ON THE FUNDAMENTAL LEMMA OF NEYMAN AND PEARSON[1]

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1. Summary and introduction. The following lemma proved by Neyman and Pearson [1] is basic in the theory of testing statistical hypotheses:

Lemma. *Let $f_1(x), \cdots, f_{m+1}(x)$ be $m + 1$ Borel measurable functions defined over a finite dimensional Euclidean space R such that $\int_R |f_i(x)| \, dx < \infty$ $(i = 1, \cdots, m + 1)$. Let, furthermore, $c_1, \cdots, c_m$ be m given constants and $\mathcal{S}$ the class of all Borel measurable subsets S of R for which*

$$\int_S f_i(x) \, dx = c_i \qquad (i = 1, \cdots, m). \tag{1.1}$$

Let, finally, $\mathcal{S}_0$ be the subclass of $\mathcal{S}$ consisting of all members S_0 of $\mathcal{S}$ for which

$$\int_{S_0} f_{m+1}(x) \, dx \geqq \int_S f_{m+1}(x) \, dx \quad \text{for all } S \text{ in } \mathcal{S}. \tag{1.2}$$

If S is a member of $\mathcal{S}$ and if there exist m constants $k_1, \cdots, k_m$ such that

$$f_{m+1}(x) \geqq k_1 f_1(x) + \cdots + k_m f_m(x) \quad \text{when} \quad x \in S, \tag{1.3}$$

$$f_{m+1}(x) \leqq k_1 f_1(x) + \cdots + k_m f_m(x) \quad \text{when} \quad x \notin S, \tag{1.4}$$

then S is a member of $\mathcal{S}_0$.

The above lemma gives merely a sufficient condition for a member S of $\mathcal{S}$ to be also a member of $\mathcal{S}_0$. Two important questions were left open by Neyman and Pearson: (1) the question of existence, that is, the question whether $\mathcal{S}_0$ is non-empty whenever $\mathcal{S}$ is non-empty; (2) the question of necessity of their sufficient condition (apart from the obvious weakening that (1.3) and (1.4) may be violated on a set of measure zero).

The purpose of the present note is to answer the above two questions. It will be shown in Section 2 that $\mathcal{S}_0$ is not empty whenever $\mathcal{S}$ is not empty. In Section 3, a necessary and sufficient condition is given for a member of $\mathcal{S}$ to be also a member of $\mathcal{S}_0$. This necessary and sufficient condition coincides with the Neyman-Pearson sufficient condition under a mild restriction.

2. Proof that $\mathcal{S}_0$ is not empty whenever $\mathcal{S}$ is not empty. Each function $f_i(x)$ determines a finite measure μ_i given by the equation

$$\mu_i(S) = \int_S f_i(x) \, dx \qquad (i = 1, 2, \cdots, m + 1). \tag{2.1}$$

[1] The main results of this paper were obtained by the authors independently of each other using entirely different methods.

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Let μ be the vector measure with the components $\mu_1, \cdots, \mu_{m+1}$; i.e., for any measurable set S the value of $\mu(S)$ is the vector $(\mu_1(S), \cdots, \mu_{m+1}(S))$. Thus, for each S the value of $\mu(S)$ can be represented by a point in the $m + 1$-dimensional Euclidean space E. A point $g = (g_1, \cdots, g_{m+1})$ of E is said to belong to the range of the vector measure μ if and only if there exists a measurable subset S of R such that $\mu(S) = g$.

It was proved by Lyapunov [2] (see also [4]) that the range M of μ is a bounded, closed and convex subset of E. Let L be the line in E which is parallel to the $(m + 1)$-th axis and goes through the point $(c_1, c_2, \cdots, c_m, 0)$. Suppose that $\mathcal{S}$ is not empty. Then the intersection M^* of L with M is not empty. Because of Lyapunov's theorem, M^* is a finite closed interval (which may reduce to a single point). There exists a subset S of R such that $\mu(S)$ is equal to the upper end point of M^*. Clearly, S is a member of $\mathcal{S}_0$.

3. Necessary and sufficient condition that a member of $\mathcal{S}$ be also a member of $\mathcal{S}_0$. Let $\nu(S)$ be the vector measure with the components $\mu_1(S), \cdots \mu_m(S)$. According to the aforementioned theorem of Lyapunov, the range N of ν is a bounded, closed and convex subset of the m-dimensional Euclidean space.

By the dimension of a convex subset Q of a finite dimensional Euclidean space we shall mean the dimension of the smallest dimensional hyperplane that contains Q. A point q of a convex set Q is said to be an interior point of Q if there exists a sphere V with center at q and positive radius such that $V \cap \Pi \subset Q$, where Π is the smallest dimensional hyperplane containing Q. Any point q that is not an interior point of Q will be called a boundary point. We shall now prove the following theorem.

THEOREM 3.1. *If $(c_1, \cdots, c_m)$ is an interior point of N, then a necessary and sufficient condition for a member S of $\mathcal{S}$ to be a member of $\mathcal{S}_0$ is that there exist m constants $k_1, \cdots, k_m$ such that* (1.3) *and* (1.4) *hold for all x except perhaps on a set of measure zero.*

PROOF. The Neyman-Pearson lemma cited in Section 1 states that our condition is sufficient. Thus, we merely have to prove the necessity of our condition. Assume that $(c_1, \cdots, c_m)$ is an interior point of N. Let c^* be the largest value for which $(c_1, \cdots, c_m, c^*) \in M$, and c^{**} the smallest value for which

$$(c_1, \cdots, c_m, c^{**}) \in M.$$

We shall first consider the case when $c^* = c^{**}$. Let $(\bar{c}_1, \cdots, \bar{c}_m)$ be any other interior point of N. We shall show that there exists exactly one real value $\bar{c}$ such that $(\bar{c}_1, \cdots, \bar{c}_m, \bar{c}) \in M$. For suppose that there are two different values $\bar{c}^*$ and $\bar{c}^{**}$ such that both $(\bar{c}_1, \cdots, \bar{c}_m, \bar{c}^*)$ and $(\bar{c}_1, \cdots, \bar{c}_m, \bar{c}^{**})$ are in M. Since $(c_1, \cdots, c_m)$ and $(\bar{c}_1, \cdots, \bar{c}_m)$ are interior points of N, there exists a point $(c_1', \cdots, c_m')$ in N such that $(c_1, \cdots, c_m)$ lies in the interior of the segment determined by $(c_1', \cdots, c_m')$ and $(\bar{c}_1, \cdots, \bar{c}_m)$. There exists a real value c' such that $(c_1', \cdots, c_m', c') \in M$. Consider the convex set T determined by the 3 points: $(\bar{c}_1, \cdots, \bar{c}_m, \bar{c}^*)$, $(\bar{c}_1, \cdots, \bar{c}_m, \bar{c}^{**})$ and $(c_1', \cdots, c_m', c')$. Obviously, $T \subset M$. But T contains points $(c_1, \cdots, c_m, h)$ and $(c_1, \cdots, c_m, h')$ with

$h \neq h'$, contrary to our assumption that $c^* = c^{**}$. Thus, for any interior point $(\bar{c}_1, \cdots, \bar{c}_m)$ of N there exists exactly one real value $\bar{c}$ such that $(\bar{c}_1, \cdots, \bar{c}_m, \bar{c}) \in M$. Since M is closed and convex, this remains true also when $(\bar{c}_1, \cdots, \bar{c}_m)$ is a boundary point of N. Thus, there exists a single valued function $\varphi(g_1, \cdots, g_m)$ such that $g_{m+1} = \varphi(g_1, \cdots, g_m)$ holds for all points $g = (g_1, \cdots, g_m, g_{m+1})$ in M. Since M is convex, φ must be linear; i.e., $\varphi(g_1, \cdots, g_m) = \sum_{i=1}^{m} k_i g_i + k_0$. Since the origin is obviously contained in M, we have $k_0 = 0$. Thus, we have $g_{m+1} = \sum_{i=1}^{m} k_i g_i$ for all points g in M. But then $f_{m+1}(x) = \sum_{i=1}^{m} k_i f_i(x)$ must hold for all x, except perhaps on a set of measure zero. Thus, for any subset S of R, the inequalities (1.3) and (1.4) are fulfilled for all x, except perhaps on a set of measure zero. This completes the proof of our theorem in the case when $c^* = c^{**}$.

We shall now consider the case when $c^{**} < c^*$. Let c be any value between c^{**} and c^*; i.e., $c^{**} < c < c^*$. We shall show that $(c_1, \cdots, c_m, c)$ is an interior point of M. For this purpose, consider a finite set of points $c^i = (c_1^i, \cdots, c_m^i)$ in $N (i = 1, \cdots, n)$ such that $c^1, \cdots, c^n$ are linearly independent, the simplex determined by $c^1, \cdots, c^n$ has the same dimension as N and contains the point $(c_1, \cdots, c_m)$ in its interior. Such points c^i in N obviously exist. There exist real values $h_i (i = 1, \cdots, n)$ such that $(c_1^i, \cdots, c_m^i, h_i) \in M$ $(i = 1, \cdots, n)$. Let T be the smallest convex set containing the points $(c_1^i, \cdots, c_m^i, h_i)$ $(i = 1, \cdots, n)$, $(c_1, \cdots, c_m, c^*)$ and $(c_1, \cdots, c_m, c^{**})$. Clearly, the dimension of T is the same as that of M and $(c_1, \cdots, c_m, c)$ is an interior point of T. Thus, $(c_1, \cdots, c_m, c)$ is an interior point of M. The point $(c_1, \cdots, c_m, c^*)$ is obviously a boundary point of M. Let $g = (g_1, \cdots, g_{m+1})$ be the generic designation of a point in the $m + 1$-dimensional Euclidean space E. Since $(c_1, \cdots, c_m, c^*)$ is a boundary point of M, there exists an m-dimensional hyperplane Π through $(c_1, \cdots, c_m, c^*)$ such that Π contains only boundary points of M and M lies entirely on one side of Π.[3] Let the equation of Π be given by

$$k_{m+1} g_{m+1} - \sum_{i=1}^{m} k_i g_i = k_{m+1} c^* - \sum_{i=1}^{m} k_i c_i. \tag{3.1}$$

Since Π contains only boundary points of M, and since $(c_1, \cdots, c_m, c)$ is not a boundary point when $c^{**} < c < c^*$, the hyperplane Π cannot be parallel to the $(m + 1)$-th coordinate axis; i.e., $k_{m+1} \neq 0$. We can assume without loss of generality that $k_{m+1} = 1$. Since M lies entirely on one side of Π, and since for $(g_1, \cdots, g_m, g_{m+1}) = (c_1, \cdots, c_m, c^{**})$ the left hand member of (3.1) is smaller than the right hand member, we must have

$$g_{m+1} - \sum_{i=1}^{m} k_i g_i \leqq c^* - \sum_{i=1}^{m} k_i c_i \tag{3.2}$$

for all $g \in M$. Let S be a subset of R such that

[3] This follows from well known results on convex bodies. See, for example, [3], p. 6.

$$(\mu_1(S), \cdots, \mu_m(S), \mu_{m+1}(S)) = (c_1, \cdots, c_m, c^*). \tag{3.3}$$

It can easily be seen that (3.2) and (3.3) can be fulfilled simultaneously only if S satisfies the conditions (1.3) and (1.4) for all x, except perhaps on a set of measure zero. This completes the proof of our theorem.

It remains to investigate the case when $(c_1, \cdots, c_m)$ is a boundary point of N. For this purpose, we shall introduce some definitions and prove some lemmas.

Let $\xi = (\xi_1, \cdots, \xi_m)$ be an m-dimensional vector with real valued components at least one of which is not zero. We shall say that ξ is maximal relative to the point $c = (c_1, \cdots, c_m)$ if

$$\sum_{i=1}^{m} \xi_i g_i \leqq \sum_{i=1}^{m} \xi_i c_i \tag{3.4}$$

for all points $(g_1, \cdots, g_m)$ in N.

We shall say that a set $\{\xi^i\}(i = 1, 2, \cdots, r; r > 1)$ of vectors is maximal relative to the point $c = (c_1, \cdots, c_m)$ if the set $\{\xi^i\}(i = 1, \cdots, r-1)$ is maximal relative to c, not all components of ξ^r are zero and

$$\sum_{j=1}^{m} \xi_j^r g_j \leqq \sum_{j=1}^{m} \xi_j^r c_j \tag{3.5}$$

holds for all points $(g_1, \cdots, g_m)$ of N for which

$$\sum_{j=1}^{m} \xi_j^i g_j = \sum_{j=1}^{m} \xi_j^i c_j \qquad (i = 1, \cdots, r-1). \tag{3.6}$$

A set of vectors $\{\xi^i\}(i = 1, \cdots, r)$ is said to be a complete maximal set relative to $c = (c_1, \cdots, c_m)$ if $\{\xi^i\}(i = 1, 2, \cdots, r)$ is maximal relative to c and no vector ξ^{r+1} exists such that ξ^{r+1} is linearly independent of the sequence $(\xi^1, \cdots, \xi^r)$ and $(\xi^1, \cdots, \xi^r, \xi^{r+1})$ is maximal relative to c.

LEMMA 3.1. *If $c = (c_1, \cdots, c_m)$ is a boundary point of N, then there exists a positive integer r and a set $\{\xi^1, \cdots, \xi^r\}$ of vectors that is a complete maximal set relative to c.*

PROOF. Since c is a boundary point of N, there exists an $(m-1)$-dimensional hyperplane Π through c such that N lies entirely on one side of Π.[3] Let the equation of Π be given by

$$\sum_{i=1}^{m} \xi_i g_i = \sum_{i=1}^{m} \xi_i c_i .$$

Since N lies entirely on one side of Π, either $\sum_{i=1}^{m} \xi_i g_i \geqq \sum_{i=1}^{m} \xi_i c_i$ for all points $(g_1, \cdots, g_m)$ in N, or $\sum_{i=1}^{m} \xi_i g_i \leqq \sum_{i=1}^{m} \xi_i c_i$ for all $(g_1, \cdots, g_m)$ in N. We put $\xi^1 = -\xi$ if $\Sigma \xi_i g_i \geqq \Sigma \xi_i c_i$ for all points $(g_1, \cdots, g_m)$ in N. Otherwise, we put $\xi^1 = \xi$. Clearly, ξ^1 is maximal relative to c. If ξ^1 is not a complete maximal set relative to c, there exists a vector ξ^2 such that ξ^2 is linearly independent of

ξ^1 and (ξ^1, ξ^2) is maximal relative to c. If (ξ^1, ξ^2) is not a complete maximal set, we can find a vector ξ^3 such that ξ^3 is linearly independent of (ξ^1, ξ^2) and (ξ^1, ξ^2, ξ^3) is a maximal set relative to c, and so on. Continuing this procedure, we shall arrive at a set $(\xi^1, \cdots, \xi^r)(r \leqq m)$ that is a complete maximal set relative to c. This completes the proof of Lemma 3.1.

LEMMA 3.2. *If $(\xi^1, \cdots, \xi^r)$ is a maximal set of vectors relative to $c = (c_1, \cdots, c_m)$ and if $\nu(S) = c$, then the following two conditions are fulfilled for all x (except perhaps on a set of measure zero):*

a) *If x is a point in R for which $\sum_{j=1}^{m} \xi_j^i f_j(x) = 0$ for $i = 1, 2, \cdots, u - 1$ and $\sum_{j=1}^{m} \xi_j^u f_j(x) > 0$ $(u = 1, 2, \cdots, r)$, then $x \in S$.*

b) *If x is a point of R for which $\sum_{j=1}^{m} \xi_j^i f_j(x) = 0$ for $i = 1, 2, \cdots, u - 1$ and $\sum_{j=1}^{m} \xi_j^u f_j(x) < 0$, then $x \notin S$.*

PROOF. Assume that $(\xi^1, \cdots, \xi^r)$ is maximal relative to c. Then, ξ^1 is maximal relative to c. This implies that for all x (except perhaps on a set of measure zero) the following condition holds: $x \in S$ when $\sum_{j=1}^{m} \xi_j^1 f_j(x) > 0$ and $x \notin S$ when $\sum_{j=1}^{m} \xi_j^1 f_j(x) < 0$. Thus, conditions (a) and (b) of our lemma must be fulfilled for $u = 1$. We shall now show that if (a) and (b) hold for $u = 1, \cdots, v$ then (a) and (b) must hold also for $u = v + 1$. For this purpose, consider the set R' of all points x for which $\sum_{j=1}^{m} \xi_j^i f_j(x) = 0$ for $i = 1, \cdots, v$. If R is replaced by R', then ξ^{v+1} is maximal relative to $c' = (c_1', \cdots, c_m')$ where $c_i' = \int_{S'} f_i(x)\, dx$ and $S' = S \cap R'$. Hence, for any x in R' (except perhaps on a set of measure zero) the following condition holds: $x \in S$ when $\sum_{j=1}^{m} \xi_j^{v+1} f_j(x) > 0$ and $x \notin S$ when $\sum_{j=1}^{m} \xi_j^{v+1} f_j(x) < 0$. But this implies that (a) and (b) hold for $u = v + 1$. This completes the proof of our lemma.

LEMMA 3.3. *Let $(\xi^1, \cdots, \xi^r)$ be a complete maximal set of vectors relative to $c = (c_1, \cdots, c_m)$, and let T be the set of all points $g = (g_1, \cdots, g_m)$ of N for which $\sum_{j=1}^{m} \xi_j^i g_j = \sum_{j=1}^{m} \xi_j^i c_j$ for $i = 1, 2, \cdots, r$. Then T is a bounded, closed and convex set and c is an interior point of T.*

PROOF. Clearly, T is a bounded, closed and convex set. Suppose that c is a boundary point of T. Then there exists a hyperplane Π of dimension $m - 1$ such that Π goes through c, Π contains only boundary points of T and T lies entirely on one side of Π[3]. Let the equation of Π be given by

$$\sum_{j=1}^{m} \xi_j g_j = \sum_{j=1}^{m} \xi_j c_j,$$

where ξ is independent of $\xi^1, \cdots, \xi^r$. Since T lies on one side of Π, we have either $\sum_{j=1}^{m} \xi_j g_j \geqq \sum_{j=1}^{m} \xi_j c_j$ for all $g = (g_1, \cdots, g_m)$ in T, or $\sum_{j=1}^{m} \xi_j g_j \leqq \sum_{j=1}^{m} \xi_j c_j$ for all g in T. Let $\xi_j^{r+1} = \xi_j (j = 1, \cdots, m)$ in the latter case, and $\xi_j^{r+1} = -\xi_j$ in the former case. Then $\sum_{j=1}^{m} \xi_j^{r+1} g_j \leqq \sum_{j=1}^{m} \xi_j^{r+1} c_j$ for all g in T. But then $(\xi^1, \cdots, \xi^r, \xi^{r+1})$ is a maximal set relative to c, contrary to our assumption that $(\xi^1, \cdots, \xi^r)$ is a complete maximal set. Thus, c must be an interior point of T and our lemma is proved.

THEOREM 3.2. *If $c = (c_1, \cdots, c_m)$ is a boundary point of N and if $(\xi^1, \cdots, \xi^r)$ is a complete maximal set of vectors relative to c, then a necessary and sufficient condition for a member S of $\mathcal{S}$ to be a member of $\mathcal{S}_0$ is that there exist m constants $k_1, \cdots, k_m$ such that for all x in R' (except perhaps on a set of measure zero) the inequalities* (1.3) *and* (1.4) *hold, where R' is the set of all points x for which*

$$\sum_{j=1}^{m} \xi_j^i f_j(x) = 0 \quad \textit{for} \quad i = 1, 2, \cdots, r.$$

PROOF. Suppose that $c = (c_1, \cdots, c_m)$ is a boundary point of N and that $(\xi^1, \cdots, \xi^r)$ is a complete maximal set of vectors relative to c. Let R^* be the set of all points x for which the following two conditions hold: (1) $\sum_{j=1}^{m} \xi_j^i f_j(x) \neq 0$ for at least one value i; (2) $\sum_{j=1}^{m} \xi_j^i f_j(x) > 0$ where i is the smallest integer for which $\sum_{j=1}^{m} \xi_j^i f_j(x) \neq 0$. For any member S of $\mathcal{S}$ let S^* denote the intersection of S with $R - R'$. It follows from Lemma 3.2 that $R^* - R^* \cap S^*$ and $S^* - R^* \cap S^*$ are sets of measure zero. Thus

$$\int_{S^*} f_i(x)\, dx = \int_{R^*} f_i(x)\, dx \qquad (i = 1, \cdots, m + 1) \tag{3.7}$$

for all $S \in \mathcal{S}$. Let

$$f_i^*(x) = f_i(x) \quad \text{for} \quad x \in R' \qquad (i = 1, \cdots, m + 1) \tag{3.8}$$

and

$$f_i^*(x) = 0 \quad \text{for} \quad x \in R - R' \quad (i = 1, 2, \cdots, m + 1). \tag{3.9}$$

Let, furthermore,

$$c_i^* = c_i - \int_{R^*} f_i(x)\, dx \qquad (i = 1, \cdots, m) \tag{3.10}$$

Let μ^*, ν^*, M^*, N^*, $\mathcal{S}^*$ and $\mathcal{S}_0^*$ have the same meaning with reference to the functions $f_1^*(x), \cdots, f_{m+1}^*(x)$ and the point $c^* = (c_1^*, \cdots, c_m^*)$ as μ, ν, M, N, $\mathcal{S}$ and $\mathcal{S}_0$ have with reference to the functions $f_1(x), \cdots, f_{m+1}(x)$ and the point $c = (c_1, \cdots, c_m)$.

It follows from Lemma 3.2 that for any subset S of R for which $\nu(S)$ is a point of the set T defined in Lemma 3.3 we have

$$\int_S f_i(x)\,dx = \int_S f_i^*(x)\,dx + \int_{R^*} f_i(x)\,dx \qquad (i = 1, \cdots, m+1).$$

Since the range of $\nu^*(S)$ is equal to N^* even when S is restricted to subsets S for which $\nu(S) \epsilon T$, the set N^* is obtained from the set T by a translation. The same translation brings the point $c = (c_1, \cdots, c_m)$ into $c^* = (c_1^*, \cdots, c_m^*)$. It then follows from Lemma 3.3 that c^* is an interior point of N^*. Application of Theorem 3.1 gives the following necessary and sufficient condition for a member S of $\mathcal{S}^*$ to be a member of $\mathcal{S}_0^*$: There exist m constants $k_1, \cdots, k_m$ such that for all x (except perhaps on a set of measure zero)

$$f_{m+1}^*(x) \geqq k_1 f_1^*(x) + \cdots + k_m f_m^*(x) \quad \text{when} \quad x \,\epsilon\, S \tag{3.11}$$

and

$$f_{m+1}^*(x) \leqq k_1 f_1^*(x) + \cdots + k_m f_m^*(x) \quad \text{when} \quad x \notin S. \tag{3.12}$$

It follows from (3.8) and (3.9) that (3.11) and (3.12) are equivalent to

$$f_{m+1}(x) \geqq k_1 f_1(x) + \cdots + k_m f_m(x) \quad \text{when} \quad x \,\epsilon\, S \cap R' \tag{3.13}$$

and

$$f_{m+1}(x) \leqq k_1 f_1(x) + \cdots + k_m f_m(x) \quad \text{when} \quad x \,\epsilon\, (R - S) \cap R'. \tag{3.14}$$

Theorem 3.2 follows from this and the fact that every member S of $\mathcal{S}$ is a member of $\mathcal{S}^*$ and that a member S of $\mathcal{S}$ is a member of $\mathcal{S}_0^*$ if and only if S is a member of $\mathcal{S}_0$.

It may be of interest to note that if the set R' is of measure zero, the members of $\mathcal{S}$ can differ from each other only by sets of measure zero; i.e., $\mathcal{S}$ consists essentially of one element. This is an immediate consequence of Lemma 3.2.

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