That this is possible is most easily seen geometrically by observing that the line $\pi' = \pi''$ separates point 1 from points 2 and 3, so that there exist weights p_{10}, p_{20}, p_{30} for points 1, 2, 3, respectively, so that the center of gravity of 1 and 2 lies on the line $\pi' = \pi''$, as does that of 1 and 3. Also the center of gravity of these three points with the assigned weights lies on the same side of $\pi' = \pi''$ as 2 and 3 while 4 lies on the opposite side. Thus we can determine p_{40} so that

$$\sum_{i=1}^{4} p_{i0}\pi'_{i0} = \sum_{i=1}^{4} p_{i0}\pi''_{i0}.$$

Finally we take

$$(2) \qquad p_i = \frac{p_{i0}}{\sum_{j=1}^{4} p_{j0}}, \qquad \pi'_i = \frac{\pi'_{i0}}{\sum_{j=1}^{4} p_j \pi'_{j0}}, \qquad \pi''_i = \frac{\pi''_{i0}}{\sum_{j=1}^{4} p_j \pi''_{j0}}.$$

Then all the conditions (a), (b), (c), (d) are satisfied. By similar reasoning it is easy to see that the parameters can be chosen so that w_1 and w_2 have arbitrary sizes α_1 and α_2, respectively.

It is possible to obtain cases where H_0 contains a continuum of simple hypotheses, for example

$$H_0(\lambda): P\{X = i\} = \lambda p'_i + (1 - \lambda)p''_i,$$

with $0 \leqq \lambda \leqq 1$, where p'_i, p''_i are obtained as in the main part of this paper. The same tests are most powerful and similar. Many interesting questions arise but they seem not to be of any real statistical importance.

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NOTE ON THE ESTIMATION OF A BIVARIATE DISTRIBUTION FUNCTION

By Paul B. Simpson

University of Oregon

A continuous cumulative probability distribution $F(x)$ can be estimated from a random sample (x_i), $i = 1, \cdots, n$, by the step function $G(x) = j/n$, where j is the number of $x_i \leq x$. In this single variable case, it is known that the probability distribution

$$(1) \qquad P\{\max_x |F(x) - G(x)| < \lambda\}$$

is the same for all distributions $F(x)$ [2]. It might be expected that a similar invariant property would hold for continuous bivariate distributions. An example shows that such is not the case.

Consider the cumulative distribution

$$F(x, y) = \frac{ax^2y}{2} + \frac{(2-a)y^2x}{2}, \qquad 0 < x < 1, \qquad 0 < y < 1. \tag{2}$$

Let

(a) (x_1, y_1) be a random observation from $F(x, y)$;

(b) $G(x, y) = 1$ for $x \geq x_1$ and $y \geq y_1$,
$G(x, y) = 0$ for $x < x_1$ or $y < y_1$;

(c) $\frac{1}{2} < \lambda < 1$;

(d) J be the set of points (x, y) fulfilling the three conditions: $F(x, y) \geq (1 - \lambda)$, $F(x, 1) < \lambda$, $F(1, y) < \lambda$.

It follows that

$$p[(x_1, y_1) \varepsilon J] = P[| F(x, y) - G(x, y) | < \lambda] \qquad \text{for all } x, y. \tag{3}$$

For $\lambda = .72$ and $a = 0$, $\quad .065 < P[(x_1, y_1) \varepsilon J] < .066.$

For $\lambda = .72$ and $a = 1$, $\quad .057 < P[(x_1, y_1) \varepsilon J] < .058.$

Thus, there are two continuous bivariate distributions for which the probabilities of the type (1) differ.

If we consider a set of points of the independent variables (x, y) lying on an increasing function y of x, we reduce the bivariate problem to a single-variable problem. Let $F(x, y)$ be a cumulative distribution, $-\infty < x < +\infty$, $-\infty < y < +\infty$, $\partial^2F/\partial x\partial y$ continuous almost everywhere. Consider a random sample (x_i, y_i), $i = 1, \cdots, n$. Let $G(x, y) = j/n$, where j is the number of observations (x_i, y_i) such that $x_i \leq x$ and $y_i \leq y$. Let $v = f(u)$ be any increasing continuous function, $-\infty < u < +\infty$, such that $v \to \pm\infty$ as $u \to \pm\infty$. Define a set of points (u_j, v_j) as follows: For given observation (x_i, y_i), if $y_i \geq f(x_i)$, let $v_i = y_i$ and $y_i = f(u_i)$. If $y_i < f(x_i)$, let $u_i = x_i$ and $v_i = f(x_i)$. Order the set such that $u_{j-1} \leq u_j$, $j = 2, \cdots, n$. Let $0 < \lambda < 1$. Since the maximum deviations of the step function j/n from the distribution $F(x, y)$ over the points $v = f(u)$ occur at the end points of the "intervals", we are interested in

$$P\{\text{greater of } \max_j | F(u_j, v_j) - j/n | \text{ and } \max_j | F(u_j, v_j) - (j+1)/n |\} < \lambda. \tag{4}$$

The probability distribution of (4) is the same for all $F(x, y)$ and equals the distribution for the single-variable case (1), when the size of the sample is the same.

To prove this let u be a vector $(u_1, \cdots, u_n)$. Let U be the set of u such that the property described in (4) holds. We have

$$P[u \varepsilon U] = n! \int_U \prod_{j=1}^{n} dF[u_j, f(u_j)]. \tag{5}$$

Let $z_j = F[u_j, f(u_j)]$, $z = (z_1, \cdots, z_n)$

$$P[u \varepsilon U] = n! \int_Z dz, \tag{6}$$

where

$$z \varepsilon Z \quad \text{if} \quad \max_j |z_j - j/n| < \lambda \quad \text{and} \quad \max_j |z_j - (j+1)/n| < \lambda.$$

Since (6) does not depend on $F(x, y)$, the probability is the same for all $F(x, y)$ with the given properties. Nor does (6) depend upon the particular choice of $f(u)$.

The expression (5) is the probability distribution of the type (1) for the single-variable distribution $F[x, f(x)]$. We can test the hypothesis that a given random sample was derived from a particular distribution by means of the maximum deviation of the distribution from the step function derived from the sample. Values of the probabilities have been tabulated by Massey [1].

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ON THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF A SEQUENCE OF MOMENT GENERATING FUNCTIONS

By W. Kozakiewicz

University of Montreal

In a previous paper ([1], pp. 61–69) the author studied the reciprocal relation between the convergence of a sequence of df's (distribution functions) and the convergence of the corresponding sequence of mgf's (moment generating functions) in the univariate case. It is the purpose of the present paper to give necessary and sufficient conditions for the convergence of a sequence $\{\varphi_n(t_1, t_2)\}$ of mgf's in two dimensions. The results can be extended to Euclidean spaces of higher dimensions.