By means of the relation in (8) one deduces readily that:

$$(13)\quad F(n, x) = \sum_{i=0}^{x} q_i^x/[(q_i - q_0) \cdots (q_i - q_{i-1})(q_i - q_{i+1}) \cdots (q_i - q_x)].$$

Jordan [1, p. 19, eq. (1)] shows this to be the xth Newton divided difference of q^n where the expansion is in terms of $(q - q_0) \cdots (q - q_x)$, for $x = 0, 1, \cdots, n$. The solution for (3) can now be written as:

$$(14)\quad P(n, x) = (q - q_0) \cdots (q - q_{x-1})F_n(x)$$

from which follows:

$$(15)\quad \sum_{x=0}^{n} P(n, x) = q^n.$$

As remarked before, by setting $q = 1$ one obtains the solution of (1) subject to the boundary conditions (2).

It is clear that when all the q_i are equal that the Bernouilli distribution should come out as a special case. Since in this case the divided difference becomes the corresponding derivative divided by the appropriate factorial, one obtains:

$$(16)\quad P(n, x) = \frac{(1 - q_0)^x}{x!} \frac{d^x q^n}{dq^x}\bigg|_{q=q_0}.$$

Upon reduction this yields the usual formula, but not in the usual way.

By choosing $p_x = \lambda_x/n$ and allowing n to increase without limit one obtains an analogue of the Poisson distribution, viz:

$$(17)\quad P(x) = (-\lambda_0) \cdots (-\lambda_x) \sum_{i=0}^{x} e^{-\lambda_i}/[(\lambda_0 - \lambda_i) \cdots (\lambda_{i-1} - \lambda_i)(\lambda_{i+1} - \lambda_i) \cdots (\lambda_x - \lambda_i)]$$

which corresponds to the expansion of $e^{-\lambda}$ about $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_x, \cdots$ when $\lambda = 0$.

REFERENCE

[1] Charles Jordan, *Calculus of Finite Differences*, Chelsea Publishing Co., New York, 2nd ed., 1947.

A GRAPHICAL DETERMINATION OF SAMPLE SIZE FOR WILKS' TOLERANCE LIMITS

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1. Summary. To determine the smallest sample size for which the minimum and the maximum of a sample are the $100\beta\%$ distribution-free tolerance limits at the probability level ϵ, one has to solve the equation

$$(1)\quad N\beta^{N-1} - (N - 1)\beta^N = 1 - \epsilon$$

given by S. S. Wilks [1]. A direct numerical solution of (1) by trial requires rather laborious tabulations. An approximate formula for the solution has been indicated by H. Scheffé and J. W. Tukey [2], however an analytic proof for this approximation does not seem to be available. The present note describes a graph which makes it possible to solve (1) with sufficient accuracy for all practically useful values of β and ϵ.

2. Construction of the graph. Substituting in (1)

$$N = \frac{\beta}{1 - \beta} x$$

we obtain

$$1 + x = (1 - \epsilon)\beta^{-\frac{\beta}{1-\beta} x}$$

and

$$\log (1 + x) = - \log \frac{1}{1 - \epsilon} + \left(\frac{\beta}{1 - \beta} \log \frac{1}{\beta}\right) x. \tag{2}$$

To solve (2) graphically, one has to find the intersection of the curve

$$y = \log (1 + x) \tag{3}$$

with the line

$$y = - \log \frac{1}{1 - \epsilon} + \left(\frac{\beta}{1 - \beta} \log \frac{1}{\beta}\right) x.$$

To prepare a graph on which this can be done, one first plots (3) once for al (Figure 1, Curve C). Then one marks the points $- \log \frac{1}{1 - \epsilon}$ on the y-axis and labels them with the values of ϵ (Figure 1, Scale I); chooses a constant $r > 0$ and marks the points $r \log \frac{1}{1 - \epsilon}$ on the x-axis (Figure 1, Scale II); chooses a constant $k > 0$, marks the points $kr \frac{\beta}{1 - \beta} \log \frac{1}{\beta}$ on the x-axis, draws vertical lines through each of these points, and labels them with the values of β (Figure 1, Scale III); draws the line $x = k$ (Figure 1, line L); marks the uniform Scale IV on the x-axis.

The graph reproduced here has been prepared with $r = 4$, $k = 5$. It can easily be verified that the instructions on the graph lead to solutions x of (2) and $N = x \frac{\beta}{1 - \beta}$ of (1).

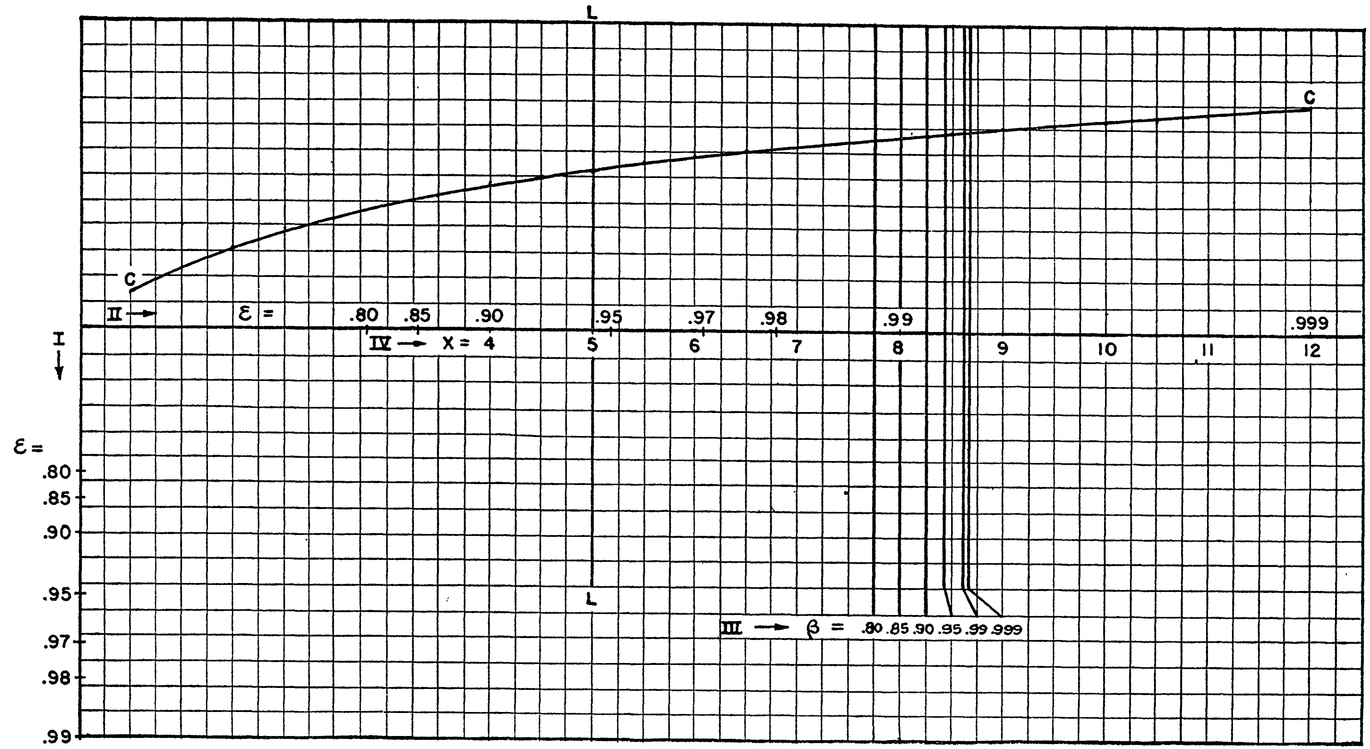

FIG. 1

To find an approximate solution of the equation $N\beta^{N-1} - (N-1)\beta^N = 1 - \epsilon$

1) connect ϵ on Scale I and ϵ on Scale II with a straight line; this line cuts vertical line marked β on Scale III at point P,
2) locate on line L the point with the ordinate of P; call this point Q,
3) connect ϵ on Scale I with Q; the connecting line cuts curve C at a point which has abscissa x on Scale IV; read off x,
4) compute $N = x \dfrac{\beta}{1 - \beta}$

3. Improvement by iterations. The graphical solution, usually accurate to two significant digits, may be improved easily by iterations. Replacing (2) by the equation

$$x = \left[\log (1 + x) + \log \frac{1}{1 - \epsilon}\right]\left(\frac{\beta}{1 - \beta} \log \frac{1}{\beta}\right)^{-1} = f(x) \tag{4}$$

one obtains iterations $x_{j+1} = f(x_j)$ which, for $.80 \leq \epsilon \leq .999$ and $.80 \leq \beta \leq .999$ converge rapidly to the solution of (2).

EXAMPLE. For $\epsilon = .99$, $\beta = .999$, one finds graphically $x_1 = 6.6$, and from (4) the iteration formula $x_{j+1} = \dfrac{\log (1 + x_j) + 2}{.4337}$ which yields the values $x_2 = 6.642$, $x_3 = 6.648$, $x_4 = 6.649$, $x_5 = 6.649$. Rounding up we obtain the sample size $N = 6.649 \cdot 999 = 6643$.

For ϵ and β between .80 and .999 all iterations obtained from (4) are on the same side of the exact solution and converge to it monotonically. Thus, in our example, from $x_1 < x_2$ we conclude that x_1 as well as all further iterations are smaller than the exact solution.

REFERENCES

[1] S. S. WILKS, *Mathematical Statistics*, Princeton University Press, 1943, p. 94.
[2] H. SCHEFFÉ AND J. W. TUKEY, "A formula for sample sizes for population tolerance limits," *Annals of Math. Stat.*, Vol. 15 (1944), p. 217.