A SECOND FORMULA FOR THE PARTIAL SUM OF HYPERGEOMETRIC SERIES HAVING UNITY AS THE FOURTH ARGUMENT

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A convergent hypergeometric series with 1 as fourth argument has been expressed by Gauss, using gamma functions, as follows:

$$(1)\quad F(\alpha, \beta, \gamma; 1) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} + \frac{\alpha(\alpha + 1) \cdot \beta(\beta + 1)}{\gamma(\gamma + 1) \cdot 1 \cdot 2} + \cdots = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}.$$

Let us denote the νth partial sum of $F(\alpha, \beta, \gamma; 1)$ by $F_\nu(\alpha, \beta, \gamma; 1)$, and let us put

$$(2)\qquad \frac{F_\nu(\alpha, \beta, \gamma; 1)}{F(\alpha, \beta, \gamma; 1)} = G_\nu(\alpha, \beta, \gamma).$$

The following equation is obvious:

$$(3)\qquad G_\nu(\alpha, \beta, \gamma) = G_\nu(\beta, \alpha, \gamma).$$

In [1] it is shown that

$$(4)\qquad G_\nu(\alpha, \beta, \gamma) = 1 - G_\alpha(\nu, \gamma - \beta - \alpha, \gamma - \alpha + \nu)$$

is valid if α is a positive integer.

If $(\gamma - \beta - \alpha)$ is a positive integer, (3) and (4) yield

$$\begin{aligned} G_\nu(\alpha, \beta, \gamma) &= 1 - G_\alpha(\gamma - \beta - \alpha, \nu, \gamma - \alpha + \nu) \\ &= G_{\gamma-\beta-\alpha}(\alpha, \beta, \alpha + \beta + \nu). \end{aligned}$$

In terms of partial sums of the hypergeometric series this becomes

$$(5)\qquad \begin{aligned} &\frac{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}{\Gamma(\gamma)\Gamma(\gamma - \beta - \alpha)} F_\nu(\alpha, \beta, \gamma; 1) \\ &\qquad = \frac{\Gamma(\alpha + \nu)\Gamma(\beta + \nu)}{\Gamma(\nu)\Gamma(\alpha + \beta + \nu)} F_{\gamma-\beta-\alpha}(\alpha, \beta, \alpha + \beta + \nu; 1), \end{aligned}$$

which is a new formula involving partial sums of hypergeometric series with 1 as fourth argument. It is more useful than (4) if $\gamma - \beta - \alpha < \alpha$ or $\gamma < 2\alpha + \beta$.

It is of theoretic interest that the arguments of the new series do not depend on the third argument γ of the original series. Therefore it is possible to develop a simple recursion formula. If we write (5) for $(\gamma - 1)$ instead of γ, the series of the second member has one term less. Subtracting these equations yields after some simplifications

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$$\begin{aligned}(6)\qquad &(\gamma - \alpha - 1)(\gamma - \beta - 1)F_\nu(\alpha, \beta, \gamma; 1) \\ &\qquad - (\gamma - \beta - \alpha - 1)(\gamma - 1)F_\nu(\alpha, \beta, \gamma - 1; 1) \\ &\qquad\qquad = \frac{\Gamma(\nu + \alpha)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\nu + \beta)}{\Gamma(\beta)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\nu + \gamma - 1)} \cdot \frac{\Gamma(1)}{\Gamma(\nu)}.\end{aligned}$$

Many recursion formulas are known for hypergeometric functions, but (6) may be the first equation of this type linking two hypergeometric partial sums of ν terms each.

In order to demonstrate the numerical advantage of the new formula (5), we restate the example of [1]. An urn may contain N balls of which a black and b white. A single ball is drawn. We note its color, return the ball into the urn and add Δ balls of the same color. The probability that the n_1th black ball appears at the latest in the n-th drawing is

$$(7)\qquad W(n) = \frac{\frac{a}{\Delta}\left(\frac{a}{\Delta} + 1\right) \cdots \left[\frac{a}{\Delta} + n_1 - 1)\right]}{\frac{N}{\Delta}\left(\frac{N}{\Delta} + 1\right) \cdots \left[\frac{N}{\Delta} + (n_1 - 1)\right]} F_{n-n_1+1}\left(n_1, \frac{b}{\Delta}, \frac{N}{\Delta} + n_1; 1\right).$$

If $\frac{a}{\Delta}$ is a positive integer (5) yields

$$\begin{aligned}(8)\qquad W(n) &= \frac{(n - n + 1)(n - n_1 + 2) \cdots n}{\left(\frac{b}{\Delta} + n - n_1 + 1\right)\left(\frac{b}{\Delta} + n - n_1 + 2\right) \cdots \left(\frac{b}{\Delta} + n\right)} \\ &\qquad \cdot F_{a/\Delta}\left(n_1 \frac{b}{\Delta}, \frac{b}{\Delta} + n + 1; 1\right).\end{aligned}$$

If we take

$$\Delta = 1, \qquad a = 1, \qquad b = N - 1,$$

we get

$$(9)\qquad W(n) = \frac{n!(N + n - n_1 - 1)!}{(n - n_1)!(N + n - 1)!}.$$

Calculating W(n), using the original formula (7), is quite tedious, but (5) sometimes simplifies the numerical work. Let us calculate the probability W(6) that the third black ball appears in the 6th drawing, if the number of the original balls is N = 10. Using formulas (7), (4), and (9) respectively we have

$$W(6) = \frac{3!9!}{12!}\left[1 + \frac{3\cdot 9}{13\cdot 1} + \frac{(3\cdot 4)(9\cdot 10)}{(13\cdot 14)(1\cdot 2)} + \frac{(3\cdot 4\cdot 5)(9\cdot 10\cdot 11)}{(13\cdot 14\cdot 15)(1\cdot 2\cdot 3)}\right] = \frac{4}{91};$$

$$W(6) = 1 - \frac{12!9!}{8!13!}\left[1 + \frac{4\cdot 1}{14\cdot 1} + \frac{(4\cdot 5)(1\cdot 2)}{(14\cdot 15)(1\cdot 2)}\right] = \frac{4}{91};$$

$$W(6) = \frac{6!12!}{3!15!} = \frac{4}{91}.$$

The time saved in using both formulas, of course, increases as the number of terms, $n - n_1 - 1$, of the original series, increases.

Let us mention that the special distribution corresponding to (9) does not have finite moments. For arbitrary values of N, a, Δ the arithmetic mean is

$$E(n) = \frac{N - \Delta}{a - \Delta} \cdot n_1, \tag{10}$$

the expectation of $n(n + 1)$ is

$$E[n(n + 1)] = \frac{(N - \Delta)(N - 2\Delta)}{(a - \Delta)(a - 2\Delta)} \cdot n_1(n_1 + 1), \tag{11}$$

and finally the variance

$$\sigma^2(n) = \frac{(N - \Delta)(N - a)[(n_1 - 1)\Delta + a]}{(a - \Delta)^2(a - 2\Delta)} \cdot n_1. \tag{12}$$

The mode can be derived from the fact that

$$w(n + 1) = w(n) \quad \text{for} \quad n = \frac{N}{a + \Delta} \cdot (n_1 - 1). \tag{13}$$

Especially we get $w(11) = w(10)$ for our numerical example.

The mean and variance do not exist for $a = \Delta = 1$, as in our example. However, it is possible to find a number n so that $W(n)$ takes any value near to unity, for instance .99. For large n and small n_1 (9) yields the approximation

$$W(n) = \frac{n(n - 1) \cdots (n - n_1 + 1)}{(N + n - 1)(N + n - 2) \cdots (N + n - n_1)} \sim \left(\frac{n - \dfrac{n_1 - 1}{2}}{N + n - \dfrac{n_1 + 1}{2}} \right)^{n_1}.$$

Hence, $W(2666) = .99$ for our example. One needs 2666 trials if one wants a 99% probability for getting three black balls. This surprising result cannot be derived from the original formula (7).

REFERENCE

[1] H. von Schelling, "A formula for the partial sums of some hypergeometric series", *Annals of Math. Stat.*, Vol. 20 (1949), pp. 120–122.