7.8) and (7.6), respectively.

If $\bar{H}_1(m)$ is true, the d-th moment ($d = 0, 1, \cdots$) of $\bar{L}_1(m)$ is

$$(7.11) \qquad E[\bar{L}_1(m)]^d = \prod_{a=1}^{h} \prod_{s=1}^{n-1} \left\{ \frac{\left(\frac{N-1}{2} + \frac{1-a}{2(n-1)} + \frac{s-1}{n-1}\right)_d}{\left(\frac{N}{2} + \frac{1-a}{2(n-1)} + \frac{s-1}{n-1}\right)_d} \right\}, \qquad (d = 0, 1, \cdots).$$

If $\bar{H}_1(m)$ is true, the exact distribution of $\bar{L}_1(m)$ is given by (6.10) where the p_j, q_j and c can be specified by means of (7.11).

7f. Relations among $L_1(mvc)$, $L_1(vc)$, and $L_1(m)$ and among $\bar{L}_1(mvc)$, $\bar{L}_1(vc)$, and $\bar{L}_1(m)$. $L_1(mvc)$ is the product of $L_1(vc)$ and $L_1(m)$ (see (7.4)); moreover, when $H_1(mvc)$ is true, the d-th moment ($d = 0, 1, \cdots$) of $L_1(mvc)$ equals the product of the d-th moments of $L_1(vc)$ and $L_1(m)$ (see (6.4), (7.2), and (7.5)). From this result and the argument given after (6.8) it follows that when $H_1(mvc)$ is true, $L_1(mvc)$ is the product of two independent chance quantities, namely, $L_1(vc)$ and $L_1(m)$. Similarly, when $\bar{H}_1(mvc)$ is true, $\bar{L}_1(mvc)$ is the product of two independent chance quantities, namely, $\bar{L}_1(vc)$ and $\bar{L}_1(m)$.

7g. Exact distributions of single sample criteria in special cases. For a sample of size N and a partition $(1^b, n_1, \cdots, n_h)$ of the t variates of Π (see section 2) let the cumulative distribution function (c.d.f.) of $L_1(mvc)$, when $H_1(mvc)$ is true, be

$$(7.12) \qquad F(u \mid 1^b, n_1, \cdots, n_h \mid N) = \text{Prob}\,\{L_1(mvc) \le u\};$$

also, let $F(y \mid 1^b, n_1, \cdots, n_h \mid N)$ and $F(z \mid 1^b, n_1, \cdots, n_h \mid N)$ be the c.d.f.'s of $L_1(vc)$ and $L_1(m)$ when $H_1(vc)$ and $H_1(m)$ are true, respectively. Let $F(\bar{u} \mid n^h \mid N)$, $F(\bar{y} \mid n^h \mid N)$, and $F(\bar{z} \mid n^h \mid N)$ be the c.d.f.'s of $\bar{L}_1(mvc)$, $\bar{L}_1(vc)$, and $\bar{L}_1(m)$ when $\bar{H}_1(mvc)$, $\bar{H}_1(vc)$ and $\bar{H}_1(m)$ are true, respectively.

It can be shown that

$$F(u \mid 1^b, 2 \mid N) = I_u[(N - b - 2)/2, (b + 2)/2],$$

$$F(u \mid 1^b, 3 \mid N) = I_{\sqrt{u}}[N - b - 3, b + 3],$$

$$F(y \mid 1^b, 2 \mid N) = I_y[(N - b - 2)/2, (b + 1)/2],$$

$$
\begin{aligned}
(7.13)\quad & F(y \mid 1^b, 3 \mid N) = I_{\sqrt{y}}[N - b - 3, b + 2], \\
& F(z \mid 1^b, n \mid N) = I_{z'}[(N - 1)(n - 1)/2, (n - 1)/2], [z' = z^{1/(n-1)}], \\
& F(\bar{u} \mid 2^2 \mid N) = I_{\sqrt{\bar{u}}}[N - 4, 3], \\
& F(\bar{y} \mid 2^2 \mid N) = I_{\sqrt{\bar{y}}}[N - 4, 2], \\
& F(\bar{z} \mid n^2 \mid N) = I_{\bar{z}'}[(N - 1)(n - 1) - 1, n - 1], \qquad [\bar{z}' = \bar{z}^{1/2(n-1)}],
\end{aligned}
$$

where $I_x(P, Q)$ is defined in (6.5).

Distributions of the criteria in certain cases where the normal distribution is *completely symmetric* (see section 2) are given in [5].

7h. *Asymptotic distributions of the single sample criteria.* When the sample size, N, is large, we may use a theorem [6] (see also [9, pp. 151–2]) concerning the approximate distribution of the likelihood-ratio criterion. For large N the distributions of the quantities $-N \ln L_1(mvc)$, $-N \ln L_1(vc)$, and $-N \ln L_1(m)$ (when $H_1(mvc)$, $H_1(vc)$, and $H_1(m)$, respectively, are true) are approximately chi-square distributions with $(1/2)\,[t(t + 3) - b(b + 3) - h(h + 5)] - hb$, $(1/2)[t(t + 1) - b(b + 1) - h(h + 3)] - hb$, and $t - b - h$ degrees of freedom, respectively. Also, for large N the distributions of the quantities $-N \ln \bar{L}_1(mvc)$, $-N \ln \bar{L}_1(vc)$, and $-N \ln \bar{L}_1(m)$ (when $\bar{H}_1(mvc)$, $\bar{H}_1(vc)$, and $\bar{H}_1(m)$, respectively, are true) are approximately chi-square distributions with $[t(t + 3)/2 - h(h + 2)]$, $[t(t + 1)/2 - h(h + 1)]$, and $t - h$ degrees of freedom, respectively.

8. k-Sample Criteria. In this section solutions of problems (i) and (ii) (see section 1) are given for the three H_k and the three $\bar{H}_k$ hypotheses (all stated in section 2).

A test of any of these hypotheses is based on k simple, random samples ($k \geq 2$) from k compound-symmetric, normal t-variate distributions. The probability density function, Q, of the k samples, say, $O_{N_p}(p = 1, \cdots, k;\ N_p > b + h)$ is

$$
(8.1)\qquad Q = \pi^{-N't/2}\left[\prod_{p=1}^{k} \mid G_{ij,p} \mid^{N_p/2}\right] \times \exp\,[-\sum_{i,j,p,a_p} G_{ij,p}(X_{ia_p} - m_{i,p})(X_{ja_p} - m_{j,p})],
$$

$(N' = \sum_{p=1}^{k} N_p\ ;\ i, j = 1, \cdots, t)$, where X_{ia_p} is the a_p-th sample value of the i-th variate in the p-th population ($a_p = 1, \cdots, N_p$), $m_{i,p}$ is the mean (expected value) of the i-th variate in the p-th population, and $(1/2) \mid\mid G_{ij,p} \mid\mid^{-1}$ is the variance-covariance matrix of the variates in the p-th population (see (3.1)). For a given set of k samples Q is the likelihood function of the parameters $G_{ij,p}$ and $m_{i,p}$ ($i, j = 1, \cdots, t;\ p = 1, \cdots, k$).

The six hypotheses under consideration (see section 2) can be restated in terms of $G_{ij,p}$ and $m_{i,p}$; e.g., $H_k(MVC \mid mvc)$ asserts that $m_{i,1} = m_{i,2} = \cdots = m_{i,k}$ and $\| G_{ij,1} \| = \| G_{ij,2} \| = \cdots = \| G_{ij,k} \|$ given that for all p the vector $(m_{1,p}, \cdots, m_{t,p})$ is block symmetric and the matrix $\| G_{ij,p} \|$ is block symmetric (of type I) for a preassigned partition $(1^b, n_1, \cdots, n_h)$ of the t variates (see sections 2 and 3).

8a. *Expressions for the criteria.* Let $\lambda_k(MVC \mid mvc), \cdots, \bar{\lambda}_k(M \mid mvc)$ represent the likelihood-ratio criteria for the six hypotheses $H_k(MVC \mid mvc), \cdots, \bar{H}_k(M \mid mVC)$ respectively, and let $L_k(MVC \mid mvc), \cdots, \bar{L}_k(M \mid mVC)$ be the sample criteria for the respective hypotheses. We choose the L_k as follows:

$$
\begin{aligned}
L_k(MVC \mid mvc) &= [\lambda_k(MVC \mid mvc)]^2, \\
L_k(VC \mid mvc) &= [\lambda_k(MC \mid mvc)]^2, \\
L_k(M \mid mVC) &= [\lambda_k(M \mid mVC)]^{2/N'}, \\
&= \left\{\frac{L_k(MVC \mid mvc)}{L_k(VC \mid mvc)}\right\}^{1/N'};
\end{aligned}
\tag{8.2}
$$

the expressions for $\bar{L}_k(MVC \mid mvc)$, $\bar{L}_k(VC \mid mvc)$, and $\bar{L}_k(M \mid mVC)$ are the same as those in (8.2) with λ_k replaced by $\bar{\lambda}_k$. The λ_k and $\bar{\lambda}_k$ can be obtained explicitly by straightforward application of the likelihood-ratio method (see the paragraph preceding section 4a).

8b. *Moments of the k-sample criteria.* The exact distribution of any of the k-sample criteria, when the corresponding null hypothesis is true, is given in (6.10), where the quantities p_j, q_j, and c can be specified by means of the moment expressions given below. The moments have been obtained by means of the operators discussed in Section 5.

For each of the following six moment expressions the null hypothesis, corresponding to the sample criterion involved, is assumed to be true:

$$
\begin{aligned}
E[L_k(MVC \mid mvc)]^d &= \left\{\frac{\prod_{p=1}^{k}\prod_{r=1}^{q}\prod_{u_p=1}^{N_p}\left(\frac{1}{2} - \frac{r}{2N_p} + \frac{(u_p - 1)}{N_p}\right)_d}{\prod_{r=1}^{q}\prod_{u=1}^{N'}\left(\frac{1}{2} - \frac{r}{2N'} + \frac{(u - 1)}{N'}\right)_d}\right\} \\
&\quad \times \left\{\frac{\prod_{p=1}^{k}\prod_{a=1}^{h}\prod_{u'_p=1}^{N_p(n_a-1)}\left(\frac{1}{2} + \frac{(u'_p - 1)}{N'(n_a - 1)}\right)_d}{\prod_{a=1}^{h}\prod_{u'=1}^{N'(n_a-1)}\left(\frac{1}{2} + \frac{(u' - 1)}{N'(n_a - 1)}\right)_d}\right\};
\end{aligned}
\tag{8.3}
$$

$$
E[L_k(VC \mid mvc)]^d = \left\{\frac{\prod_{p=1}^{k}\prod_{r=1}^{q}\prod_{u_p=1}^{N_p}\left(\frac{1}{2} - \frac{r}{2N_p} + \frac{(u_p - 1)}{N_p}\right)_d}{\prod_{r=1}^{q}\prod_{u=1}^{N'}\left(\frac{1}{2} - \frac{(k + r - 1)}{2N'} + \frac{(u - 1)}{N'}\right)_d}\right\}
$$

$$\times \left\{ \frac{\prod_{p=1}^{k} \prod_{a=1}^{h} \prod_{u'_p=1}^{N_p(n_a-1)} \left(\frac{1}{2} + \frac{(u'_p - 1)}{N_p(n_a - 1)} \right)_d}{\prod_{a=1}^{h} \prod_{u'=1}^{N'(n_a-1)} \left(\frac{1}{2} + \frac{(u' - 1)}{N'(n_a - 1)} \right)_d} \right\};$$

$$E[L_k(M \mid mVC)]^d = \prod_{r=1}^{q} \left\{ \frac{\left(\frac{N' - k + 1 - r}{2} \right)_d}{\left(\frac{N' - r}{2} \right)_d} \right\};$$

$$E[\bar{L}_k(MVC \mid mvc)]^d = \left\{ \frac{\prod_{p=1}^{k} \prod_{a=1}^{h} \prod_{u_p=1}^{N_p} \left(\frac{1}{2} - \frac{a}{2N_p} + \frac{(u_p - 1)}{N_p} \right)_d}{\prod_{a=1}^{h} \prod_{u=1}^{N'} \left(\frac{1}{2} - \frac{a}{2N'} + \frac{u - 1}{N'} \right)_d} \right\}$$

$$\times \left\{ \frac{\prod_{p=1}^{k} \prod_{a=1}^{h} \prod_{u'_p=1}^{N_p(n-1)} \left(\frac{1}{2} + \frac{1 - a}{2N_p(n - 1)} + \frac{(u'_p - 1)}{N_p(n - 1)} \right)_d}{\prod_{a=1}^{h} \prod_{u'=1}^{N'(n-1)} \left(\frac{1}{2} + \frac{1 - a}{2N'(n - 1)} + \frac{(u' - 1)}{N'(n - 1)} \right)_d} \right\};$$

$$E[\bar{L}_k(VC \mid mvc)]^d = \left\{ \frac{\prod_{p=1}^{k} \prod_{a=1}^{h} \prod_{u_p=1}^{N_p} \left(\frac{1}{2} - \frac{a}{2N_p} + \frac{(u_p - 1)}{N_p} \right)_d}{\prod_{a=1}^{h} \prod_{u=1}^{N'} \left(\frac{1}{2} - \frac{(k + a - 1)}{2N'} + \frac{(u - 1)}{N'} \right)_d} \right\}$$

$$\times \left\{ \frac{\prod_{p=1}^{k} \prod_{a=1}^{h} \prod_{u'_p=1}^{N_p(n-1)} \left(\frac{1}{2} + \frac{1 - a}{2N_p(n - 1)} + \frac{(u'_p - 1)}{N_p(n - 1)} \right)_d}{\prod_{a=1}^{h} \prod_{u'=1}^{N'(n-1)} \left(\frac{1}{2} + \frac{1 - a}{2N'(n - 1)} + \frac{(u' - 1)}{N'(n - 1)} \right)_d} \right\};$$

$$E[\bar{L}_k(M \mid mVC)]^d = \prod_{a=1}^{h} \left\{ \frac{\left(\frac{N' - k + 1 - a}{2} \right)_d}{\left(\frac{N' - a}{2} \right)_d} \right\},$$

where $d = 0, 1, \cdots$ and $(T)_d$ is defined in (6.4).

8c. *Comments on the criteria.* By an argument similar to that used in section 7f it follows from (8.3) that when $H_k(MVC \mid mvc)$ is true $L_k(MVC \mid mvc)$ is the product of two independently distributed chance quantities, namely, $L_k(VC \mid mvc)$ and $[L_k(M \mid mVC)]^{N'}$. The same assertion holds true if we replace each L by $\bar{L}$ and H by $\bar{H}$.

Exact distributions of the k-sample criteria, when the corresponding null hypotheses are true, can be obtained explicitly for special values of k and special compound symmetries; but owing to lack of space we shall not consider them in this paper.

When the sample size N' is large, the exact distributions of

$$-\ln L_k(MVC \mid mvc), \quad -\ln L_k(VC \mid mvc), \quad -N' \ln L_k(M \mid mVC),$$
$$-\ln \bar{L}_k(MVC \mid mvc), \ -\ln \bar{L}_k(VC \mid mvc),$$

and $-N' \ln \bar{L}_k(M \mid mVC)$ (if the corresponding null hypotheses, respectively, are true) are approximately chi-square distributions with

$$(k-1)\left[\frac{b(b+3)}{2} + hb + \frac{h(h+5)}{2}\right],$$
$$(k-1)[b(b+1)/2 + hb + h(h+3)/2],$$

$q(k-1)$, $h(h+2)(k-1)$, $h(h+1)(k-1)$, and $h(k-1)$ degrees of freedom, respectively.

9. Illustrative examples. The first of the following two examples[2] illustrates the use of $L_1(mvc)$, $L_1(vc)$, and $L_1(m)$ in a psychometrics experiment; the second example illustrates the use of $\bar{L}_1(mvc)$, $\bar{L}_1(vc)$, and $\bar{L}_1(m)$ in a medical-research experiment (see section 1).

EXAMPLE 1. In an experiment to establish methods of obtaining reader reliability in regard to essay scoring, 126 examinees were given a three-part English Composition examination. Each part required that the examinee write an essay, and for each examinee four scores were obtained on the following four things, respectively: (1) the part-2 and part-3 essays together, (2) the original part-1 essay, (3) a long-hand copy of the part-1 essay, (4) a carbon copy of the long-hand copy in (3). Scores were assigned by a group of "English Readers" using procedures designed to counterbalance certain experimental conditions. The score on (1) serves as a criterion. The experimenter asks whether on the basis of the sample (of size 126) the quantities associated with (2), (3), and (4) can be considered as interchangeable among themselves and interchangeable with respect to their relation to the criterion (1).

Let X_1, X_2, X_3, and X_4 be the scores on (1), (2), (3), and (4), respectively. It is assumed that (X_1, X_2, X_3, X_4) has a normal 4-variate distribution and that the set of scores $(X_{1\alpha}, X_{2\alpha}, X_{3\alpha}, X_{4\alpha})$ $(\alpha = 1, \cdots, 126)$ obtained from the essays is a random sample of values of (X_1, X_2, X_3, X_4). The following three questions will be considered (see section 2), where the grouping of the four variates is (1, 3): (a) Is the sample consistent with the hypothesis $H_1(mvc)$? (b) Is the sample consistent with the hypothesis $H_1(vc)$? (c) Is the sample consistent with the hypothesis $H_1(m)$? In the particular experiment under discussion (a) is the experimenter's question.

[2] Mr. L. R. Tucker (Educational Testing Service, Princeton, New Jersey) and Captain J. Allan Rafferty, M.D. (Air University School of Aviation Medicine, Randolph Field, Texas) kindly gave the author the data for Examples 1 and 2, respectively.

The sample means and variance-covariance matrix are as follows:

	X_1	X_2	X_3	X_4
	77.8976	20.9425	23.4544	18.0384
	20.9425	25.0704	12.4363	11.7257
	23.4544	12.4363	28.2021	9.2281
	18.0384	11.7257	9.2281	22.7390
Means	28.0556	14.9048	15.4841	14.4444

This matrix is (1/126) $\| v_{ij} \|$ $(i, j = 1, \cdots, 4)$ (see (4.3)). The sample criteria $L_1(mvc)$, $L_1(vc)$, and $L_1(m)$ will be used to answer questions (a), (b), and (c), respectively. The values of the criteria can be computed from the values of $| v_{ij} |$, $| v'_{ij} |$, and $| \bar{v}_{ij} |$ (see (4.9), (7.1), (7.3)), where v'_{ij} is given in (4.7) and $\bar{v}_{ij}$ is given below (7.1). The $\bar{v}_{ij}(i \neq 1 \neq j)$ are evaluated by simple averaging of certain elements in $\| v_{ij} \|$. Both $| v'_{ij} |$ and $| \bar{v}_{ij} |$ have the block pattern of (3.2) and can be expressed in the simplified form of (3.3), where $h = 1$ and $n_1 = 3$; the simplified form of $| v'_{ij} |$ can also be obtained from (4.10) and (4.11). From the data above it is found that

$$L_1(mvc) = | v_{ij} | / | v'_{ij} | = .9214,$$

$$L_1(vc) = | v_{ij} | / | \bar{v}_{ij} | = .9568,$$

$$L_1(m) = | \bar{v}_{ij} | / | v'_{ij} | = .9630.$$

The second, fourth, and fifth formulas in (7.13) (for $N = 126$, $b = 1$, $n = 3$) give the distributions of $L_1(mvc)$, $L_1(vc)$, and $L_1(m)$, respectively (when the hypothesis with which the criterion is associated is true). By direct computation with expressions for the Incomplete Beta Function ratios the per cent points corresponding to the observed values of $L_1(mvc)$, $L_1(vc)$, and $L_1(m)$ are found to be .26, .49, and .09, respectively. Thus at the 5% significance level the answer to any given one of the three questions (a), (b), (c) is yes. Critical values of $L_1(mvc)$, $L_1(vc)$, and $L_1(m)$ for various significance levels can be obtained from [3] by interpolation.

EXAMPLE 2. In an experiment to study certain properties of the blood of asphyxiated dogs, the $\%CO_2$ and hematocrit of 10 asphyxiated dogs were measured four minutes and seven minutes after asphyxiation. Let X_1 and X_3 be $\%CO_2$ and hematocrit four minutes after asphyxiation, respectively, and X_2 and X_4 be $\%CO_2$ and hematocrit seven minutes after asphyxiation, respectively. It is assumed that (X_1, X_2, X_3, X_4) has a normal 4-variate distribution and that the set of measurements $(X_{1\alpha}, X_{2\alpha}, X_{3\alpha}, X_{4\alpha})$ $(\alpha = 1, \cdots, 10)$ obtained from the 10 dogs is a random sample of values of (X_1, X_2, X_3, X_4). The following questions will be considered, where the grouping is (2^2): (a) Is the sample consistent with the hypothesis $\bar{H}_1(mvc)$? (b) Is the sample consistent with the hypothesis $\bar{H}_1(vc)$? (c) Is the sample consistent with the hypothesis $\bar{H}_1(m)$? In the particular experiment under discussion (a) is the experimenter's question.

The sample means and sums of squares and cross-products are as follows:

	X_1	X_2	X_3	X_4
	294.916	313.908	−89.364	−69.282
	313.908	363.689	−130.422	−69.261
	−89.364	−130.422	210.356	241.688
	−69.282	−69.261	241.688	515.789
Means	50.780	53.590	41.180	43.890.

This matrix is $\| v_{ij} \|$ $(i, j = 1, \cdots, 4)$ (see (4.3)). The sample criteria $\bar{L}_1(mvc)$, $\bar{L}_1(vc)$, and $\bar{L}_1(m)$ will be used to answer questions (a), (b), and (c), respectively. The values of these criteria can be computed from the data above (see (7.6), (7.8), and (7.10)) and are found to be:

$$\bar{L}_1(mvc) = | v_{ij} | / | \bar{v}'_{ij} | = .09107,$$

$$\bar{L}_1(vc) = | v_{ij} | / | \bar{v}_{ij} | = .3259,$$

$$\bar{L}_1(m) = | \bar{v}_{ij} | / | \bar{v}'_{ij} | = .2794.$$

The sixth, seventh, and eighth formulas in (7.13) (for $N = 10$, $n = 2$) give the distributions of $\bar{L}_1(mvc)$, $\bar{L}_1(vc)$, and $\bar{L}_1(m)$, respectively (when the hypothesis with which the criterion is associated is true). From [1] it is found that the observed values of $\bar{L}_1(mvc)$, $\bar{L}_1(vc)$, and $\bar{L}_1(m)$ correspond to the 1.2, 12.4, and .6 per cent points, respectively, of the distributions referred to above. Thus at the 5% significance level the answer to questions (a) and (c) is no and to (b) is yes. The critical values of $\bar{L}_1(mvc)$, $\bar{L}_1(vc)$, and $\bar{L}_1(m)$ for various significance levels can be found from [3].

More than one of the sample criteria may be of interest in regard to a given sample (see [5] pp. 267–268). For example, in an experiment such as that described in Example 1 suppose the answer to question (a) is no. The experimenter might then consider question (b); if the answer is no, the inconsistency of the sample with $H_1(mvc)$ might be regarded as due to the variances or covariances. If the answer to (b) is yes, the experimenter might then consider (c); if the answer here is no, the inconsistency of the sample with $H_1(mvc)$ might be regarded as due to the means. If, however, the answer here is yes, further study might be required to "explain" the inconsistency.

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