A MULTIVARIATE GAMMA-TYPE DISTRIBUTION[1]

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Introduction. Mehler has shown that the two-variate probability density function (pdf) for correlated variates, each of which has a marginal Guassian distribution, can be expressed as a series bilinear in Hermite polynomials:

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\{-\tfrac{1}{2}(x^2 - 2\rho xy + y^2)/(1-\rho^2)\}$$
$$= \frac{1}{2\pi} \exp\{-\tfrac{1}{2}(x^2 + y^2)\}\left[1 + \rho H_1(x)H_1(y) + \frac{\rho^2}{2!} H_2(x)H_2(y) + \cdots\right].$$

Kibble [5] has extended this result to any number of variables and noticed a small difference between the generalization and the particular case due to Mehler.

It is known that Mehler's series is not an isolated result, there being a similar series bilinear in Laguerre polynomials, discussed by Hardy [3], Watson [6], and Kibble [4], and series bilinear in certain other other polynomials, discussed by Campbell [2], and by Aitken and Gonin [1]. All these results can be generalized for any number of variables in much the same way as Kibble has generalized Mehler's result. These generalizations are contained in Krishnamoorthy's thesis "Multivariate Distribution Functions" (in the library of the University of Madras). In the present paper the generalization involving Laguerre polynomials is given.

1. Notation and summary. It was shown by Kibble [4] that a two-variate distribution function, in which each of the variates x_i, $i = 1, 2$, has the frequncy function

$$\phi(x_i) = \frac{x_i^{p-1} e^{-x_i}}{\Gamma(p)}, \tag{1.1}$$

may be represented by

$$\phi(x_1)\phi(x_2)\left[1 + \frac{\rho^2}{p} L_1(x_1, p)L_1(x_2, p) + \frac{\rho^4}{2!p(p+1)} L_2(x_1, p)L_2(x_2, p) + \cdots\right],$$

where $L_r(x, p)$, $p > 0$, is the generalized Laguerre polynomial of degree r satisfying

$$L_r(x, p) \equiv r! L_r^{(p-1)}(x) = \frac{\left(-\frac{d}{dx}\right)^r [x^r\phi(x)]}{\phi(x)}. \tag{1.2}$$

[1] Sections 1 to 4 of this paper, deriving the distributions, were written by the first author; Section 5, on the convergence of certain series, was contributed by the second author.

It is the object of this paper to extend Kibble's result to n variables, assuming (i) that the variates have each a marginal Gamma-type distribution given by (1.1) with the same parameter p; (ii) that the variates have Gamma-type distributions with different parameters. The extension in case (i) appears in (3.7) and that in case (ii) appears in (4.1). The convergence of series obtained in either case is established in Section 5.

An outline of the procedure followed may be given thus. We obtain, in (2.2), the moment-generating function (mgf) for the joint distribution of $\xi_i = \frac{1}{2}x_i^2$, $(i = 1, 2, \cdots, n)$, where each x_i has a normal distribution with zero mean. From this we get the mgf for the distribution of the sums of squares in a sample of m from a normal correlated n-variate distribution, and thence, in (2.3), a possible mgf for an n-variate distribution in which each variate has a Gamma-type distribution. Finally we obtain from (2.3) the n-variate distribution in (3.7) by a process which is essentially the inversion of the Laplace transform,

$$(1.3) \qquad (1 - \alpha)^{-p}\left(\frac{\alpha}{1 - \alpha}\right)^r = \int_0^\infty e^{\alpha x} f_r(x, p)\, dx,$$

where

$$f_r(x, p) = \frac{L_r(x, p)\phi(x)}{p^{(r)}}, \qquad p^{(r)} = p(p + 1) \cdots (p + r - 1).$$

It will be noticed that (1.3) is true for $r = 0$ if we define (as we may) $L_0(x, p) = 1$, $p^0 = 1$.

2. An mgf for an n-variate Gamma-type distribution. Let $\| \rho_{ij} \|$ defined by

$$\| \rho_{ij} \| = \begin{Vmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{Vmatrix},$$

where $\rho_{ij} = \rho_{ji}$, be a positive definite matrix. Then the normal correlated n-variate distribution having zero means and $\| \rho_{ij} \|$ for its variance-covariance matrix is given by

$$(2.1) \qquad dF = \frac{| \rho^{ij} |^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n}} \exp\left\{-\tfrac{1}{2} \sum_{i,j=1}^{n} \rho^{ij} x_i x_j\right\} dx_1\, dx_2 \cdots dx_n$$

in the usual notation, where $\| \rho^{ij} \|$ is the inverse of $\| \rho_{ij} \|$. Denoting the mgf for a distribution of ξ_i, $i = 1, 2, \cdots, n$, having any pdf, by

$$G(\alpha_1, \alpha_2, \cdots, \alpha_n) = E\,\{\exp \sum_i \alpha_i \xi_i\},$$

we find that, in the case $\xi_i = \frac{1}{2}x_i^2$ where x_i have the joint distribution (2.1), the mgf is

$$G_{\frac{1}{2}}(\alpha_1, \alpha_2, \cdots, \alpha_n) = E\{\exp \tfrac{1}{2} \sum \alpha_i x_i^2\}$$

$$= \frac{|\rho^{ij}|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n}} \int \exp\{-\tfrac{1}{2} \sum (\rho^{ij} - \delta_{ij}\alpha_i)x_i x_j\}\, d\bar{x},$$

where $-\infty \leq x_i \leq \infty$, $i = 1, 2, \cdots, n$, $\int \cdots d\bar{x}$ denotes an integration with respect to all x_i, and δ_{ij} is the Kronecker delta defined as zero if $i \neq j$ and unity otherwise. Therefore

$$G_{\frac{1}{2}}(\alpha_1, \alpha_2, \cdots, \alpha_n) = |\rho^{ij}|^{\frac{1}{2}} |\rho^{ij} - \delta_{ij}\alpha_i|^{-\frac{1}{2}} = |\delta_{ij} - \rho_{ij}\alpha_i|^{-\frac{1}{2}}, \tag{2.2}$$

provided that $\| \rho^{ij} - \delta_{ij}\alpha_i \|$ is positive definite. It follows that the mgf for the distribution of the sums of squares in a sample of m from a normal correlated n-variate distribution is obtained by raising the expression in (2.2) to the mth power, and furthermore, that the replacement of $m/2$ by p leads to a possible mgf for an n-variate distribution in which each variate x_i has the frequency function (1.1). Therefore *a possible mgf for an n-variate Gamma-type distribution defined as above is obtained from* (2.2) *when* $-\frac{1}{2}$ *in the power on the right side of* (2.2) *is changed to* $-p$. *The expression for the mgf is*

$$G_p(\alpha_1, \alpha_2, \cdots, \alpha_n) = |\delta_{ij} - \rho_{ij}\alpha_i|^{-p}$$

$$= \begin{vmatrix} 1-\alpha_1 & -\rho_{12}\alpha_2 & \cdots & -\rho_{1n}\alpha_n \\ -\rho_{12}\alpha_1 & 1-\alpha_2 & \cdots & -\rho_{2n}\alpha_n \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ -\rho_{1n}\alpha_1 & -\rho_{2n}\alpha_2 & \cdots & 1-\alpha_n \end{vmatrix}^{-p} \tag{2.3}$$

$$= \{(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_n)\}^{-p} \begin{vmatrix} 1 & -\rho_{12}\beta_2 & \cdots & -\rho_{1n}\beta_n \\ -\rho_{12}\beta_1 & 1 & \cdots & -\rho_{2n}\beta_n \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ -\rho_{1n}\beta_1 & -\rho_{2n}\beta_2 & \cdots & 1 \end{vmatrix}^{-p},$$

where $\beta_i = \alpha_i/(1 - \alpha_i)$, $i = 1, 2, \cdots, n$. It is convenient to write (2.3) in the form

$$G(\alpha_1, \alpha_2, \cdots, \alpha_n) = \{(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_n)\}^{-p}\{g(\beta_1, \beta_2, \cdots, \beta_n)\}^{-p}, \tag{2.4}$$

where $g(\beta_1, \beta_2, \cdots, \beta_n)$ is the determinant of β's in (2.3).

3. A series for an n-variate Gamma-type distribution. Expanding the g in (2.4) by Maclaurin's theorem for a function of n variables, we get

$$g(\beta_1, \beta_2, \cdots, \beta_n) = g_0 + \left(\sum_i \beta_i \frac{\partial}{\partial \beta_i}\right) g_0 \tag{3.1}$$
$$+ \frac{1}{2!}\left(\sum_i \beta_i \frac{\partial}{\partial \beta_i}\right)^{[2]} g_0 + \cdots + \frac{1}{n!}\left(\sum_i \beta_i \frac{\partial}{\partial \beta_i}\right)^{[n]} g_0,$$

where $g_0 = g(0, 0, \cdots, 0)$ and $(\Sigma_i \beta_i(\partial/\partial\beta_i))^{[r]}g_0$ is the result of first expanding $(\Sigma \beta_i(\partial/\partial\beta_i))^r$ regarding the operators $\partial/\partial\beta_i$ as algebraical numbers, then giving the operators their proper roles in the expanded form of $(\Sigma \beta_i(\partial/\partial\beta_i))^r$, and finally putting $\beta_1 = \beta_2 = \cdots = \beta_n = 0$ in the partial derivatives of g which we get when the expanded form is applied to g.

Clearly the expansion of $g(\beta_1, \beta_2, \cdots, \beta_n)$ in (3.1) does not contain terms linear in the β's or terms such as $\beta_1^{p_1} \cdots \beta_i^{p_i} \cdots$ with any $p_i > 1$. In fact

$$g_0 = 1, \qquad \frac{\partial g_0}{\partial \beta_1} = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ -\rho_{12} & 1 & \cdots & 0 \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ -\rho_{1n} & 0 & \cdots & 1 \end{vmatrix} = 0,$$

where, of course, the partial derivation is performed before we put $\beta_1 = \beta_2 = \cdots = \beta_n = 0$; and similarly $\partial g_0/\partial\beta_2$, $\partial g_0/\partial\beta_3, \cdots, \partial g_0/\partial\beta_n$ are all zero, so that $(\Sigma_{i=1}^n \partial/\partial\beta_i)g_0 = 0$.

Further

$$\frac{\partial^2 g_0}{\partial\beta_1\partial\beta_2} = \begin{vmatrix} 0 & -\rho_{12} & 0 & \cdots & 0 \\ -\rho_{12} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ -\rho_{1n} & -\rho_{2n} & & & 1 \end{vmatrix} = \begin{vmatrix} 0 & \rho_{12} \\ \rho_{12} & 0 \end{vmatrix} = -C_{12} \quad \text{say,}$$

$$\frac{\partial^3 g_0}{\partial\beta_1\partial\beta_2\partial\beta_3} = -\begin{vmatrix} 0 & \rho_{12} & \rho_{13} \\ \rho_{12} & 0 & \rho_{23} \\ \rho_{13} & \rho_{23} & 0 \end{vmatrix} = -C_{123} \quad \text{say,}$$

$$\vdots$$

$$\frac{\partial^n g_0}{\partial\beta_1\partial\beta_2\cdots\partial\beta_n} = (-1)^n \begin{vmatrix} 0 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 0 & & \rho_{2n} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \rho_{1n} & \rho_{2n} & \cdots & 0 \end{vmatrix} = -C_{123\cdots n} \quad \text{say.}$$

Hence (3.1) can be written

$$g(\beta_1, \beta_2, \cdots, \beta_n) = 1 - \Big(\sum_{i<j} C_{ij}\,\beta_i\,\beta_j + \sum_{i<j<k} C_{ijk}\,\beta_i\,\beta_j\,\beta_k + \cdots + C_{123\cdots n}\,\beta_1\,\beta_2 \cdots \beta_n\Big) = 1 - B \text{ say}. \tag{3.2}$$

Using (3.2) in (2.4) and expanding $(1 - B)^{-p}$ formally by the binomial theorem, we get

$$G_p(\alpha_1, \alpha_2, \cdots, \alpha_n) = \{(1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_n)\}^{-p} \cdot \left\{\sum_0^\infty \frac{p(p+1)\cdots(p+r-1)}{r!} B^r\right\} = \sum_0^\infty \frac{p(p+1)\cdots(p+r-1)}{r!} B^*_r, \tag{3.3}$$

where

$$B^*_r = \{(1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_n)\}^{-p} B^r.$$

Expanding B^r by the multinomial theorem, we can express B^*_r as a finite sum of the form

$$B^*_r = \prod_{i=1}^n (1 - \alpha_i)^{-p} \left(\sum K \beta_1^{a_1} \beta_1^{a_2} \cdots \beta_n^{a_n}\right) = \sum K \left\{\prod_{i=1}^n (1 - \alpha_i)^{-p} \left(\frac{\alpha_i}{1 - \alpha_i}\right)^{-a_i}\right\} \tag{3.4}$$

where K is a polynomial in $C_{ij}, \cdots, C_{12\cdots n}$, and $a_1, \cdots, a_n$ are nonnegative integers of which not more than $n - 2$ are zero.

It is now plain from (3.4) and (1.3) that B^*_r can be expressed as a Laplace transform:

$$B^*_r = \int_{0 \leqq x_i \leqq \infty} e^{\Sigma \alpha_i x_i} \sum K \prod_{i=1}^n f_{a_i}(x_i, p)\, d\bar{x},$$

of which the determining function is

$$\sum K \prod_{i=1}^n f_{a_i}(x_i, p) = \phi(x_1)\phi(x_2) \cdots \phi(x_n) \sum K \prod_{i=1}^n \frac{L_{a_i}(x_i, p)}{p^{(a_i)}} \tag{3.5}$$

$$= \phi(x_i)\phi(x_2) \cdots \phi(x_n) \left\{\sum_{i<j} C_{ij} \frac{L(x_i, p)}{p} \frac{L(x_j, p)}{p} + \cdots + C_{12\cdots n} \frac{L(x_1, p)}{p} \cdots \frac{L(x_n, p)}{p}\right\}^r, \tag{3.6}$$

where $\{\cdots\}^r$ is a symbol for the rth power of a multinomial, in expanding which we suppose that

$$\left\{\frac{L(x, p)}{p}\right\}^m \left\{\frac{L(x, p)}{p}\right\}^n = \left\{\frac{L(x, p)}{p}\right\}^{m+n}$$

for all positive integers m, n, and after expanding which we set

$$\left\{\frac{L(x, p)}{p}\right\}^m \equiv \frac{L_m(x, p)}{p^{(m)}}.$$

Finally we can replace B_r^* in the series of (3.3) by its determining function in (3.6) and obtain the form

$$\begin{aligned}\phi(x_1)\phi(x_2) \cdots \phi(x_n) \sum_0^\infty \frac{p^{(r)}}{r!}\left\{\sum_{i<j} C_{ij} \frac{L(x_i, p)}{p} \frac{L(x_j, p)}{p}\right. \\ \left. + \cdots + C_{123\cdots n} \frac{L(x_i, p)}{p} \cdots \frac{L(x_n, p)}{p}\right\}^r,\end{aligned} \tag{3.7}$$

where $\phi(x_i)$ is defined by (1.1), for a distribution function having $G_p(\alpha_1, \alpha_2, \cdots, \alpha_n)$ defined as in (2.3) for its mgf. The convergence of series (3.7) is proved, with a certain restriction on the ρ's, in Section 5. Consequently, *with this restriction as regards convergence, we can take* (3.7) *to be an n-variate distribution function in which each variate* x_i, $i = 1, 2, \cdots, n$ *has the distribution function* $\phi(x_i)$ *in* (1.1).

Remark on the series (3.7). If there are only two β's present in any term of (3.4), this being their least number possible, they will be raised to the same degree r, and therefore the corresponding term of (3.5) will have Laguerre polynomials of the same degree r. If, however, more than two β's are present in a term of (3.4), their degrees may be different and consequently also the degrees of the Laguerre polynomials in the corresponding term of (3.5). Hence the n-variate Gamma-type distribution symbolically denoted by (3.7) has the property that (i) any term in its expansion involving two variables contains Laguerre polynomials of the same degree in those variables, while (ii) a term involving more than two variables may contain Laguerre polynomials of different degrees in the variables. It is known [5] that an analogous property is possessed by the extension to n variates of Mehler's series in Hermite polynomials.

4. A generalization of Section 3. If we take instead of the mgf in (2.4) the more general mgf

$$(1 - \alpha_1)^{-p_1}(1 - \alpha_2)^{-p_2} \cdots (1 - \alpha_n)^{-p_n}\{g(\beta_1, \beta_2, \cdots, \beta_n)\}^{-p}$$

and repeat the reasoning of Section 3, we shall obtain, in the symbolic notation of (3.7), the following series (whose convergence is established in Section 5 under the condition on the ρ's already referred to):

$$\begin{aligned}\phi(x_1)\phi(x_2) \cdots \phi(x_n) \sum_0^\infty \frac{p^{(r)}}{r!}\left\{\sum_{i<j} C_{ij} \frac{L(x_i, p_i)}{p_i} \frac{L(x_j, p_j)}{p_j}\right. \\ \left. + \cdots + C_{12\cdots n} \frac{L(x_1, p_1)}{p_1} \cdots \frac{L(x_n, p_n)}{p_n}\right\}^r,\end{aligned} \tag{4.1}$$

$$\phi(x_i) = \frac{x_i^{p_i-1}e^{-x_i}}{\Gamma(p_i)}, \qquad i = 1, 2, \cdots, n. \tag{4.2}$$

This series, under the condition which secures its convergence, may be regarded as an n-variate distribution function in which each variate x_i, $i = 1, 2, \cdots, n$ has the distribution function $\phi(x_i)$ of (4.2).[2]

5. Addendum: the convergence of the series in (3.7) and (4.1). The object of this addendum is to establish, under a suitable condition, the convergence of the series in (3.7) and (4.1) The proof of the convergence depends on the following lemma.

LEMMA. *In the symbolic notation of* (3.6), *for* $r \geq 1$

$$\left|\left\{\frac{L(x, p)}{p}\right\}^r\right| \equiv \left|\frac{L_r(x, p)}{p^{(r)}}\right| < \begin{cases} K(x, p)r^{\frac{1}{2}(-p+\frac{1}{2})}, & 0 < p < \frac{1}{2} \\ K(x, p), & p \geq \frac{1}{2}, \end{cases}$$

where $K(x, p)$ is a constant depending on x and p.

PROOF. From the well known result $\Gamma(x + a)/\Gamma(x) \sim x^a$ as $x \to \infty$, where a is a constant, we get

$$\frac{p^{(r)}}{r!} = \frac{p(p + 1) \cdots (p + r - 1)}{r!} = \frac{\Gamma(p + r)}{\Gamma(p)\Gamma(r + 1)} \sim \frac{r^{p-1}}{\Gamma(p)} \quad \text{as } r \to \infty. \tag{5.1}$$

From a formula of Fejér [7], Hille [8] has deduced that

$$\frac{L_r(x, p)}{r!} = \frac{1}{\sqrt{\pi}} e^{\frac{1}{2}x}x^{-\frac{1}{2}(p-\frac{1}{2})}r^{\frac{1}{2}(p-\frac{3}{2})} \cos\left[2\sqrt{rx} - \pi\left(\frac{1}{4} + \frac{p - 1}{2}\right)\right] + O\,[r^{\frac{1}{2}(p-2)}], \quad r \to \infty. \tag{5.2}$$

Combining (5.2) with (5.1), we conclude that

$$\left|\frac{L_r(x, p)}{p^{(r)}}\right| = \left|\frac{L_r(x, p)/r!}{p^{(r)}/r!}\right| < A(x, p)r^{-\frac{1}{2}(p-\frac{1}{2})}, \qquad r > r_0, \tag{5.3}$$

where $A(x, p)$ is a constant which depends on x and p. Further, once r_0 is fixed,

$$\left|\frac{L_r(x, p)}{p^{(r)}}\right| < B(x, p), \qquad r \leq r_0, \tag{5.4}$$

where $B(x, p)$ is also a constant which depends on x and p. Equations (5.3) and (5.4) together yield the result stated in the lemma where $K = \max(A, B)$.

THEOREM. *The series in* (3.7),

$$\sum \frac{p^{(r)}}{r!} t_r \equiv \sum \frac{p^{(r)}}{r!}\left\{\sum C_{ij} \frac{L(x_i, p)}{p} \frac{L(x_j, p)}{p} + \cdots + C_{123\cdots n} \frac{L(x_1, p)}{p} \cdots \frac{L(x_n, p)}{p}\right\}^r,$$

is absolutely convergent provided that

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[2] Thanks are due to Dr. P. Kesava Menon and Prof. C. T. Rajagopal for helping to settle certain points of detail.

$$\sigma \equiv \sum_{i,j} | C_{ij}| + \sum_{i,j,k} | C_{ijk} | + \cdots + | C_{123\cdots n} | < 1. \tag{5.5}$$

PROOF. We have, in symbolic notation,

$$t_r = \sum \lambda_{m_2, m_2', \cdots, m_n} \left\{ C_{ij} \frac{L(x_i, p)}{p} \frac{L(x_j, p)}{p} \right\}^{m_2} \left\{ C_{i'j'} \frac{L(x'_i, p)}{p} \frac{L(x'_j, p)}{p} \right\}^{m_2'} \cdots \left\{ C_{123\cdots n} \frac{L(x_1, p)}{p} \cdots \frac{L(x_n, p)}{p} \right\}^{m_n}, \tag{5.6}$$

where one at least of the suffixes i', j' is different from i, j (similar statements being true of the C's with 3, 4, $\cdots$ suffixes), and

$$\lambda_{m_2, m_2', \cdots m_n} = \frac{r!}{m_2! \, m_2'! \cdots m_n!}, \qquad m_2 + m_2' + \cdots + m_n = r.$$

First suppose that $p \geq \frac{1}{2}$. Then (5.6) gives, by virtue of the lemma,

$$| t_r | \leq \Sigma \lambda_{m_2, m_2', \cdots, m_n} K(x_1, p) K(x_2, p) \cdots K(x_n, p) \, | C_{ij} |^{m_2} \, | C_{i'j'} |^{m_2'} \cdots | C_{123\cdots n} |^{m_n}. \tag{5.7}$$

Therefore, writing

$$\kappa = \max \{K(x_1, p), K(x_2, p), \cdots, K(x_n, p)\}$$

we get from (5.7)

$$\begin{aligned} | t_r | &\leq \kappa^n \Sigma \lambda_{m_2, m_2', \cdots, m_n} | C_{ij} |^{m_2} | C_{i'j'} |^{m_2'} \cdots | C_{123\cdots n} |^{m_n} \\ &= \kappa^n (\Sigma | C_{ij} | + \Sigma | C_{ijk} | + \cdots + | C_{123\cdots n} |)^r \equiv \kappa^n \sigma^r. \end{aligned}$$

And so $(p^{(r)}/r!) \, | t_r | \leq u_r \equiv (p^{(r)}/r!) \kappa^n \sigma^r$, where Σu_r is known to be convergent for $\sigma < 1$, and hence $\Sigma p^{(r)} t_r / r!$ is absolutely convergent for $\sigma < 1$.

In the case $p < \frac{1}{2}$, it is obvious from the lemma that

$$\frac{p^{(r)}}{r!} | t_r | \leq v_r \equiv \frac{p^{(r)}}{r!} \kappa^n r^{\frac{1}{2}(\frac{1}{2} - p)n} \sigma^r,$$

where

$$v_r^{1/r} = \left[\frac{p^{(r)}}{r!} \right]^{1/r} \kappa^{n/r} \left[r^{1/r} \right]^{\frac{1}{2}(\frac{1}{2} - p)n} \sigma \rightarrow \sigma \text{ as } r \rightarrow \infty.$$

Consequently, by Cauchy's root-test, Σv_r is convergent for $\sigma < 1$, and so again $\Sigma p^{(r)} t_r / r!$ is absolutely convergent for $\sigma < 1$.

A sufficient condition for the convergence of the series in the theorem, simpler in form than (5.5), *is*

$$N \rho^* < 1, \tag{5.8}$$

where N is the result of replacing every one of the ρ's in the C's by unity and ρ^* is the maximum of the terms in the ρ's when we omit the numerical coefficients of the terms.

A sufficient condition for the convergence of the series (4.1) *is again either* (5.5) *or* (5.8) since, arguing exactly as above, we find that

$$| \text{ the } (r+1)^{th} \text{ term of the series (4.1) } | \leq \prod_{i=1}^{n} \phi(x_i).\ \frac{p^{(r)}}{r!}\, \kappa^n r^{\Sigma_i \frac{1}{2}(\frac{1}{2}-p_i)} \sigma^r,$$

where the summation in the power of r is for all p_i which are less than $\frac{1}{2}$.

Note. The case $n = 2$ makes the series in the theorem identical with a series obtained by W. F. Kibble [4] for a two-variate Gamma-type distribution. Kibble's proof of the convergence is, however defective[3] since he assumes that

$$\frac{L_r(x, p)}{p^{(r)}} \sim \frac{L_{r-1}(x, p)}{p^{(r-1)}}, \qquad r \to \infty,$$

is a consequence of (5.2).

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