TEST CRITERIA FOR HYPOTHESES OF SYMMETRY OF A REGRESSION MATRIX[1]

By Uttam Chand[2]

University of North Carolina and Boston University

Summary. Hotelling's [1] theoretical findings in mathematical economics on the rational behavior of buyers in maximizing their net profit indicate that the matrix of the first partial derivatives of a set of related demand functions would be symmetric and negative definite. It is the object of this paper to determine whether the assumption of symmetry will be tenable in the light of the particular set of observations. The study of test functions for the property of definiteness as a whole will form the subject of a forthcoming paper. The present investigation assumes that the demand functions are regression functions and, therefore, results obtained in this paper do not cover all types of demand functions. The test function U proposed here for the hypothesis of symmetry is invariant under all contragredient transformations. The distribution of U depends on unknown nuisance parameters. The likelihood ratio under the hypothesis of symmetry leads to a multilateral matric equation which represents $\frac{1}{2}\, p(p+1)$ equations of the third degree in $\frac{1}{2}\, p(p+1)$ unknown regression coefficients for the p-variate case. It has not been possible to establish the existence of a non-trivial solution of this equation, and it is, therefore, not being given here.

1. Introduction. Let p_i denote the price of the ith commodity and q_i the quantity consumed at that price. Consider $p_i = f_i(q_1, q_2, \cdots)$ a set of demand functions and let $u = u(q_1, q_2, \cdots)$ represent the gross receipts of a purchaser of goods. Under the assumption that each entrepreneur tries to maximize his net profit $\pi = u - \Sigma p_i q_i$, Hotelling [1] in an important contribution concerning the theoretical nature of supply and demand functions showed that if the entrepreneur is working in a steady economic state in which there is no restriction on his money expenditure, then the matrix of the first partial derivatives of prices on quantities would be symmetric, that is,

$$\frac{\partial p_i}{\partial q_j} = \frac{\partial p_j}{\partial q_i}$$

and that for a true maximum such a matrix would be negative definite, that is,

$$\frac{\partial p_i}{\partial q_i} < 0, \qquad \frac{\partial(p_i, p_j)}{\partial(q_i, q_j)} > 0, \qquad \frac{\partial(p_i, p_j, p_k)}{\partial(q_i, q_j, q_k)} < 0, \cdots .$$

[1] This paper was presented at the Cleveland meeting of the Institute on December 27, 1948.

[2] The author wishes to express his grateful appreciation to Professors Harold Hotelling and William G. Madow for guidance in this research.

It would thus appear that the inequalities arising out of the negative definiteness of the matrix generalize the conditions that a demand curve shall decline.

No suitable statistical tests have existed for testing the hypothesis of symmetry and negative definiteness of the matrix referred to in the previous paragraph. Henry Schultz [2] was first to consider such a question and the present paper has grown out of his statistical attempts. To verify Hotelling's laws on the basis of a particular set of data consider

$$p_i = f_i(q_1, q_2, \cdots) + u_i, \tag{1.1}$$

a system of demand equations where p_i and q_j denote current prices and quantities and where u_i is a stochastic variable. We shall assume that the quantities are fixed and prices are determined by demand. For example, some government agency could conduct actual experiments fixing alternative sets of quantities and observing what prices the choice of buyers would lead to. In such a situation we shall, therefore, be justified in assuming demand functions to be regression functions. In general the quantities are determined by a certain type of supply function under the prevailing market mechanism. Suppose the supply functions are given by

$$p_i = Q_i(q_1, q_2, \cdots) + v_i, \tag{1.2}$$

where v_i is a stochastic variable and u's and v's have a more or less specified joint probability law. If (1.1) and (1.2) are to hold simultaneously their solutions, if they exist, will be the only observable values of prices and quantities; and, therefore, quantities such as $\partial q_i/\partial p_j$ cannot in general be estimated and consequently no question of testing symmetry could be raised. However we could conceive of a different type of supply functions from those in (1.2) containing other independently determined variables besides the p's and being of such a stochastic type that the equations (1.1) would be regression equations [12]. For the purpose of this investigation we shall assume that the demand equations are regression equations such that the mathematical expectation of p_i is equal to f_i and since not all demand functions are regression equations, the results of the present investigation are not applicable to all types of demand equations.

Since we are studying certain properties of correlated variables any proposed statistical criterion must satisfy the property of invariance under linear transformations of prices and quantities. The fact of the transformation of quantities being not independent of that of prices will further restrict us to the consideration of such relations as are invariant under a linear transformation of one set of variates contragredient to those of the other ([3], pp. 108–109). The importance of such a class of relations was first suggested by Hotelling in a series of papers [4], [5], [6]. Examples of such a "value preserving" class of transformations may be found in the mixing of different grades of wheat or the combination of raw materials and labor into finished products such that the total value remains unchanged.

The statistic U (Section 4) proposed here for the hypothesis of symmetry for the case of two related commodities is invariant under all contragredient transformations. It is exact in the sense that its probability distribution law is precisely determined under the hypothesis. Certain practically useful relations between this statistic and Student's t will be indicated. This test has in addition the property of being an unbiased test in the sense of Neyman and Pearson. We consider its p-variate generalization in Section 4.4.

2. Probability model. Let $Y = \| y_{i\alpha} \|$ be a $p \times N$ sample matrix from a normal multivariate parent having $\sigma = \| \sigma_{ij} \|$ as the dispersion matrix and $\eta = \| \eta_{i\alpha} \| = \beta X$ as the corresponding matrix of expectations where $\beta = \| \beta_{ij} \|$ is the population regression matrix and $X = \| x_{i\alpha} \|$ is the matrix of nonrandom observations on the fixed variates (e.g., $y_1, \cdots, y_p$ may denote prices and $x_1, \cdots, x_p$ the quantities consumed at these prices). Let $g = \| g_{ij} \| = XY'$, where $g_{ij} = \Sigma_\alpha x_{i\alpha} y_{j\alpha}$ and where Y' is the transpose of Y. Set $a = XX'$ and $c = \| c_{ij} \| = a^{-1}$ $(i, j = 1, \cdots, p; \alpha = 1, \cdots, N; p \leq N)$. We shall assume without loss of generality that $y_{i\alpha}$'s and $x_{i\alpha}$'s are either measured from their means or from polynomial means if the y's are subject to a time trend. It will be noticed that the symmetry and definiteness of the matrix of partial derivatives of prices on quantities is equivalent to the symmetry and definiteness of the regression matrix β.

3. Contragredient transformation of the two sets of variates. Let $f = \| f_{ij} \|$ be a $p \times p$ nonsingular matrix and let the columns x of the matrix X be subjected to the transformation f; we write $x^* = fx$. If the columns y are transformed into columns y^* in such a way that $y^{*\prime} x^* = y' x$ for every x and y, then the transformation of the y's is uniquely determined, viz., $y^* = f'^{-1} y$. We say, under these circumstances, that the columns x on the one hand, and the columns y on the other hand, are transformed contragrediently under f. For the mathematical expectation of y^*'s we have $E(y^*) = \beta^* x^*$ where $\beta^* = f'^{-1} \beta f^{-1}$. Consequently $\beta^{*\prime} = \beta^*$ implies $\beta' = \beta$ and conversely. Thus we notice that the symmetry of the matrix β is preserved by this type of transformation. Since the property of definiteness is invariant under any nonsingular linear transformation, the hypotheses of symmetry and definiteness are invariant and we might as well consider the properties of the matrix β^*. If we denote by $\sigma^* = \| \sigma^*_{ij} \|$ the covariance matrix of the y^*'s, we have $\sigma^* = f'^{-1} \sigma f^{-1}$ and consequently the ratio of the determinants $| \beta |$ and $| \sigma |$ is an absolute invariant. We now state the following theorem:

THEOREM I. *If σ is a positive definite matrix and β a real symmetric matrix and the two are cogrediently transformed, there exists a nonsingular linear transformation which will reduce σ to an identity matrix and β to a diagonal matrix.*

PROOF. We have $\beta = f' \beta^* f$ and $\sigma = f' \sigma^* f$ and the proof follows from a theorem given in [3] (p. 171). We shall make use of this result in Sections 4 and 5.

4. Hypothesis of symmetry of the regression matrix β.

4.1. *The statistic U.* We shall show that for the bivariate case the statistic U

now to be presently defined provides an exact and unbiased test for $H_0 : \beta_{12} = \beta_{21}$ against the set of alternatives which do not specify anything except $\beta_{12} \neq \beta_{21}$. Consider

$$U = (b_{12} - b_{21})^2(c_{22}s_{11} + c_{11}s_{22} - 2c_{12}s_{12})^{-1},$$

where

(i) The sample regression coefficients b_{ij} are normally distributed with means β_{ij} and $E(b_{kj} - \beta_{kj})(b_{mi} - \beta_{mi}) = \sigma_{km}c_{ij}$.

(ii) The s_{ij}'s are the unbiased estimates, each based on (say) n degrees of freedom, of σ_{ij} and follow the Wishart [7] law. Actually we have

$$ns_{ij} = \sum_{\alpha=1}^{N} (y_{i\alpha} - Y_{i\alpha})(y_{j\alpha} - Y_{j\alpha}),$$

where $Y_{i\alpha}$'s are sample regression functions.

(iii) The c_{ij}'s have been previously defined (Section 2).

Under the assumption of the conditional bivariate normal law for the $y_{i\alpha}$'s (Section 2), the residuals of $y_{i\alpha}$'s from their respective sample regression functions are normally distributed. If the $y_{i\alpha}$'s are subject to a time trend, as may very often be the case in economics, it will be more appropriate to consider the model

$$E(y_{i\alpha}) = \alpha_0 + \alpha_1\xi_1(t) + \alpha_2\xi_2(t) + \cdots + \beta_{i1}(x_{1\alpha} - \bar{x}_1) + \beta_{i2}(x_{2\alpha} - \bar{x}_2),$$

where the $\xi(t)$'s are known polynomials in time. Under such a model also residuals are known to be normally distributed. Consequently we might as well have assumed such a model which will thus only affect the number of degrees of freedom available for the estimates s_{ij}.

THEOREM II. *If x and y are transformed contragrediently, the statistic U is an absolute invariant.*

PROOF. Set $s = \| s_{ij} \|$ and $b = \| b_{ij} \|$. Under the contragredient transformation of x and y (Section 3) we have $b = f'b^*f$; $s = f's^*f$; and $c = f'c^*f$. If we perform this transformation on U and simplify, we notice that the numerator and denominator of U are relative invariants of weight -2 and consequently U is an absolute invariant.

4.2. *Distribution of U under the null hypothesis.* Since U is an absolute invariant under the contragredient transformation of x and y we may derive the distribution of U taking σ to be an identity matrix and β to be a diagonal matrix (Theorem I) in the parametric space.

The numerator and denominator of U are distributed independently of one another [8]. Let $Z = c_{22}s_{11} + c_{11}s_{22} - 2c_{12}s_{12}$. This is a positive definite quadratic form in normally distributed variates. Let u_α and v_α represent residuals of y_1 and y_2 from the corresponding sample regression functions. There exists an orthogonal transformation of the N variables u_α and v_α which will simultaneously yield $s_{11} = \Sigma_1^n u_\alpha^{*2}/n$; $s_{22} = \Sigma_1^n v_\alpha^{*2}/n$ and $s_{12} = \Sigma_1^n u_\alpha^* v_\alpha^*/n$, where u^* and v^* are normally and independently distributed with 0 means and a common variance for each set u^*, v^*. Consider now the orthogonal transformation

$$u'_\alpha = u^*_\alpha \cos\theta - v^*_\alpha \sin\theta,$$
$$v'_\alpha = u^*_\alpha \sin\theta + v^*_\alpha \cos\theta,$$

where θ is so determined that $nZ = d_1\Sigma_1^n u'^2_\alpha + d_2\Sigma_1^n v'^2_\alpha$; then

$$d_1 = \tfrac{1}{2}\{c_{11} + c_{22} + [(c_{11} - c_{22})^2 + 4c_{12}^2]^{\frac{1}{2}}\},$$
$$d_2 = \tfrac{1}{2}\{c_{11} + c_{22} - [(c_{11} - c_{22})^2 + 4c_{12}^2]^{\frac{1}{2}}\}.$$

Consequently $U = \chi^2_{1,1}(C_1\chi^2_{2,n} + C_2\chi^2_{3,n})^{-1}$ where the χ^2's have independent χ^2-distributions with degrees of freedom as indicated in the second subscripts and

$$C_1 = (2n)^{-1}\{1 + [1 - 4\,|\,c\,|/(c_{11} + c_{22})^2]^{\frac{1}{2}}\},$$
$$C_2 = (2n)^{-1}\{1 - [1 - 4\,|\,c\,|/(c_{11} + c_{22})^2]^{\frac{1}{2}}\},$$

and $|\,c\,| = c_{11}c_{22} - c_{12}^2$.

The distribution of a quantity similar to U was first obtained by Hsu [9]. An independent derivation of the distribution of the quantity $\tau = \xi(\lambda_1\chi_1^2 + \lambda_2\chi_2^2)^{-\frac{1}{2}}$, where λ_1, λ_2 are certain positive constants and ξ is $N(0, 1)$, will also be found in [10]. Robbins and Pitman [11] have obtained general results for the distribution of the ratio of mixtures of χ^2's, of which the form (4.2.1) given below is a particular case.

We have the following two forms for the frequency function of U:

$$g_0(U) = [\mathrm{B}(n, \tfrac{1}{2})U^{\frac{1}{2}}]^{-1}(C_2/C_1)^{n/2}C_2^{\frac{1}{2}}(1 + C_2U)^{-n-\frac{1}{2}} \cdot F\left(n + \tfrac{1}{2}, \tfrac{1}{2}n, n, \frac{1 - C_2/C_1}{1 + C_2U}\right) \tag{4.2.1}$$

for any value of n, and

$$g_0(U) = U^{-\frac{1}{2}}\sum_{h=0}^{\frac{1}{2}n-1}\Big(\Gamma(\tfrac{1}{2}n)\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}n - h)\Gamma(h + 1)\Big)^{-1}\Gamma(\tfrac{1}{2}(n + 1) - h) \cdot\Gamma(\tfrac{1}{2}n + h)(C_1 - C_2)^{-\frac{1}{2}n-h} \cdot[(-1)^hC_1^{\frac{1}{2}(n+1)}C_2^h(1 + C_1U)^{-\frac{1}{2}(n+1)+h} + (-1)^{\frac{1}{2}n}C_2^{\frac{1}{2}(n+1)}C_1^h(1 + C_2U)^{-\frac{1}{2}(n+1)+h}], \tag{4.2.2}$$

for n even [10].

We notice that since $C_1 + C_2 = 1/n$, the distribution of U essentially depends on C_1 or C_2 and is, therefore, precisely determined by n and the quantity $c/(\mathrm{tr}\ c)^2$ $(=a/(\mathrm{tr}\ a)^2) = w$ (say). If $w > \frac{1}{4}$, C_1 and C_2 are both imaginary. When the matrix c is a 2×2 matrix, the truth of the relation $0 \leq w \leq \frac{1}{4}$ can also be verified independently. The relation (4.2.1) is not defined when $C_2 = 0$, i.e., when $w = 0$; however it is clear from the form of U that it is distributed as Student's t^2 with n degrees of freedom. If $w = \frac{1}{4}$, $C_1 = C_2 = 1/(2n)$, and U has the t^2 distribution with $2n$ degrees of freedom. We shall refer to this again in the next section where we examine the overall behavior of the probability of Type I error of U with respect to w.

4.3. *Probability of Type I error of U.* To derive $P = P(U \geq U_0)$ corresponding to the form of the frequency function (4.2.1) we put $\zeta = (1 + C_2U)^{-1}$ and after integration obtain

$$P = (C_2/C_1)^{\frac{1}{2}n} \sum_{h=0}^{\infty} \Gamma(\tfrac{1}{2}n + h)[\Gamma(\tfrac{1}{2}n)\Gamma(h + 1)]^{-1}(1 - C_2/C_1)^h \cdot I_{\zeta_0}(n + h, \tfrac{1}{2}), \tag{4.3.1}$$

where $I_{\zeta_0}(p, q)$ is the incomplete beta ratio and $\zeta_0 = (1 + C_2U_0)^{-1}$. The series (4.3.1) consists of positive terms and is absolutely and uniformly convergent. Corresponding to the form (4.2.2) for even degrees of freedom we similarly obtain

$$P = \sum_{h=0}^{\frac{1}{2}n-1} \Gamma(\tfrac{1}{2}n + h)(\Gamma(\tfrac{1}{2}n)\Gamma(h + 1))^{-1}(C_1 - C_2)^{-\frac{1}{2}n-h} \cdot[(-1)^h C_1^{\frac{1}{2}n} C_2^h I_{\zeta_0'}(\tfrac{1}{2}n - h, \tfrac{1}{2}) + (-1)^{\frac{1}{2}n} C_2^{\frac{1}{2}n} C_1^h I_{\zeta_0}(\tfrac{1}{2}n - h, \tfrac{1}{2})], \tag{4.3.2}$$

where $\zeta_0' = (1 + C_1U_0)^{-1}$.

Consider the series (4.3.1). Following Robbins and Pitman [11] if we set

$$d_h = (C_2/C_1)^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n + h)(1 - C_2/C_1)^h[\Gamma(\tfrac{1}{2}n)\Gamma(h + 1)]^{-1},$$

so that $\Sigma_0^\infty d_h = 1$, we have

$$0 \leq P - \sum_{h=0}^{p} d_h I_{\zeta_0}(n + h, \tfrac{1}{2}) \leq \left(1 - \sum_{h=0}^{p} d_h\right) I_{\zeta_0}(n + 2(p + 1), \tfrac{1}{2}).$$

For any given U_0 this inequality sets an upper bound to the error committed in P in stopping at the $(p + 1)$st term of the series (4.3.1) which has been found to be slowly convergent. Whenever n is even and not large, the finite form (4.3.2) is to be preferred for computational purposes.

We now state the following theorem concerning the dependence of the probability of Type I error of U on the variable parameter w:

THEOREM III. *For any n and fixed U_0, $P(U \geq U_0 \mid H_0)$ is a monotone decreasing function of the variable parameter w.*

PROOF. We shall prove this result by considering the derivative of P with respect to w. From (4.3.1) we obtain

$$\frac{dP}{dw} = (C_2/C_1)^{\frac{1}{2}n-1}(1 - 4w)^{-\frac{1}{2}} \sum_{h=0}^{\infty} \Gamma(\tfrac{1}{2}n + h)[\Gamma(\tfrac{1}{2}n)\Gamma(h + 1]^{-1}$$

$$\cdot\left[4(1 + (1 - 4w)^{\frac{1}{2}})^{-2} I_{\zeta_0}(n + h, \tfrac{1}{2})\{\tfrac{1}{2}n(1 - C_2/C_1)^h - h(C_2/C_1)(1 - C_2/C_1)^{h-1}\} - 2\frac{C_2/C_1(1 - C_2/C_1)^h \zeta_0^{n+h}(1 - \zeta_0)^{\frac{1}{2}}(1 - (1 - 4w)^{\frac{1}{2}})^{-1}}{B(n + h, \tfrac{1}{2})}\right],$$

which may actually be shown to represent a derivative. Following Hsu ([9], pp. 14–15) the series

$$\sum\Gamma(\tfrac{1}{2}n + h)(\Gamma(\tfrac{1}{2}n)\Gamma(h + 1))^{-1}[\tfrac{1}{2}n(1 - C_2/C_1)^h - h(C_2/C_1)(1 - C_2/C_1)^{h-1}] \cdot I_{\zeta_0}(n + h, \tfrac{1}{2})$$

can be shown to be equivalent to

$$\sum(\tfrac{1}{2}n + h)(\Gamma(\tfrac{1}{2}n)\Gamma(h + 1))^{-1}\Gamma(\tfrac{1}{2}n + h)(1 - C_2/C_1)^h\eta_h,$$

where

$$\eta_h = (n + h)^{-1}\zeta_0^{n+h}(1 - \zeta_0)^{\frac{1}{2}}/\mathrm{B}(n + h, \tfrac{1}{2}) = I_{\zeta_0}(n + h, \tfrac{1}{2}) - I_{\zeta_0}(n + h + 1, \tfrac{1}{2}).$$

After some simplification we obtain

$$\frac{dP}{dw} = (1 - 4w)^{-\frac{1}{2}}(C_2/C_1)^{\frac{1}{2}n-1}(nC_1)^{-2} \sum_{h=0}^{\infty} \Gamma(\tfrac{1}{2}n + h)(\Gamma(\tfrac{1}{2}n)\Gamma(h + 1))^{-1}(1 - C_2/C_1)^h \cdot[\tfrac{1}{2}n(1 - 2nC_1) + h(1 - nC_1)]\eta_h. \tag{4.3.3}$$

The terms of the series (4.3.3) will be negative in the beginning but will finally become positive. Let the $(r + 1)$st term be the first positive term. Since η_h is a monotone decreasing function of h we have

$$\begin{aligned}\frac{dP}{dw} &< \eta_r(1 - 4w)^{-\frac{1}{2}}(C_2/C_1)^{\frac{1}{2}n-1}(nC_1)^{-2} \\ &\quad \cdot\sum_{h=0}^{\infty} \Gamma(\tfrac{1}{2}n + h)(\Gamma(\tfrac{1}{2}n)\Gamma(h + 1))^{-1}(1 - C_2/C_1)^h \\ &\quad \cdot[\tfrac{1}{2}n(1 - 2nC_1) + h(1 - nC_1)] \\ &= \eta_r(1 - 4w)^{-\frac{1}{2}}(C_2/C_1)^{\frac{1}{2}n-1}(2nC_1^2)^{-1} \\ &\quad \cdot[(1 - 2nC_1)(C_2/C_1)^{-\frac{1}{2}n} + (1 - C_2/C_1)(1 - nC_1)(C_2/C_1)^{-\frac{1}{2}n-1}] \\ &= 0.\end{aligned}$$

This proves the theorem except for the end point $w = 0$ of the interval $0 \leq w \leq \frac{1}{4}$, for which the series (4.2.1) and consequently (4.3.1) are not defined. To cover this point we need only to note that the cumulative distribution function (cdf) of the statistic $U = \chi_{1,1}^2((1/n)\chi_{2,n}^2 + C_2(\chi_{3,n}^2 - \chi_{2,n}^2))^{-1}$ is a continuous function of C_2 and that when $C_2 \to 0$, the cdf of U tends to the cdf of Student's t^2 for n degrees of freedom.

Having established the monotone nature of P with respect to w we are now in a position to assert that U could be regarded as Student's t^2 with degrees of freedom lying between n and $2n$.

4.4 *A p-variate generalization of U.* The reader will at once recognize the following technique for obtaining the generalization of U to the p-variate case to be similar to that of obtaining Hotelling's T from the ordinary Student's t. Consider

$$B_{12} = \alpha_1 b_{12} + \alpha_2 b_{13} + \alpha_3 b_{23} + \cdots + \alpha_{\frac{1}{2}p(p-1)} b_{p-1,p},$$

$$B_{21} = \alpha_1 b_{21} + \alpha_2 b_{31} + \alpha_3 b_{32} + \cdots + \alpha_{\frac{1}{2}p(p-1)} b_{p,p-1}.$$

Let A denote the sample covariance matrix of the $\frac{1}{2}p(p - 1)$ symmetric differences. Define row vectors

$$\alpha = (\alpha_1, \alpha_2, \cdots), \qquad b_1 = (b_{12}, b_{13}, \cdots), \qquad b_2 = (b_{21}, b_{31}, \cdots)$$

and let α', b_1', b_2' denote the corresponding column vectors. If we regard α's as constants, then

$$\mathfrak{U} = \frac{(B_{12} - B_{21})^2}{\text{Estimated var}(B_{12} - B_{21})} = \frac{[\alpha(b_1 - b_2)']^2}{\alpha A \alpha'}$$

is also distributed like U. If we determine α's so as to maximize $\mathfrak{U}$ we at once find that $\alpha \propto (b_1 - b_2)A^{-1}$ and we have

$$\mathfrak{U} = (b_1 - b_2)A^{-1}(b_1 - b_2)',$$

which reduces to U when $p = 2$. It can be shown after a very laborious simplification that $\mathfrak{U}$ is also invariant under all contragredient transformations. The distribution of $\mathfrak{U}$ is still under investigation.

4.5. *The power function of U and its unbiased character.* If we let

$$\delta = (\beta_{12} - \beta_{21})(c_{22}\sigma_{11} + c_{11}\sigma_{22} - 2c_{12}\sigma_{12})^{-\frac{1}{2}},$$

we shall presently see that except for the noncentral χ^2 in the numerator the non null distribution of U is similar to its null distribution. We shall first indicate the results that can at most be accomplished in the non null case by the contragredient transformation of y and x. Noting that under this type of transformation β and σ are transformed cogrediently we state:

LEMMA 1. *If $\beta_{12} \neq \beta_{21}$, there does not exist a nonsingular cogredient transformation f which will reduce β to a diagonal matrix and σ to an identity matrix.*

PROOF. Suppose there exists an f such that

$$f\sigma f' = I, \qquad f\beta f' = D,$$

where I is an identity and D a diagonal matrix. Therefore $\beta = f^{-1}Df'^{-1}$; $\beta' = f^{-1}Df'^{-1}$, yielding $\beta = \beta'$, which is contrary to the hypothesis.

LEMMA 2. *If $\beta_{12} \neq \beta_{21}$, there exists a nonsingular transformation f which reduces σ to $\sigma^* = \begin{Vmatrix} 1 & \rho \\ \rho & 1 \end{Vmatrix}$, β to another nonsymmetric matrix β^* and which leaves the standardised "distance" δ between the two alternatives invariant.*

PROOF. Such a transformation is given by $f = \begin{Vmatrix} \sigma_{11}^{-\frac{1}{2}} & 0 \\ 0 & \sigma_{22}^{-\frac{1}{2}} \end{Vmatrix}$. This completes the proof.

We may thus derive the non null distribution of U assuming $\sigma_{11} = \sigma_{22} = 1$. We shall presently see that the power function of U depends only on one nuisance parameter ρ.

To reduce the positive definite form Z in the denominator of U to a linear combination of two independently distributed χ^2's we proceed as follows:

(i) There exists an orthogonal transformation which will simultaneously yield $s_{11} = \Sigma_1^n z_{1\alpha}^2/n$; $s_{22} = \Sigma_1^n z_{2\alpha}^2/n$; $s_{12} = \Sigma_1^n z_{1\alpha} z_{2\alpha}/n$, where $z_{1\alpha}$ and $z_{2\alpha}$ follow a certain bivariate law.

(ii) The transformation

$$z_{1\alpha}^* = (1 - \rho^2)^{-\frac{1}{2}}(z_{1\alpha} - \rho z_{2\alpha})$$
$$z_{2\alpha}^* = z_{2\alpha}$$

further reduces Z to a quadratic form in normally and independently distributed variates.

(iii) A proper choice of θ in the orthogonal transformation

$$z_{1\alpha}' = z_{1\alpha}^* \cos\theta - z_{2\alpha}^* \sin\theta$$
$$z_{2\alpha}' = z_{1\alpha}^* \sin\theta + z_{2\alpha}^* \cos\theta$$

ensures the vanishing of sample covariance of $z_{1\alpha}'$ and $z_{2\alpha}'$ and we obtain $nz = q_1\Sigma_1^n z_{1\alpha}'^2 + q_2\Sigma_1^n z_{2\alpha}'^2$, where q_1 and q_2 depend upon ρ and the elements of the matrix c.

Finally we have $U = \chi_{1,1}'^2(\gamma_1\chi_{2,n}^2 + \gamma_2\chi_{3,n}^2)^{-1}$, where χ'^2 is a noncentral χ^2 and

$$\gamma_1 = (2n)^{-1}[1 + (1 - 4\,|\,c\,|(1 - \rho^2)(c_{11} + c_{22} - 2\rho c_{12})^{-2})^{\frac{1}{2}}],$$
$$\gamma_2 = (2n)^{-1}[1 - (1 - 4\,|\,c\,|(1 - \rho^2)(c_{11} + c_{22} - 2\rho c_{12})^{-2})^{\frac{1}{2}}].$$

We observe that if the covariance matrix is an identity matrix, the values of γ_1 and γ_2 check with the values of C_1 and C_2 (Section 4.3).

Following Hsu [9] we obtain the following forms for the non null frequency function and power function of U:

$$g(U) = e^{-\frac{1}{2}\delta^2}(\gamma_1/\gamma_2)^{\frac{1}{2}n} \sum_{r=0}^{\infty} (\tfrac{1}{2}\delta^2)^r \frac{\gamma_2^{r+\frac{1}{2}} U^{r-\frac{1}{2}}(1 + \gamma_2 U)^{-n-r-\frac{1}{2}}}{\Gamma(r + 1)\mathrm{B}(n, r + \frac{1}{2})} \cdot F\left(n + r + \tfrac{1}{2}, \tfrac{1}{2}n, n, \frac{1 - \gamma_2/\gamma_1}{1 + \gamma_2 U}\right) \tag{4.5.1}$$

and

$$\beta(\delta, \rho, n) = e^{-\frac{1}{2}\delta^2}(\gamma_2/\gamma_1)^{\frac{1}{2}n} \sum_{h=0}^{\infty}\sum_{r=0}^{\infty} (\tfrac{1}{2}\delta^2)^r \Gamma(\tfrac{1}{2}n + h)(1 - \gamma_2/\gamma_1)^h \cdot [\Gamma(\tfrac{1}{2}n)\Gamma(r + 1)\Gamma(h + 1)]^{-1} I_{a_0}(n + h, r + \tfrac{1}{2}) \tag{4.5.2}$$

where F denotes the hypergeometric function and $a_0 = (1 + \gamma_2 U_0)^{-1}$. Because of the fixed relation $\gamma_1 + \gamma_2 = 1/n$ either of the above two results could be expressed in terms of γ_1 or γ_2 and consequently ρ is the only nuisance parameter present in (4.5.1) and (4.5.2).

To show that U provides an unbiased test for the hypothesis $\beta_{12} = \beta_{21}$ we state the following theorem:

THEOREM IV. *For any n and fixed ρ the power function $\beta(\delta, \rho, n)$ is a monotone increasing function of the standardised "distance" δ beween the two alternatives.*

PROOF. Consider the double series

$$\sum_{r=0}^{\infty} \sum_{h=0}^{\infty} (\tfrac{1}{2}\delta^2)^r \Gamma(\tfrac{1}{2}n + h)(1 - \gamma_2/\gamma_1)^h [\Gamma(\tfrac{1}{2}n)\Gamma(r + 1)\Gamma(h + 1)]^{-1} I_{a_0}(n + h, r + \tfrac{1}{2})$$

which is dominated by

$$\sum_{r=0}^{\infty} (\gamma_1/\gamma_2)^{\frac{1}{2}n} (\tfrac{1}{2}\delta^2)^r / r!$$

This latter series has infinite radius of convergence and consequently we can differentiate (4.5.2) term by term. Setting $\frac{1}{2}\delta^2 \doteq \Delta^*$ and differentiating we obtain after simplification

$$\frac{\partial\beta(\delta, \rho, n)}{\partial\Delta^*} = (\gamma_2/\gamma_1)^{\frac{1}{2}n} e^{-\Delta^*} \sum_{h=0}^{\infty} \sum_{r=0}^{\infty} \Gamma(\tfrac{1}{2}n + h)(1 - \gamma_2/\gamma_1)^h \Delta^{*r}$$

$$\cdot (\Gamma(\tfrac{1}{2}n)\Gamma(h + 1)\Gamma(r + 1))^{-1} [I_{a_0}(n + h, r + \tfrac{3}{2}) - I_{a_0}(n + h, r + \tfrac{1}{2})].$$

Since $I_{a_0}(n + h, r + \frac{3}{2}) - I_{a_0}(n + h, r + \frac{1}{2}) > 0$, therefore $\partial\beta(\delta, \rho, n)/\partial\Delta^* > 0$. This proves the theorem and establishes the unbiased character of the test based on U.

REFERENCES

[1] H. HOTELLING, "Edgeworth's taxation paradox and the nature of demand and supply functions," *Jour. Polit. Economy*, Vol. 40 (1932), pp. 577–616.

[2] H. SCHULTZ, *The Theory and Measurement of Demand*, University of Chicago Press, 1938, ch. 18.

[3] M. BOCHER, *Introduction to Higher Algebra*, The Macmillan Company, New York, 1907.

[4] H. HOTELLING, "Relations between two sets of variates," *Biometrika*, Vol. 28 (1936), pp. 321–377.

[5] H. HOTELLING, "Spaces of statistics and their metrization," *Science*, Vol. 67 (1928), pp. 149–150.

[6] H. HOTELLING, "Commodity transformations and matrices," *Annals of Math. Stat.*, Vol. 10 (1939), p. 88.

[7] J. WISHART, "The generalized product-moment distribution in samples from a normal multivariate population," *Biometrika*, Vol. 20 (1928), pp. 32–52.

[8] S. S. WILKS, *Mathematical Statistics*, Princeton University Press, 1946, pp. 245–247.

[9] P. L. HSU, "Contribution to the theory of Student's t-test as applied to the problem of two samples," *Stat. Res. Memoirs*, Vol. 2 (1938), pp. 1–24.

[10] UTTAM CHAND, "Distributions related to the comparison of two means and two regression coefficients," *Annals of Math. Stat.*, Vol. 21 (1950), pp. 507–522.

[11] H. ROBBINS AND E. J. G. PITMAN, "Application of the method of mixtures to quadratic forms in normal variates," *Annals of Math. Stat.*, Vol. 20 (1949), pp. 552–560.

[12] TJALLING C. KOOPMANS, ed., *Statistical Inference in Dynamic Economic Models*, Cowles Commission Monograph 10, John Wiley and Sons, New York, 1950