To prove this let u be a vector $(u_1, \cdots, u_n)$. Let U be the set of u such that the property described in (4) holds. We have

$$P[u \varepsilon U] = n! \int_U \prod_{j=1}^{n} dF[u_j, f(u_j)]. \tag{5}$$

Let $z_j = F[u_j, f(u_j)]$, $z = (z_1, \cdots, z_n)$

$$P[u \varepsilon U] = n! \int_Z dz, \tag{6}$$

where

$$z \varepsilon Z \quad \text{if} \quad \max_j |z_j - j/n| < \lambda \quad \text{and} \quad \max_j |z_j - (j+1)/n| < \lambda.$$

Since (6) does not depend on $F(x, y)$, the probability is the same for all $F(x, y)$ with the given properties. Nor does (6) depend upon the particular choice of $f(u)$.

The expression (5) is the probability distribution of the type (1) for the single-variable distribution $F[x, f(x)]$. We can test the hypothesis that a given random sample was derived from a particular distribution by means of the maximum deviation of the distribution from the step function derived from the sample. Values of the probabilities have been tabulated by Massey [1].

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ON THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF A SEQUENCE OF MOMENT GENERATING FUNCTIONS

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In a previous paper ([1], pp. 61–69) the author studied the reciprocal relation between the convergence of a sequence of df's (distribution functions) and the convergence of the corresponding sequence of mgf's (moment generating functions) in the univariate case. It is the purpose of the present paper to give necessary and sufficient conditions for the convergence of a sequence $\{\varphi_n(t_1, t_2)\}$ of mgf's in two dimensions. The results can be extended to Euclidean spaces of higher dimensions.

Let $\{F_n(x_1, x_2)\}$ be a sequence of df's. For $x_i \geq 0$, let

$$M_n(x_1, x_2) = \iint_{|u_i| \geq x_i} dF_n(u_1, u_2),$$

and designate by $M(x_1, x_2)$ the least upper bound of the sequence $\{M_n(x_1, x_2)\}$.

THEOREM 1. *If the sequence $\{F_n(x_1, x_2)\}$ converges on an everywhere dense set and if there exist numbers α_1, α_2 such that for $|t_i| < \alpha_i$,*

$$M(x_1, x_2) \leq K \exp(-|t_1| x_1 - |t_2| x_2), \tag{1}$$

where K is independent of x_1 and x_2, then

(a) *there exists a df $F(x_1, x_2)$ such that*

$$\lim_{n\to\infty} F_n(x_1, x_2) = F(x_1, x_2) \tag{2}$$

at each point of continuity of $F(x_1, x_2)$,

(b) *the mgf's of $F(x_1, x_2)$ and $F_n(x_1, x_2)$, say $\varphi(t_1, t_2)$ and $\varphi_n(t_1, t_2)$, exist for $|t_i| < \alpha_i$,*

(c) $\lim_{n\to\infty} \varphi_n(t_1, t_2) = \varphi(t_1, t_2)$ *for $|t_i| < \alpha_i$ and uniformly in each interval $|t_i| \leq \beta_i < \alpha_i$.*

To prove (a), notice that there exists a function $F(x_1, x_2)$, continuous to the right and with nonnegative second difference, such that (2) holds at each continuity point of $F(x_1, x_2)$. From (1) we see that F is a df.

Now let $\beta_i < \gamma_i < \alpha_i$ $(i = 1, 2)$, and denote by R_z, for $z \geq 0$, the region $|x_i| < z$. Let k and l be integers such that $l > k > 0$. Then, from (1), we find that, for $|t_i| \leq \beta_i$,

$$\iint_{R_l - R_k} \exp(t_1 x_1 + t_2 x_2)\, dF_n(x_1, x_2) < C\{\exp[(\beta_1 - \gamma_1)k] + \exp[(\beta_2 - \gamma_2)k]\}, \tag{3}$$

where C is independent of k and l.

The relations (2) and (3) imply the truth of (b) and (c). Thus Theorem 1 is proved.

THEOREM 2. *Let $\{F_n(x_1, x_2)\}$ be a sequence of df's and let $\{\varphi_n(t_1, t_2)\}$ be the corresponding sequence of mgf's. If $\varphi_n(t_1, t_2)$ exist for $|t_i| < \alpha_i$, and if there exists a finite valued function $\varphi(t_1, t_2)$ such that $\lim_{n\to\infty} \varphi_n(t_1, t_2) = \varphi(t_1, t_2)$, $|t_i| < \alpha_i$, then*

(a) *the inequality* (1) *holds for $|t_i| < \alpha_i$,*

(b) *there exists a df $F(x_1, x_2)$ such that* (2) *holds at each continuity point of $F(x_1, x_2)$,*

(c) *for $|t_i| < \alpha_i$, the mgf of $F(x_1, x_2)$ exists and equals $\varphi(t_1, t_2)$,*

(d) $\lim_{n\to\infty} \varphi_n(t_1, t_2) = \varphi(t_1, t_2)$ *uniformly for $|t_i| \leq \beta_i < \alpha_i$* $(i = 1, 2)$.

To prove (a), note that for $|t_i| < \alpha_i$, $x_1 \geq 0$, we have

$$\iint_{u_i \geq x_i} dF_n(u_1, u_2) \leq \exp(-|t_1|x_1 - |t_2|x_2) \tag{4}$$

$$\iint_{u_i \geq x_i} \exp(|t_1|u_1 + |t_2|u_2)\, dF_n(u_1, u_2) \leq M_0 \exp(-|t_1|x_1 - t_2|x_2),$$

where $\varphi_n(|t_1|, |t_2|) \leq M_0$. Such a number $M_0 = M_0(t_1, t_2)$ exists since $\{\varphi_n(|t_1|, |t_2|)\}$ converges for $|t_i| < \alpha_i$. This gives an estimate for $M_n(x_1, x_2)$, which shows that (a) holds. The Helly selection principle ([2], pp. 60–62 and 83) leads to (b). The relations (c) and (d) follow immediately from Theorem 1.

From Theorems 1 and 2 we obtain

THEOREM 3. *Let $\{F_n(x_1, x_2)\}$ be a sequence of df's and let $\{\varphi_n(t_1, t_2)\}$ be the corresponding sequence of mgf's which are all assumed to exist for $|t_i| < \alpha_i$. Then the necessary and sufficient condition for the convergence of $\{\varphi_n(t_1, t_2)\}$ for $|t_i| < \alpha_i$ is that the relations* (a) *and* (b) *of Theorem* 2 *be satisfied.*

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A NOTE ON THE MAXIMUM VALUE OF KURTOSIS

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In "A note on skewness and kurtosis," J. E. Wilkins (*Annals of Math. Stat.* Vol. 15 (1944), pp. 333–335) gave a short and elegant proof of the inequality for skewness and kurtosis

$$\beta_2 \geq \beta_1^2 + 1. \tag{1}$$

Then he gave an upper bound, depending on the population size N, for the skewness:

$$\max \beta_1 = (N - 2)/(N - 1)^{\frac{1}{2}}. \tag{2}$$

Now we shall derive an upper bound for the kurtosis. It will appear that the sign "=" in (1) is valid for the upper bounds, and the two maximum values indeed arise in the same "extreme" population.

To find the maximum value of the kurtosis β_2 we consider the function Σx_i^4 in the x-space, where $\Sigma x_i^2 = N$ and $\Sigma x_i = 0$. We have to maximize $\Sigma x_i^4 - \lambda \Sigma x_i^2 - \mu \Sigma x_i$. The maximizing values are given by the N equations, found by differentiation with respect to x_i

$$4x_i^3 - 2\lambda x_i - \mu = 0, \tag{3}$$

[1] "Aspirant" of the Belgian National Foundation for Scientific Research.