CORRECTION TO "ASYMPTOTIC FORMULAS FOR SIGNIFICANCE LEVELS OF CERTAIN DISTRIBUTIONS"

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Professor Henry Scheffé has recently pointed out to me an error in my paper "Asymptotic formulas for significance levels of certain distributions," which appeared in *Annals of Math. Stat.*, Vol. 14 (1943), pp. 56–62. In the determination of the significance levels of Student's t distribution, appeal was made to a theorem of Cramér which requires independent random variables. The variables defined at the top of page 61, however, *cannot* be taken as independent, so that the theorem does not apply.

The asymptotic formula (following the notation of the paper)

$$t_{p,n} = y_p + \frac{y_p^3 + y_p}{4n} + o\left(\frac{1}{n}\right),$$

where

$$\Phi(y_p) = 1 - p, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-v^2/2}\, dv,$$

is nevertheless correct. This may be shown directly from the distribution function

$$G_n(x) = \tfrac{1}{2} + \frac{1}{\sqrt{n\pi}} \frac{\Gamma(\frac{1}{2}(n+1))}{\Gamma(\frac{1}{2}n)} \int_0^x \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} dt.$$

Writing

$$\begin{aligned}\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} &= \exp\left[-\frac{n+1}{2} \log\left(1 + \frac{t^2}{n}\right)\right] \\ &= \exp\left[-\frac{n+1}{2}\left(\frac{t^2}{n} - \frac{t^4}{2n^2} + \frac{t^6}{3(n+\theta t^2)^3}\right)\right], \qquad |\theta| < 1,\end{aligned}$$

and using Stirling's formula, it follows that $G_n(x)$ can be written in the form

$$\begin{aligned}G_n(x) &= \tfrac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\left[1 + \frac{t^4 - 2t^2 - 1}{4n} + \frac{1}{n^2} Q_n(t)\right] dt \\ &= \Phi(x_p) - \frac{x^3 + x}{4n\sqrt{2\pi}} e^{-x^2/2} + \frac{1}{n^2\sqrt{2\pi}} \int_0^x Q_n(t) e^{-t^2/2}\, dt,\end{aligned}$$

where $Q_n(t)$ is a bounded function of t and n in each finite interval.

Let $t_{p,n} = y_p + a_n$, where $a_n = o(1)$. Then $G_n(t_{p,n}) = \Phi(y_p) = 1 - p$, and we have

$$na_n \frac{\Phi(y_p + a_n) - \Phi(y_p)}{a_n} = \frac{(y_p + a_n)^3 + (y_p + a_n)}{4\sqrt{2\pi}} e^{-\frac{1}{2}(y_p+a_n)^2} + O\left(\frac{1}{n}\right),$$

so that

$$\lim_{n\to\infty} na_n = \frac{y_p^3 + y_p}{4}.$$

This is the required result.