It can also be proved, by considering the limiting form of the recurrence relation (19), that the frequency function f_n is asymptotically normal. The main difficulty of proving this fact lies in showing that the frequency function actually possesses a limiting form; and the proof is rather too long to be given here.

REFERENCES

[1] R. Deltheil, *Probabilités Géométriques. Traité du Calcul des Probabilités et de ses Applications: Tome II, Fasciscule II*, Gauthier-Villars, 1926.
[2] K. Pearson, *Tables of the Incomplete Beta-Function*, Cambridge University Press, 1934.
[3] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946.

A NOTE ON THE ASYMPTOTIC SIMULTANEOUS DISTRIBUTION OF THE SAMPLE MEDIAN AND THE MEAN DEVIATION FROM THE SAMPLE MEDIAN

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Consider a random sample of $2k + 1$ values from a one-dimensional distribution of the continuous type with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x) = F'(x)$. Let the mean, standard deviation and median of the distribution be denoted by m, σ and θ respectively (θ assumed to be unique). We shall suppose that in some neighborhood of $x = \theta$, $f(x)$ has a continuous derivative $f'(x)$.

If we arrange the sample values in ascending order of magnitude:

$$x_1 < x_2 < \cdots < x_{2k+1},$$

there is a unique sample median x_{k+1} which we shall denote by ξ. The mean deviation from the sample median is then defined by

$$M = \frac{1}{2k} \sum_{i=1}^{2k+1} | x_i - \xi |.$$

In the material that follows we shall assume that the sample items have been ordered only to the extent that k of them are less than ξ and k of them are greater than ξ.

We then have the following

Theorem. *Let $f(x)$ be a pdf with finite second moment, continuous at $x = \theta$ with $f(\theta) \neq 0$. Then the simultaneous distribution of ξ and M is asymptotically normal. The means of the limiting distribution are θ, the population median, and u', the mean deviation from the population median, while the asymptotic variances are $1/4f^2(\theta)2k$ and $((m - \theta)^2 + \sigma^2 - u'^2)/2k$. The asymptotic expression for the correlation coefficient is $(m - \theta)/\sqrt{(m - \theta)^2 + \sigma^2 - u'^2}$.*

Proof: Let $u = (M - u')\sqrt{2k}$ and $v = (\xi - \theta)\sqrt{2k}$, where $u' = E \mid x - \theta \mid$. Then the simultaneous characteristic function of the two random variables u

and v is given by the following:

$$\begin{aligned}\phi(t_1, t_2) &= E[e^{it_1u+it_2v}] \\ &= E[e^{it_1(M-u')\sqrt{2k}+it_2(\xi-\theta)\sqrt{2k}}] \\ &= E \exp\left[it_1\left(\frac{1}{2k}\sum_{i=1}^{2k+1}|x_i-\xi| - u'\right)\sqrt{2k} + it_2(\xi-\theta)\sqrt{2k}\right] \\ &= \frac{(2k+1)!}{(k!)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\xi}\cdots\int_{-\infty}^{\xi}\int_{\xi}^{\infty}\cdots\int_{\xi}^{\infty} \\ &\quad \cdot \exp\left[it_1\left\{\frac{\sum_{i=k+2}^{2k+1}x_i - \sum_{j=1}^{k}x_j}{2k} - u'\right\}\sqrt{2k} + it_2(\xi-\theta)\sqrt{2k}\right] \\ &\quad f(x_1)f(x_2)\cdots f(x_k)f(x_{k+2})\cdots f(x_{2k+1})f(\xi) \\ &\quad dx_{2k+1}\cdots dx_{k+2}dx_k\cdots dx_1 d\xi \\ &= \frac{(2k+1)!}{(k!)^2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\xi}\exp\left\{-\frac{it_1}{\sqrt{2k}}(x+u')\right\}f(x)\,dx\right]^k \\ &\quad \left[\int_{\xi}^{\infty}\exp\left\{\frac{it_1}{\sqrt{2k}}(x-u')\right\}f(x)\,dx\right]^k e^{it_2(\xi-\theta)\sqrt{2k}}f(\xi)\,d\xi.\end{aligned}$$

Upon making the substitution $\xi = \theta + y/\sqrt{2k}$, the above expression can be reduced to the following form:

$$\begin{aligned}(1)\quad \phi(t_1, t_2) = \frac{(2k+1)!}{\sqrt{2k}(k!)^2}\int_{-\infty}^{\infty}\Bigg\{\Bigg[&\int_{-\infty}^{\theta}\exp\left[-\frac{it_1}{\sqrt{2k}}(x+u')\right]f(x)\,dx \\ &+ \int_{\theta}^{\theta+(y/\sqrt{2k})}\exp\left[-\frac{it_1}{\sqrt{2k}}(x+u')\right]f(x)\,dx\Bigg] \\ \cdot \Bigg[&\int_{\theta}^{\infty}\exp\left[\frac{it_1}{\sqrt{2k}}(x-u')\right]f(x)\,dx \\ &- \int_{\theta}^{\theta+(y/\sqrt{2k})}\exp\left[\frac{it_1}{\sqrt{2k}}(x-u')\right]f(x)\,dx\Bigg]\Bigg\}^k \\ \cdot\ & e^{it_2y}f\left(\theta+\frac{y}{\sqrt{2k}}\right)dy.\end{aligned}$$

Now

$$\int_{-\infty}^{\theta}\exp\left[-\frac{it_1}{\sqrt{2k}}(x+u')\right]f(x)\,dx = \tfrac{1}{2} - \frac{it_1}{\sqrt{2k}}\int_{-\infty}^{\theta}(x+u')f(x)\,dx - \frac{t_1^2}{2(2k)}\int_{-\infty}^{\theta}(x+u')^2f(x)\,dx + \frac{\zeta_1(2k, t_1)}{2k};$$

and

$$\int_\theta^\infty \exp\left[\frac{it_1}{\sqrt{2k}}(x - u')\right] f(x)\,dx = \tfrac{1}{2} + \frac{it_1}{\sqrt{2k}}\int_\theta^\infty (x - u')f(x)\,dx - \frac{t_1^2}{2(2k)}\int_\theta^\infty (x - u')^2 f(x)\,dx + \frac{\zeta_2(2k, t_1)}{2k},$$

where for every fixed t_1, $\zeta_1(2k, t_1)$ and $\zeta_2(2k, t_1) \to 0$ as $k \to \infty$. Similarly, under the substitution $x = (z/\sqrt{2k}) + \theta$,

$$\int_\theta^{\theta+(y/\sqrt{2k})} \exp\left[\frac{it_1}{\sqrt{2k}}(x - u')\right] f(x)\,dx = \frac{1}{\sqrt{2k}}\int_0^y f\left(\frac{z}{\sqrt{2k}} + \theta\right) dz + \frac{it_1}{2k}\int_0^y \left(\frac{z}{\sqrt{2k}} + \theta - u'\right) f\left(\frac{z}{\sqrt{2k}} + \theta\right) dz + \frac{\zeta_3(2k, t_1)}{2k};$$

and

$$\int_\theta^{\theta+(y/\sqrt{2k})} \exp\left[-\frac{it_1}{\sqrt{2k}}(x + u')\right] f(x)\,dx = \frac{1}{\sqrt{2k}}\int_0^y f\left(\frac{z}{\sqrt{2k}} + \theta\right) dz - \frac{it_1}{2k}\int_0^y \left(\frac{z}{\sqrt{2k}} + \theta + u'\right) f\left(\frac{z}{\sqrt{2k}} + \theta\right) dz + \frac{\zeta_4(2k, t_1)}{2k},$$

where $\zeta_3(2k, t_1)$ and $\zeta_4(2k, t_1) \to 0$ as $k \to \infty$ for each fixed t_1. Substituting these expressions in (1) and performing the indicated multiplications we find after some calculation that (1) can be reduced to the following form:

$$\phi(t_1, t_2) = \int_{-\infty}^\infty \frac{(2k+1)!}{\sqrt{2k}(k!)^2 2^{2k}}\left[1 - \frac{t_1^2(\sigma^2 - u'^2) - 4it_1(m - \theta)yf\left(\frac{z_1}{\sqrt{2k}} + \theta\right)}{2k} + \frac{-4\left\{yf\left(\frac{z_1}{\sqrt{2k}} + \theta\right)\right\}^2 + \zeta(2k, t_1)}{2k}\right]^k e^{it_2y} f\left(\theta + \frac{y}{\sqrt{2k}}\right) dy,$$

where $0 < z_1 < y$ and $\zeta(2k, t_1) \to 0$ for every fixed t_1 as $k \to \infty$. Now taking the limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \phi(t_1, t_2) = \int_{-\infty}^\infty \sqrt{\frac{2}{\pi}} \exp\left[-\frac{t_1^2}{2}(\sigma^2 - u'^2) + \frac{4it_1(m - \theta)f(\theta)y}{2} - \frac{4[f^2(\theta)]y^2}{2} + it_2 y\right] f(\theta)\,dy.$$

Upon performing the integration,

$$\lim_{k\to\infty} \phi(t_1, t_2) = \exp\left[-\tfrac{1}{2}\left\{t_1^2[(m - \theta)^2 + \sigma^2 - u'^2] + \frac{2t_1t_2(m - \theta)}{2f(\theta)} + \frac{t_2^2}{4f^2(\theta)}\right\}\right].$$

Since $\sigma^2 > u'^2$, this is the characteristic function for two variables which are normally distributed. Thus, the simultaneous distribution of ξ and M is asymptotically normal. It is of interest to note that, if the pdf $f(x)$ is symmetric, the correlation coefficient is zero, and M and ξ are asymptotically independent. We might also note that $\phi(t_1, 0)$ is the characteristic function for the mean deviation from the sample median. Thus, the random variable M is asymptotically normal with asymptotic mean and variance u' and $((m - \theta)^2 + \sigma^2 - u'^2)/2k$ respectively.

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REFERENCES

[1] H. CRAMÉR, *Mathematical Methods of Statistics*, Princeton University Press, 1946.
[2] R. K. ZEIGLER, "On the mean deviation from the median," unpublished thesis, State University of Iowa.

NOTE ON THE EXTENSION OF CRAIG'S THEOREM TO NON-CENTRAL VARIATES

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A theorem due to A. T. Craig [1] and H. Hotelling [3] concerning the distribution of real quadratic forms in normal variates is extended to the case of non-central normal variates with equal variance.

The following notation is used: A, A_1, A_2 are real symmetric matrices, L is an orthogonal matrix, Γ is a diagonal matrix of latent roots, and X, Y, M and U are column vectors.

THEOREM. *Let $X' = (x_1, \cdots, x_n)$ be a set of normally and independently distributed variates with equal variance σ^2 and means $M' = (m_1, \cdots, m_n)$. Then, a necessary and sufficient condition that a real symmetric quadratic form $Q(X) = X'AX$ of rank r be distributed as $\sigma^2\chi^2$, where*

$$p(\chi^2, r, \lambda^2) = \tfrac{1}{2}e^{-\lambda^2}(\chi^2/2)^{(r-2)/2}\, e^{-\chi^2/2} \tag{1}$$
$$\sum_{j=0}^{\infty} (\lambda^2\chi^2/2)^j/j!\,\Gamma[(r - 2j)/2],$$

is that $A^2 = A$. If $Q(X)/\sigma^2$ is distributed by $p(\chi^2, r, \lambda^2)$, then $\lambda^2 = Q(M)/2\sigma^2$.

Further, let $Q_1(X) = X'A_1X$ and $Q_2(X) = X'A_2X$ be real symmetric quadratic forms of ranks r_1 and r_2. Then a necessary and sufficient condition that $Q_1(X)$ and $Q_2(X)$ be statistically independent is that $A_1A_2 = 0$.

PROOF. The theorem is proved by establishing the equivalence and factorization of moment generating functions [4]. The moment generating function of