It is interesting to note the simplification of the independence condition given in [2, 3] which is possible when the forms are assumed to be non-negative. It may also be of interest to note that the condition for independence given in the present theorem is identical with the corresponding condition for two linear forms. (In fact, the latter condition has been used in the above proof.) Further we observe that if Q_2 is the square of a linear form with mean 0, we get a necessary and sufficient condition for independence between a linear form and a non-negative quadratic form. The corresponding condition when Q_1 is not supposed to be non-negative has been given in [4].

As an application consider a quadratic form Q in normally correlated variables. Let it be known that Q has a χ^2-distribution with f degrees of freedom. If further

$$Q = Q_1 + Q_2 + \cdots + Q_s, \tag{6}$$

where the Q_i's are non-negative and mutually uncorrelated quadratic forms, then each Q_i has a χ^2-distribution with f_i degrees of freedom, say, and $\Sigma f_i = f$. The proof with the aid of the above theorem is almost immediate. We thus get another formulation of the theorem of Cochran [1] on the decomposition of a quadratic form.

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A FORMULA FOR THE PARTIAL SUMS OF SOME HYPERGEOMETRIC SERIES

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Let an urn contain N balls of which are a black and b white. A single ball is drawn. We note its color, return the ball into the urn and add Δ balls of the same color. The probability $w(n_1)$ to obtain n_1 black balls in n trials is given by a formula due to F. Eggenberger and G. Pólya [1]:

[1] Opinions or conclusions contained in this paper are those of the author. They are not to be construed as necessarily reflecting the views or endorsement of the Navy Department.

$$(1)\quad w(n_1) = \binom{n}{n_1} \frac{a(a+\Delta)\cdots[a+(n_1-1)\Delta]\cdot b(b+\Delta)\cdots[b+(n-n_1-1)\Delta]}{N(N+\Delta)\cdots[N+(n-1)\Delta]}$$

$$(n \text{ fixed}, \qquad n_1 \text{ variable}).$$

Now, we fix n_1 and ask for the probability that the n_1th black ball appears at the nth drawing. We find

$$(2)\quad w(n) = \binom{n-1}{n_1-1} \frac{a(a+\Delta)\cdots[a+(n_1-1)\Delta].b(b+\Delta)\cdots[b+(n-n_1-1)\Delta]}{N(N+\Delta)\cdots[N+(n-1)\Delta]}$$

$$(n_1 \text{ fixed}, \qquad n \text{ variable})$$

This function is the $(n - n_1 + 1)$th element of the series

$$\frac{\frac{a}{\Delta}\left(\frac{a}{\Delta}+1\right)\cdots\left[\frac{a}{\Delta}+(n_1-1)\right]}{\frac{N}{\Delta}\left(\frac{N}{\Delta}+1\right)\cdots\left[\frac{N}{\Delta}+(n_1-1)\right]}\cdot F\left(n_1, \frac{b}{\Delta}, \frac{N}{\Delta}+n_1; 1\right).$$

Consequently, the probability that the n_1th black ball appears at the latest in the nth drawing reads, with an obvious abbreviation,

$$W(n) = \sum_{i=n_1}^{n} w(i)$$

$$(3)\quad = \frac{\frac{a}{\Delta}\left(\frac{a}{\Delta}+1\right)\cdots\left[\frac{a}{\Delta}+(n_1-1)\right]}{\frac{N}{\Delta}\left(\frac{N}{\Delta}+1\right)\cdots\left[\frac{N}{\Delta}+(n_1-1)\right]}\cdot F_{n-n_1+1}\left(n_1, \frac{b}{\Delta}, \frac{N}{\Delta}+n_1; 1\right).$$

Now, we assume the n_1th black ball did not appear in the nth drawing. What is the alternative? The $(n - n_1 + 1)$th white ball must have appeared in the nth drawing at latest. The corresponding probability is according to the equation (3)

$$\overline{W}(n) = \sum_{i=n-n_1+1}^{n} \overline{w}(i)$$

$$(4)\quad = \frac{\frac{b}{\Delta}\left(\frac{b}{\Delta}+1\right)\cdots\left[\frac{b}{\Delta}+(n-n_1)\right]}{\frac{N}{\Delta}\left(\frac{N}{\Delta}+1\right)\cdots\left[\frac{N}{\Delta}+(n-n_1)\right]} F_{n_1}\left(n-n_1+1, \frac{a}{\Delta}, \frac{N}{\Delta}+n-n_1+1; 1\right).$$

The relation (4) originates from (3) by writing b instead of a and $(n - n_1 + 1)$ instead of n_1. The alternatives add to certainty:

$$(5)\quad W(n) + \overline{W}(n) = 1.$$

Change the notations in the following manner:

$$(6) \qquad n_1 \to \alpha, \quad \frac{b}{\Delta} \to \beta; \quad \frac{N}{\Delta} + n_1 \to \gamma; \quad n - n_1 + 1 \to \nu.$$

From (6.1) and (6.4) find by addition

$$(7) \qquad n \to \nu + \alpha - 1.$$

From (6.1) and (6.3)

$$(8) \qquad \frac{N}{\Delta} \to \gamma - \alpha.$$

From (6.2) and (8)

$$(9) \qquad \frac{a}{\Delta} = \frac{N}{\Delta} - \frac{b}{\Delta} \to \gamma - \alpha - \beta.$$

Formula (5) reads now

$$(10) \qquad \begin{aligned} &\frac{(\gamma - \alpha - \beta)(\gamma - \alpha - \beta + 1) \cdots (\gamma - \beta - 1)}{(\gamma - \alpha)(\gamma - \alpha + 1) \cdots (\gamma - 1)} \cdot F_\nu(\alpha, \beta, \gamma; 1) \\ &\quad + \frac{\beta(\beta + 1) \cdots (\beta + \nu - 1)}{(\gamma - \alpha)(\gamma - \alpha + 1) \cdots (\gamma - \alpha + \nu - 1)} \\ &\qquad \cdot F_\alpha(\nu, \gamma - \beta - \alpha, \gamma - \alpha + \nu; 1) = 1. \end{aligned}$$

$F_\nu(\alpha, \beta, \gamma; 1)$ denotes the partial sum of the first ν elements of the hypergeometric series $F(\alpha, \beta, \gamma; 1)$. It is to be mentioned that α is a positive integer necessarily as follows from (6.1). Since

$$W(\infty) = 1 = \frac{(\gamma - \alpha - \beta)(\gamma - \alpha - \beta + 1) \cdots (\gamma - \beta - 1)}{(\gamma - \alpha)(\gamma - \alpha + 1) \cdots (\gamma - 1)} F_\infty(\alpha, \beta, \gamma; 1),$$

the relation (10) can be written

$$(11) \qquad \frac{F_\nu(\alpha, \beta, \gamma; 1)}{F_\infty(\alpha, \beta, \gamma; 1)} + \frac{F_\alpha(\nu, \gamma - \beta - \alpha, \gamma - \alpha + \nu; 1)}{F_\infty(\nu, \gamma - \beta - \alpha, \gamma - \alpha + \nu; 1)} = 1,$$

where ν and α are positive integers.

This result is not interesting from the standpoint of pure mathematics since the sum $F(\alpha, \beta, \gamma; 1)$ is known. But the relation is useful for the statisticians. In calculating the function $W(n)$ they need a sum of n_1 elements instead of $(n - n_1 + 1)$. If n_1 is small (and this holds in practical applications), the exact calculation of $W(n)$ is possible for every n.

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