RELATIONS BETWEEN VARIOUSLY DEFINED EFFECTS AND INTERACTIONS IN ANALYSIS OF VARIANCE

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1. Summary. From an algebraic point of view the analysis of variance tests of effects and interactions can be based on the minimum values of a certain quadratic expression in which the "h-matrix" (defined in Section 3) is fundamental. The arbitrariness in the choice of this matrix reflects the arbitrariness in the definition of effects and interactions. The paper considers the dependence of the result of these tests on the h-matrix used and expresses the answer by the two theorems of Section 4, which are proved in the subsequent sections.

2. Introduction. The sums of squares which appear in an analysis of variance when the significance of effects and/or interactions is tested can be obtained by taking the minima with regard to values $a_{k_1 \cdots k_s}$ of such expressions as

$$(1) \qquad \sum_{i_1=1}^{n_1} \cdots \sum_{i_s=1}^{n_s} g_{i_1 \cdots i_s}\Big[y_{i_1 \cdots i_s} - \sum_{k_1} \cdots \sum_{k_s} a_{k_1 \cdots k_s} h_{k_1 \cdots k_s}(i_1 \cdots i_s)\Big]^2,$$

where the $y_{i_1 \cdots i_s}$ are the means of $g_{i_1 \cdots i_s}$ observed values for levels i_t of variables x_t $(t = 1, \cdots, s)$, respectively, and the values h form a nonsingular matrix which will be described in detail in the next section. The summation inside the bracket in (1) is carried out over sets $(k_1 \cdots k_s)$ of subscripts, $0 \leq k_t \leq n_t - 1$, which depend on the aggregate of effects and interactions to be tested. If all $n_1 \cdots n_s$ possible sets appeared in (1), the minimum would, of course, be zero. To each test there belongs a set of k's which is left out of the combinations of subscripts in (1) according to the following rule:

The interaction of order $(t - 1)$ between $x_1, \cdots, x_t$ is tested by omitting all $a_{k_1 \cdots k_t 0 \cdots 0}$ for which $k_1 \cdots k_t \neq 0$. (A main effect is equivalent to an interaction of order zero.) An aggregate of interactions is tested by leaving out all combinations referring to any of its several components [2].

As an illustration, let us take $s = 2$ and $g_{i_1 \cdots i_s} = 1$. We choose the following (orthogonal) matrix of $h_{k_1 k_2}(i_1 i_2)$:

$$\begin{array}{rl} (i_1\, i_2) = & \begin{array}{rrrrrr} 11 & 12 & 21 & 22 & 31 & 32 \end{array} \\ (k_1\, k_2) = \begin{array}{r} 00 \\ 01 \\ 10 \\ 11 \\ 20 \\ 21 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -2 & -2 \\ 1 & -1 & 1 & -1 & -2 & 2 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}. \end{array}$$

If we test for the first main effect, then we retain a_{00}, a_{01}, a_{11}, and a_{21}, and we obtain for the minimum, after straightforward calculations,

$$\frac{(y_{11}+y_{12})^2+(y_{21}+y_{22})^2+(y_{31}+y_{32})^2}{2}-\frac{(\Sigma\Sigma\, y_{i_1i_2})^2}{6}.$$

Similarly, testing for the interaction, we would retain a_{00}, a_{01}, a_{10}, a_{20}, and obtain

$$\Sigma\Sigma\, y_{i_1i_2}^2-\tfrac{1}{2}[(y_{11}+y_{12})^2+(y_{21}+y_{22})^2+(y_{31}+y_{32})^2]$$
$$-\tfrac{1}{3}[(y_{11}+y_{21}+y_{31})^2+(y_{12}+y_{22}+(y_{32})^2]+(\Sigma\Sigma\, y_{i_1i_2})^2/6.$$

If we had taken general weights $g_{i_1\cdots i_s}$, but still using the same values for the h-matrix, then we should have obtained results which are equivalent to those given by Yates' "method of weighted squares of means" [5].

3. Assumptions and definitions. Let the "h-matrix" $h_{k_1\cdots k_s}(i_1\cdots i_s)$ ($i_t = 1, \cdots, n_t$; $k_t = 0, 1, \cdots, n_t - 1$) be such that all the elements in the same row have equal sets of subscripts and all the elements in the same column have equal sets of arguments. It will be assumed that this matrix satisfies the following conditions:

CONDITION A.

$$\sum_{i_1=1}^{n_1}\cdots\sum_{i_s=1}^{n_s} w_{i_1\cdots i_s}\, h_{k_1\cdots k_s}(i_1\cdots i_s)\, h_{m_1\cdots m_s}(i_1\cdots i_s)\neq 0$$

if simultaneously $k_t = m_t$, *and* $= 0$ *otherwise. The* $w_{i_1\cdots i_s}$ *are positive weights.* It follows that the h-matrix is not singular.

CONDITION B. *If any* $k_t = 0$, *then* $h_{k_1\cdots k_t\cdots k_s}$ *is independent of* i_t.

In particular, if the h's are orthogonal polynomials of degrees k_t in i_t, then Conditions A and B hold by definition.

It has been shown [1] that these two conditions can be satisfied simultaneously only if the weights are "proportionate," i.e., if $w_{i_1\cdots i_s}/\sum_{i_t=1}^{n_t} w_{i_1\cdots i_s}$ is independent of all i_m ($m \neq t$) for all t.

From Condition A can be derived the following lemma, which will be used at a later state:

LEMMA. *If* $k_t \neq 0$, *then* $\sum_{i_t=1}^{n_t} w_{i_1\cdots i_s}\, h_{k_1\cdots k_s}(i_1\cdots i_s) = 0$.

PROOF. We assume that $t = 1$; this clearly does not restrict the generality of the argument, since it may be repeated identically for any other value of t. From Condition A we have the equations:

$$\sum_{i_1=1}^{n_1}\cdots\sum_{i_s=1}^{n_s} w_{i_1\cdots i_s}\, h_{k_1\cdots k_s}(i_1\cdots i_t)\, h_{0m_2\cdots m_s}(i_1\cdots i_s) = 0$$

for all $m_2, \cdots, m_s$, since k_1 is assumed to be different from zero. If we regard the $n_2 \cdots n_s$ expressions $\sum_{i_1=1}^{n_t} w_{i_1\cdots i_s}\, h_{k_1\cdots k_s}(i_1\cdots i_s)$ for all $i_2, \cdots, i_s$ as unknown values, we have the same number of linear homogeneous equations for them. The determinant of the system is orthogonal and hence not zero. It follows that the unknown values must be zero, and thus the lemma is proved.

All sets $(k_1 \cdots k_t 0 \cdots 0)$ with $k_1 \cdots k_t \neq 0$ form a "block," which we denote by $((k_1 \cdots k_t))$. The meaning of $((k_{m_1} \cdots k_{m_t}))$ is immediately obvious. Every set of subscripts belongs to one and only one block. If we consider a particular block and then omit one or more values from within the double brackets denoting it, another block is obtained, which we call a "sub-block" of the former.

4. The problem. Even with Conditions A and B to be satisfied, there remains still an arbitrariness in the choice of the h-matrix, and this reflects an arbitrariness in the definition of interactions [3]. If the h's are such that Condition A is satisfied with $w_{i_1 \cdots i_s} = g_{i_1 \cdots i_s}$ for all i_t, then the computation of the minimum of (1) becomes very simple, but clearly this cannot be a reason for choosing the h-matrix accordingly [4]. However, in this paper we shall be concerned with another aspect of the situation: we wish to find out whether two different h-matrices can lead to the same minimum value and, if so, under what conditions. The answer depends on the particular test carried out and is expressed by the following two theorems.

THEOREM 1. *If two h-matrices satisfy Condition A with regard to the same weights, then both lead to the same minimum of* (1), *whatever the aggregate of interactions tested.*

THEOREM 2. *If the aggregate is such that for each retained block of subscripts all its sub-blocks are also retained, then the minimum of* (1) *is independent of the h-matrix* (*even if the latter is not orthogonal with regard to any weights*).

It follows from the latter theorem that when only the highest order interaction (that of order $s - 1$) is tested, and hence all sets of subscripts except those constituting the block $((k_1 \cdots k_s))$ are retained, any h-matrix leads to the same conclusion.

5. Transformation of the problem. In what follows we shall denote, where no misunderstanding can arise, the various sets $(i_1 \cdots i_s)$ by $I_1, \cdots, I_N$, and the sets $(k_1 \cdots k_s)$ by $K_1, \cdots, K_M$. Here $N = n_1 \cdots n_s$, and M is the number of retained sets of subscripts, e.g., $M = (n_1 - 1) \cdots (n_t - 1)$ if only the block $((k_1 \cdots k_t))$ were retained. We have $N > M$, except in the trivial case where the minimum of (1) is zero.

Let us now imagine that we have two h-matrices, the elements of which are denoted by h and h' respectively. If for any given set of $a_{K_i} (i = 1, \cdots, M)$ we can find a set a'_{K_i} so that

$$\sum_{i=1}^{M} a_{K_i} h_{K_i}(I_t) = \sum_{i=1}^{M} a'_{K_i} h'_{K_i}(I_t) \tag{2}$$

for all $I_t (t = 1, \cdots, N)$, then clearly the set of values which (1) can assume is identical with that of a similar expression when h is replaced by h'. Hence the minima of the two expressions will also be the same.

It follows that different h-matrices will lead to the same minimum of (1) if they and the retained blocks of subscripts are such that (2) can be solved for the a'_{K_i}, assuming that the a_{K_i} are given. In (2) there are N equations for the M

unknowns a'_{K_i}. It will be possible, therefore, to solve the set only if not more than M of the equations are linearly independent.

Regarding, to begin with, only the left-hand side (l.h.s.) of (2), we can certainly select M sets of arguments $I_1, \cdots, I_M$ so that the determinant $| h_{K_i}(I_t) | \neq 0$ (since the complete h-matrix is not singular). Hence for any further argument J, say, we can solve the system of linear equations

$$
\begin{aligned}
h_{K_1}(J) &= \sum_{t=1}^{M} C_{I_t} h_{K_1}(I_t), \\
&\cdots\cdots\cdots\cdots\cdots \\
h_{K_M}(J) &= \sum_{t=1}^{M} C_{I_t} h_{K_M}(I_t).
\end{aligned} \tag{3}
$$

This gives $h_{K_i}(J)$ as a linear combination of $h_{K_i}(I_1), \cdots, h_{K_i}(I_M)$ which is the same for all $i = 1, \cdots, M$. Therefore the l.h.s. of those equations in (2) in which $I_t = J$ will again be the same linear combination of the l.h.s. of the equations in which the arguments are $I_1, \cdots, I_M$, respectively. Consequently the whole equation, written for J, will be the very same linear combination of the equations for the I_t severally, if it can be shown that the C_{I_t} which we find from (3) are equally applicable to the r.h.s. of (2), i.e., to the h'_{K_i}. Since the two matrices are, by the assumptions of Theorem 1, orthogonal with regard to the same weights, it is sufficient to prove that the C_{I_t}, which are the solutions of (3), although possibly dependent on the weights $w_{i_1 \cdots i_s}$, do not otherwise depend on the h-matrix considered.

6. Proof of Theorem 1. In general, the sets of subscripts $K_1, \cdots, K_M$ will not all belong to the same block. We consider first the block out of which K_1 is taken and assume that it consists of $K_1, \cdots, K_P$ ($P \leq M$). It is no restriction of generality to assume further that this is the block $((k_1 \cdots k_m))$, so that $P = (n_1 - 1) \cdots (n_m - 1)$.

We fix our attention on one single set belonging to this block, say $(\bar{k}_1, \cdots, \bar{k}_m, 0, \cdots, 0)$. Conditions A and B imply linear relations between the $h_{\bar{k}_1 \cdots \bar{k}_m 0 \cdots 0}(i_1 \cdots i_s)$, and we shall now establish how many of these values can be chosen independently, thereby fixing all others implicitly. If $h_{\bar{k}_1 \cdots \bar{k}_m 0 \cdots 0}$ is known for $(\bar{\imath}_1, \cdots, \bar{\imath}_s)$ where the $\bar{\imath}_t$ are some fixed values, then, by virtue of Condition B, it is also known for all $(\bar{\imath}_1, \cdots, \bar{\imath}_m, i_{m+1}, \cdots, i_s)$ where the $i_{m+1}, \cdots, i_s$ are arbitrary. We need therefore only investigate relations between the $n_1 \cdots n_m$ values $h_{\bar{k}_1 \cdots \bar{k}_m 0 \cdots 0}(i_1 \cdots i_m \bar{\imath}_{m+1} \cdots \bar{\imath}_s)$. These are not all independent either, since our lemma gives, for $r = 1, \cdots, m$,

$$\sum_{i_r=1}^{n_r} w_{i_1 \cdots i_s} h_{\bar{k}_1 \cdots \bar{k}_m 0 \cdots 0}(i_1 \cdots i_s) = 0 \tag{4}$$

for all $i_1, \cdots, i_{r-1}, i_{r+1}, \cdots, i_s$.

Thus only $(n_1 - 1) \cdots (n_m - 1) = P$ values among the $h_{\bar{k}_1 \cdots \bar{k}_m 0 \cdots 0}(i_1 \cdots i_s)$ will be independent, and it is easy to indicate how such a set can be found. In the matrix $\| h_{K_j}(I_t) \|$ ($j = 1, \cdots, P$; $t = 1, \cdots, M$) there must be a square sub-matrix of order P which is not singular, since otherwise the determinant of (3) would be zero, contrary to our assumptions. Let this sub-matrix be $\| h_{K_j}(I_u) \|$ ($u = t_1, \cdots, t_P$; j as before). Then it follows that the $h_{\bar{k}_1 \cdots \bar{k}_m 0 \cdots 0}(I_u)$ constitute a set of values which can arbitrarily be selected. Indeed, if they were dependent, by virtue of (4) and Condition B, then identical linear relations would hold for all K_j, i.e., for all rows of the matrix $\| h_{K_j}(I_u) \|$, which would hence be singular.

We may, then, rewrite the first P equations in (3) by expressing all $h(I_t)$ on the r.h.s. in terms of $I_{t_1}, \cdots, I_{t_P}$ as arguments. The coefficients will be linear combinations of C_{I_t} and of the weights, which appear in (4). Since the l.h.s., i.e., $h(J)$, with subscripts of h as before, can also be expressed as a linear combination of these same $h(I_u)$, by virtue of (4) and Condition B, we see that the C_{I_t} must satisfy P identities, in which only the weights $w_{i_1 \cdots i_s}$ are parameters.

All blocks to which the $K_1, \cdots, K_M$ belong can be treated in the same way and thus we obtain altogether M equations from which the $C_{I_t}(t = 1, \cdots, M)$ can be obtained. They will depend on the weights, but not otherwise on the h-matrix. This completes the proof of Theorem 1.

7. Proof of Theorem 2. We turn now to Theorem 2, and assume that the sets of subscripts retained in (1) are those of the blocks $B_0, B_1, \cdots, B_n$ and of all their sub-blocks. We can at once indicate the sets of arguments $I_1, I_2, \cdots$ equal in number to the retained sets of subscripts, and such that $h(j_1 \cdots j_s)$ can be expressed as a linear combination of $h(I_1), h(I_2), \cdots$ for all sets of subscripts considered. For this purpose we take, for each retained set $(k_1 \cdots k_s)$, the set of arguments $(k_1 + 1, \cdots, k_s + 1)$. Thus there will be the same number $(=M)$ of sets of arguments as there are sets of retained subscripts. We note in particular, that if any of the $k_t = 0$, the corresponding j_t will be unity. This will be the case in respect of all sets of a block, if it is true for any set in it.

To simplify our formulae, we introduce the following notation: If $(J) \equiv (j_1 \cdots j_s)$, then $(J)_i$ is the result of replacing by unity all those j_t which correspond to a $k_t = 0$ in block B_i. Further, $(J)_{ij}$ is the result of replacing by unity all those j_t which correspond to a $k_t = 0$ either in B_i or in B_j, or, in other words, those j_t which do not correspond to the largest common sub-block of B_i and B_j. The notation $(J)_{ij \cdots k}$ is similarly defined. Now if K_i is any set of subscripts in the block B_i, then it follows from Condition B that

$$h_{K_i}(j_1 \cdots j_s) = h_{K_i}(j_1 \cdots j_s)_i,$$

and, more generally,

$$h_{K_i}(j_1 \cdots j_s)_{j \cdots k} = h_{K_i}(j_1 \cdots j_s)_{ij \cdots k},$$

since all those additional arguments 1 in $(j_1 \cdots j_s)_{ij \cdots k}$ which do not already appear in $(j_1 \cdots j_s)_{j \cdots k}$ correspond to zeros in the subscripts of B_i. Moreover, these relations remain true if we take, instead of B_i, any of its sub-blocks, since such a sub-block contains all those zeros which were in B_i (and some more).

We shall now prove that the relation

$$(5) \qquad h_K(J) = \sum_{i=0}^{n} h_K(J)_i - \sum_{i \neq j=0}^{n} h_K(J)_{ij} + \cdots + (-1)^n h_K(J)_{01\cdots n},$$

(J) being an arbitrary set of arguments, holds for all K out of $B_0, B_1, \cdots, B_n$ and also out of any of their sub-blocks. This is a linear relation of the type which we need for the proof of Theorem 2 and we see that all sets of arguments appearing on the r.h.s. are among those $I_1, I_2, \cdots$ which we have initially selected as a basis. Hence, if we prove that this relation holds for any K out of the blocks and sub-blocks considered, then Theorem 2 follows.

First let K be a set out of B_0. Equation (5) can be written as follows:

$$h_K(J) = h_K(J)_0 + \sum_{i=1}^{n} h_K(J)_i - \sum_{i=1}^{n} h_K(J)_{0i} - \sum_{i \neq j=1}^{n} h_K(J)_{ij} + \sum_{i \neq j=1}^{n} h_K(J)_{0ij} + \cdots + (-1)^n h_K(J)_{01\cdots n}.$$

Now we have $h_K(J) = h_K(J)_0$. Moreover, the second term on the r.h.s. cancels with the third, the fourth with the fifth, and so on until all terms are exhausted. This proves relation (5) for K out of B_0. But it is evident that the proof could equally well have been carried out for any other of the given blocks or for any of their sub-blocks. This completes the proof of Theorem 2. It will be noticed that no weights appear in (5), so that under the given conditions the theorem holds even for matrices which are not orthogonal (in the sense of Condition A) with regard to any set of weights.

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