INDEPENDENCE OF NON-NEGATIVE QUADRATIC FORMS IN NORMALLY CORRELATED VARIABLES

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In a recent paper by the author [5] the following theorem has been mentioned without proof. Though the theorem is very simple and easy to prove the author has not found it elsewhere in the literature.

Theorem. *If two non-negative quadratic forms in normally correlated variables with zero means are uncorrelated the two forms are independent.*

To prove the theorem, let the two forms be

$$(1) \qquad Q_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \qquad Q_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j,$$

where the x_i's are normally correlated and all have mean 0. By a well-known theorem on quadratic forms we can reduce Q_1 and Q_2 to the forms

$$(2) \qquad Q_1 = \sum_{i=1}^{n} c_i y_i^2, \qquad Q_2 = \sum_{i=1}^{n} d_i z_i^2,$$

where the y_i's and z_i's are linear functions of the x_i's. In the $2n$-dimensional normal distribution of the y_i's and the z_i's, let ρ_{ij} be the covariance of y_i and z_j. It is then easily shown that the covariance of y_i^2 and z_j^2 is $2\rho_{ij}^2$, and hence that

$$(3) \qquad \text{cov}\,(Q_1, Q_2) = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} c_i d_j \rho_{ij}^2.$$

As the forms are supposed to be non-negative all coefficients in (2) are non-negative. If Q_1 and Q_2 are uncorrelated, each term on the right hand of (3) must vanish. Consequently, if $c_i \neq 0$ and $d_j \neq 0$, we must have $\rho_{ij} = 0$. This means that all y_i's in Q_1 with non-zero coefficients are independent of all z_i's in Q_2 with non-zero coefficients. Hence Q_1 and Q_2 are independent. Q.E.D.

To see if Q_1 and Q_2 are uncorrelated we need an expression for the covariance of the two forms in terms of the coefficients in (1) and the variances and covariances of the original variables x_i. Let A and B be the matrices of the two forms (1). Clearly we may suppose A and B to be symmetric. Let the variance-covariance matrix of the x_i's be L. By straightforward calculations we find

$$(4) \qquad \text{cov}\,(Q_1, Q_2) = 2\, Tr\, ALBL.$$

Here we have used $Tr\, M$ to denote the "trace," i.e. the sum of the diagonal elements in a square matrix M. In case of independent variables with variance 1, we get

$$(5) \qquad \text{cov}\,(Q_1, Q_2) = 2\, Tr\, AB.$$

The formulae (4) and (5) are given in [5].

It is interesting to note the simplification of the independence condition given in [2, 3] which is possible when the forms are assumed to be non-negative. It may also be of interest to note that the condition for independence given in the present theorem is identical with the corresponding condition for two linear forms. (In fact, the latter condition has been used in the above proof.) Further we observe that if Q_2 is the square of a linear form with mean 0, we get a necessary and sufficient condition for independence between a linear form and a non-negative quadratic form. The corresponding condition when Q_1 is not supposed to be non-negative has been given in [4].

As an application consider a quadratic form Q in normally correlated variables. Let it be known that Q has a χ^2-distribution with f degrees of freedom. If further

$$Q = Q_1 + Q_2 + \cdots + Q_s, \tag{6}$$

where the Q_i's are non-negative and mutually uncorrelated quadratic forms, then each Q_i has a χ^2-distribution with f_i degrees of freedom, say, and $\Sigma f_i = f$. The proof with the aid of the above theorem is almost immediate. We thus get another formulation of the theorem of Cochran [1] on the decomposition of a quadratic form.

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A FORMULA FOR THE PARTIAL SUMS OF SOME HYPERGEOMETRIC SERIES

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Let an urn contain N balls of which are a black and b white. A single ball is drawn. We note its color, return the ball into the urn and add Δ balls of the same color. The probability $w(n_1)$ to obtain n_1 black balls in n trials is given by a formula due to F. Eggenberger and G. Pólya [1]:

[1] Opinions or conclusions contained in this paper are those of the author. They are not to be construed as necessarily reflecting the views or endorsement of the Navy Department.