# A PROPERTY OF SOME TESTS OF COMPOSITE HYPOTHESES

BY C. M. STEIN

*University of Chicago*

In all common statistical tests, a result significant at the 1 per cent level is necessarily significant at the 5 per cent level. In this note we show that this statement is not true for all statistical tests. More precisely, for any $\alpha_1$, $\alpha_2$ satisfying $0 < \alpha_1 < \alpha_2 < 1$, we construct a composite hypothesis $H_0$ and a simple hypothesis $H_1$ such that there are sets $w_1$, $w_2$ in the sample space which are the unique most powerful critical regions of size $\alpha_1$, $\alpha_2$, respectively, for testing $H_0$ against $H_1$. Furthermore, $w_1$, $w_2$ are similar regions. But $w_2$ does not contain $w_1$.

Let $X$ be a random variable which can take one of the four values 1, 2, 3, 4. Let $H_0$ consist of two simple hypotheses $H_0'$ and $H_0''$, where $H_0'$ states that $P\{X = i\} = p_i'$, and $H_0''$ states that $P\{X = i\} = p_i''$; and let $H_1$ state that $P\{X = i\} = p_i$ for $i = 1, 2, 3, 4$. Later, we shall determine appropriate positive values for the $p_i$, $p_i'$, $p_i''$. Let $\pi_i' = p_i'/p_i$, $\pi_i'' = p_i''/p_i$. By a slight modification of the Neyman-Pearson lemma [1] (see also [2]), the region $w_1$ consisting of the points $x = 1$ and $x = 2$, and the region $w_2$ consisting of the points $x = 1$ and $x = 3$, are both most powerful critical regions and similar if and only if

$$\text{(a)} \qquad \begin{aligned} p_1\pi_1' + p_2\pi_2' &= p_1\pi_1'' + p_2\pi_2'', \\ p_1\pi_1' + p_3\pi_3' &= p_1\pi_1'' + p_3\pi_3''; \end{aligned}$$

(b) there exist constants $a_1$, $a_2$, $b_1$, $b_2 \geq 0$ with $a_1 + b_1 > 0$, $a_2 + b_2 > 0$, such that $a_1\pi_1' + b_1\pi_1''$, $a_1\pi_2' + b_1\pi_2''$ are both less than or equal to $a_1\pi_3' + b_1\pi_3''$, $a_1\pi_4' + b_1\pi_4''$, and $a_2\pi_1' + b_2\pi_1''$, $a_2\pi_3' + b_2\pi_3''$ are both less than or equal to $a_2\pi_2' + b_2\pi_2''$, $a_2\pi_4' + b_2\pi_4''$. Expressed geometrically in the $(\pi', \pi'')$-plane, if $a_1$, $a_2$, $b_1$, $b_2 > 0$ and "less than" holds in all the above relations (which will always be the case in our construction), this means that the line joining points 2 and 3 intersects both axes at positive values, and the point 1 is inside and point 4 outside the triangle formed by this line and the coordinate axes. Of course $H_0'$, $H_0''$, $H_1$ are all probability distributions and all of the points 1, 2, 3, 4 are to have positive probabilities, so that we want

$$\text{(c)} \qquad \sum p_i = 1, \qquad \sum p_i\pi_i' = 1, \qquad \sum p_i\pi_i'' = 1;$$

$$\text{(d)} \qquad p_i > 0, \qquad \pi_i' > 0, \qquad \pi_i'' > 0.$$

We shall show that conditions (a), (b), (c), (d) can be satisfied in a great variety of ways. Choose $\pi_{i0}'$, $\pi_{i0}''$ so that $\pi_{10}'' > \pi_{10}'$, $\pi_{20}' > \pi_{20}''$, $\pi_{30}' > \pi_{30}''$, $\pi_{40}'' > \pi_{40}'$, and (b) is satisfied when $\pi_i'$, $\pi_i''$ are replaced by $\pi_{i0}'$, $\pi_{i0}''$, respectively. Let $p_{10}$ be an arbitrary nonnegative number. Choose $p_{20}$, $p_{30} \geq 0$ so that

$$\text{(1)} \qquad \begin{aligned} p_{10}\pi_{10}' + p_{20}\pi_{20}' &= p_{10}\pi_{10}'' + p_{20}\pi_{20}'', \\ p_{10}\pi_{10}' + p_{30}\pi_{30}' &= p_{10}\pi_{10}'' + p_{30}\pi_{30}''. \end{aligned}$$

That this is possible is most easily seen geometrically by observing that the line $\pi' = \pi''$ separates point 1 from points 2 and 3, so that there exist weights $p_{10}$, $p_{20}$, $p_{30}$ for points 1, 2, 3, respectively, so that the center of gravity of 1 and 2 lies on the line $\pi' = \pi''$, as does that of 1 and 3. Also the center of gravity of these three points with the assigned weights lies on the same side of $\pi' = \pi''$ as 2 and 3 while 4 lies on the opposite side. Thus we can determine $p_{40}$ so that

$$\sum_{i=1}^{4} p_{i0}\pi'_{i0} = \sum_{i=1}^{4} p_{i0}\pi''_{i0}.$$

Finally we take

$$(2) \qquad p_i = \frac{p_{i0}}{\sum_{j=1}^{4} p_{j0}}, \qquad \pi'_i = \frac{\pi'_{i0}}{\sum_{j=1}^{4} p_j\pi'_{j0}}, \qquad \pi''_i = \frac{\pi''_{i0}}{\sum_{j=1}^{4} p_j\pi''_{j0}}.$$

Then all the conditions (a), (b), (c), (d) are satisfied. By similar reasoning it is easy to see that the parameters can be chosen so that $w_1$ and $w_2$ have arbitrary sizes $\alpha_1$ and $\alpha_2$, respectively.

It is possible to obtain cases where $H_0$ contains a continuum of simple hypotheses, for example

$$H_0(\lambda): P\{X = i\} = \lambda p'_i + (1 - \lambda)p''_i ,$$

with $0 \leqq \lambda \leqq 1$, where $p'_i$, $p''_i$ are obtained as in the main part of this paper. The same tests are most powerful and similar. Many interesting questions arise but they seem not to be of any real statistical importance.

REFERENCES

[1] J. Neyman and E. S. Pearson, "Contributions to the theory of testing statistical hypotheses, part I," *Stat. Res. Memoirs*, Vol. 1 (1936), pp. 1–37.

[2] E. L. Lehmann, "Some principles of the theory of testing hypotheses," *Annals of Math. Stat.*, Vol. 21 (1950), pp. 1–26.

# NOTE ON THE ESTIMATION OF A BIVARIATE DISTRIBUTION FUNCTION

By Paul B. Simpson

*University of Oregon*

A continuous cumulative probability distribution $F(x)$ can be estimated from a random sample $(x_i)$, $i = 1, \cdots, n$, by the step function $G(x) = j/n$, where $j$ is the number of $x_i \leq x$. In this single variable case, it is known that the probability distribution

$$(1) \qquad P\{\max_x |F(x) - G(x)| < \lambda\}$$