ON THE POWER FUNCTION OF TESTS OF RANDOMNESS BASED ON RUNS UP AND DOWN

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1. Summary. It is shown that various statistics based on the number of runs up and down have an asymptotic multivariate normal distribution under a number of different alternatives to randomness. The concept of likelihood ratio statistics is extended to give a method for deciding what function of these runs should be used, and it is shown that the asymptotic power of these tests depends only on the covariance matrix, calculated under the hypothesis of randomness, and the expected values, calculated under the alternative hypothesis. A general method is given for calculating these expected values when the observations are independent, and these calculations are carried through for a constant shift in location from one observation to the next and for normal and rectangular populations.

2. Introduction. Let the vector random variable $X^{(n)} = X_1, \cdots, X_n$ have the joint cumulative distribution function $F^{(n)} = F^{(n)}(x_1, \cdots, x_n)$. Throughout this paper we will suppose that $F^{(n)}$ is continuous. Let Ω_n be the class of all continuous $F^{(n)}$, and let ω_n be the class of all $F^{(n)}$ of the form $F^{(n)} = \prod F^{(1)}(x_i)$, where $F^{(1)}$ is some continuous univariate distribution function. By the hypothesis of randomness, H_0, we mean the hypothesis that $F^{(n)}$, known to belong to Ω_n, actually belongs to ω_n. The statistical problem is to test H_0 on the basis of one observation $x^{(n)}$ on $X^{(n)}$.

Many methods of testing this hypothesis have been proposed. The most usual procedure has been for the statistician to devise some statistic whose distribution under the null hypothesis could be obtained without too much trouble. Then if extreme values of this statistic were observed, the hypothesis of randomness was rejected. Occasionally the appropriateness of the statistic would be considered. A common type of reasoning is that such and such a test classifies as random a set of numbers that are "obviously" nonrandom, or vice versa. Now suppose we replace the original observations by their ranks. Then under the hypothesis of randomness all sequences of ranks are equally likely and each is as "random" as the next. On the other hand, if we look long enough, we will find something very peculiar and nonrandom about any given sequence, and can prove that the probability of this peculiarity arising by chance is very small. The difficulty is that randomness is not a property of a sequence of numbers, but of the process that produced them, that is, of $F^{(n)}$. Hence what we really want is a test with a high probability of rejecting H_0 whenever $F^{(n)} \notin \omega_n$. Unfortunately no such test exists. In fact, given any critical region of size α, there exists $F^{(n)} \notin \omega_n$ for which the probability of the critical region is zero. Two ways may be found out of this dilemma. The more satisfying from a theoretical point of view is to restrict $F^{(n)}$ to a class of alternatives especially feared, and to choose a statistic with reasonably good power against these alternatives. The second method is to restrict ourselves arbitrarily to a definite class of statistics which has desirable properties such as convenience, and then to choose an optimum statistic from this class. Both approaches will be used in this paper. Whichever approach is chosen, it would be desirable to have a method of constructing a good test. We will exhibit one such method based on the second approach. However, in most cases no method of constructing a good test is known. It then becomes necessary to investigate the behavior of the power function of a number of previously devised tests, and to choose the one having the most desirable power function for the purpose in hand. In the present paper a start will be made in this direction for statistics based on runs up and down. These statistics have been independently discovered by a number of different authors and have been widely advocated for testing randomness.

The continuity of $F^{(n)}$ insures that, under H_0, Prob $\{X_i = X_j\} = 0$ for all $i \neq j$. This will also be true for $F^{(n)} \in \Omega_n$, for the type of distributions ordinarily considered. We will therefore assume that the observations $(x_1, \cdots, x_n)$ are distinct. Let B^* be the sequence of signs ($+$ or $-$) of the differences $(x_{i+1} - x_i)$ for $i = 1, \cdots, n - 1$. A sequence of p successive $+$ ($-$) signs not immediately preceded or followed by a $+$ ($-$) sign is called a run up (down) of length p. The term "runs up and down" (or u-runs) applies to both runs up and runs down. As an example, if the observations are (5 7 3 4 8 1), then $B^* = (+ - + + -)$, there are four u-runs: one run up of length one, one up of length two, and two down of length one.

Let s be the number of runs up, s_p the number of runs up of length p, and s'_p the number of runs up of length p or more in B^*. Let t, t_p, and t'_p be similarly defined for runs down, and let $r = s + t$, $r_p = s_p + t_p$, $r'_p = s'_p + t'_p$. Let k equal the total number of $+$ signs in B^*. The r's, s's, t's and k will be called u-run statistics. Levene and Wolfowitz [1] have given the exact covariance matrix and expected values of the r's, and Moore and Wallis [2] have given $E(k) = \frac{1}{2}(n - 1)$ and $\sigma^2(k) = (n + 1)/12$.

3. Asymptotic distributions. When H_0 is true, certain recurrence relations for the exact distribution of a single u-run statistic are known, and Gleissberg [3] has tabulated the exact value of Prob $(r - 1 > x)$ for $n \leq 25$ (Wallis and Moore [4] having given this for $n \leq 12$), but no usable exact distribution function is known or is likely to be found. Hence it is important to have asymptotic formulas. Wolfowitz [5] proved that under H_0 any fixed set of u-run statistics have a joint multivariate normal distribution in the limit. (If the set chosen are linearly dependent in the limit, their joint limit distribution will be degenerate.) We will indicate Wolfowitz's proof for the total runs r. There is no essential difficulty in generalizing to a set of u-run statistics.

Let the sequence of observations be broken up into subsequences

$$x_{(j-1)\alpha+1}, \quad x_{(j-1)\alpha+2}, \cdots, x_{j\alpha} \qquad (j = 1, \cdots, \beta), \tag{3.1}$$

where $\alpha = n^{2/3}$, $\beta = n^{1/3}$ approximately. Let $r^{(j)}$ be the number of runs in the jth subsequence. The partitioning of the original sequence breaks up some runs and forms some new ones, but at most two runs in each subsection are affected, so that

$$\left| \sum_{j=1}^{\beta} r^{(j)} - r \right| \leq 2\beta. \tag{3.2}$$

But under H_0 we have (1) $r^{(j)}$ and $r^{(i)}$ are independent for $i \neq j$; (2) $r^{(j)}$ has the same distribution for all j; and (3) $1/\alpha\sigma^2[r^{(j)}]$ and $1/\alpha^2\mu_4[r^{(j)}]$, where $\mu_4[r^{(j)}]$ is the fourth moment of $r^{(j)}$ about its mean, can be shown to approach fixed limits $\neq 0$ as $n \to \infty$. Hence the Lyapunov theorem applies and

$$\frac{\sum\{r^{(j)} - E[r^{(j)}]\}}{\sqrt{n}}$$

is asymptotically normally distributed with zero mean and finite variance. But

$$\frac{|\sum r^{(j)} - r|}{\sqrt{n}} \leq \frac{2\beta}{\sqrt{n}} = 2n^{-1/6} \to 0, \tag{3.3}$$

so r is likewise asymptotically normal.

Apparently it has not previously been noted that randomness of the sequence $\{X_1, \cdots, X_n\}$ is not necessary for the validity of this proof. We will consider a number of alternatives under which the limit distribution of u-run statistics is normal.

(a) We will say there is a linear trend if $F^{(n)} = \prod F^{(1)}(x_i - \theta_i)$, with $\theta_i = i\theta$. Then (1), (2) and (3) above will hold. Even if θ_i is only approximately equal to $i\theta$, we will still have asymptotic normality, although condition (2) will not hold.

(b) If the scale of the distribution changes by a constant factor from one observation to the next, that is,

$$F^{(n)} = \prod F_i^{(1)} \tag{3.4}$$

with

$$F_i^{(1)}(x) = F_{i+1}^{(1)}(cx) \qquad (c > 0) \tag{3.5}$$

we have normality in the limit. If the scale increases or decreases monotonically at less than this exponential rate, the limit distribution is the same as under H_0.

(c) We will say there is a cycle of period p if

$$F^{(n)} = \prod_{j=1}^{n/p} F^{(p)}(x_{(j-1)p+1}, x_{(j-1)p+2}, \cdots, x_{jp}). \tag{3.6}$$

A special case of this is

$$F^{(n)} = \prod_{i=1}^{n} F_i^{(1)}(x_i), \tag{3.7}$$

with

$$F_i^{(1)} = F_j^{(1)} \qquad (i \equiv j \bmod p). \tag{3.8}$$

Here again conditions (1), (2) and (3) hold approximately for large n and exactly if n/p is an integer.

(d) If the X_i satisfy a stable linear stochastic difference equation, for example,

$$X_{i+1} = \beta X_i + U_{i+1} \qquad (|\beta| < 1), \tag{3.9}$$

where the random variables U_i are independent and equidistributed, the methods used by S. Bernstein [6] to prove the Central Limit Theorem for Markov chains can be used to prove asymptotic normality.

(e) The unstable stochastic difference equation $X_{i+1} = X_i + U_{i+1}$ is of special interest since the exact distributions are known. They are the same as the distributions of runs of two kinds of elements drawn from a binomial population which were given by Mood [7].

(f) If the marginal distributions of the X_i are such that we would have asymptotic normality if the X_i were independent, the asymptotic normality will still hold under the weaker condition that $X_1, \cdots, X_i$ are independent of X_j, $X_{j+1}, \cdots, X_n$ for all i and j with $j - i$ greater than some positive constant.

It is clear that these special cases are not exhaustive, but they seem to cover the most interesting possibilities. If some other $F^{(n)}$ should prove of interest, it should be fairly easy to see whether the conditions for normality are fulfilled.

4. The likelihood ratio statistic. Let $p(\xi^{(n)} \mid F^{(n)})$ be the elementary probability of the sample point $\xi^{(n)}$ when $F^{(n)}$ is the true distribution. Let

$$p_\omega(\xi^{(n)}) = \sup_{F^{(n)} \epsilon \omega_n} p(\xi^{(n)} \mid F^{(n)})$$

and

$$p_\Omega(\xi^{(n)}) = \sup_{F^{(n)} \epsilon \Omega_n} p(\xi^{(n)} \mid F^{(n)}).$$

Then the likelihood ratio statistic of Neyman and Pearson [8] is

$$\lambda = \frac{p_\omega(\xi^{(n)})}{p_\Omega(\xi^{(n)})}. \tag{4.1}$$

In general this expression has no meaning in the nonparametric case. Wolfowitz [9] adapted it to the two-sample problem by considering only the sequence, B, of the ranks of the observations. For $\xi^{(n)}$ a point in the space of permutations of B, $p_\omega(\xi^{(n)}) \equiv 1/n!$, and λ is equivalent to $p_\Omega(\xi^{(n)})$. Wolfowitz was able to obtain an approximation to $p_\Omega(\xi^{(n)})$ and suggested its use as the test statistic. Unfortunately, under these conditions the randomness hypothesis H_0 leads to $p_\omega(\xi^{(n)}) = 1/n!$ and $p_\Omega(\xi^{(n)}) = 1$ for every $\xi^{(n)}$, so that λ is a constant and cannot be used. Now suppose $\xi^{(n)}$ is further restricted to the space of all possible sequences of signs of first difference, B^*. For any rank statistic, $p_\Omega(\xi^{(n)}) \equiv 1$; thus we always have $\lambda = p_\omega(\xi^{(n)})$. But now $p_\omega(\xi^{(n)})$ is no longer a constant, and we may take the critical region

$$W_n(B^*): p_\omega(\xi^{(n)}) \leq c. \tag{4.2}$$

If we give the sequence of runs up and down in order, with their lengths, we specify the sequence B^*. The next step is to give only the frequency distribution of runs up and down. The following step is to group together all long runs and restrict ourselves to the space of the statistics B^{**}: $s_1, s_2, \cdots, s_p, s'_{p+1}, t_1, t_2, \cdots, t_q$. But in the limit these have a joint multivariate normal distribution, so in the limit λ is equivalent to

$$Q(s_1, s_2, \cdots, s_p, s'_{p+1}, t_1, t_2, \cdots, t_q),$$

where for any set of random variables $x_1, \cdots, x_\nu$,

$$Q(x_1, \cdots, x_\nu) = \sum_{i,j=1}^{\nu} \sigma^{ij}[x_i - E(x_i)][x_j - E(x_j)] \tag{4.3}$$

with

$$\| \sigma^{ij} \| = \| \sigma_{ij} \|^{-1}.$$

For our case we use the covariance matrix under H_0, and the critical region is

$$W_n(B^{**}): Q \geq C. \tag{4.4}$$

Since $|t - s| \leq 1$, t'_{q+1} need not be included in B^{**}; if it were, $\| \sigma_{ij} \|$ would be singular in the limit.

Intuitive considerations similar to those that originally led Neyman and Pearson to the likelihood ratio statistic suggest that $W_n(B^*)$ is the "best" statistic depending only on the sequence B^*. It would then follow that $W_n(B^{**})$ is less efficient; in other words, information has been lost in ignoring the long runs and the order of appearance of the runs. Still further information will be lost if runs up and runs down of the same length are combined and the statistics B^{***}: $r_1, \cdots, r_p, r'_{p+1}$ are used. While it is not practicable to use the region $W_n(B^*)$, the region $W_n(B^{**})$ can be used. In a previous paper (Levene and Wolfowitz, [1]) the covariance matrix of the r's was given. Because of the desirability of using the region $W_n(B^{**})$ the covariance matrix of the s's and t's has now been computed and is given in the Appendix to this paper. Because of the weight of the formulas and the possibilities for error in substituting numerical values of p and q, the numerical values needed for tests based on s; on s_1, s'_2, t_1; and on s_1, s_2, s'_3, t_1, t_2 are given, as are a few additional values. These values have been checked by addition, using formulas of the type

$$\sigma^2(r) = \sigma^2(s_1 + s'_2 + t_1 + t'_2), \tag{4.5}$$

where the right-hand member is to be expanded as a sum of sixteen terms. The methods used in obtaining the covariances are similar to those used in Levene and Wolfowitz [1]. The covariances of k, the total number of plus signs in B^*, with s_p and s'_p are also given. It can be shown as follows that under H_0, $\sigma(s_p, k) = -\sigma(t_p, k)$, $\sigma(s'_p, k) = -\sigma(t'_p, k)$, and $\sigma(r_p, k) = \sigma(r'_p, k) = 0$. We have $\sigma(r_p, k) =$

$\sigma(s_p, k) + \sigma(t_p, k)$. But by symmetry, under H_0, $\sigma(t_p, k) = \sigma(s_p, k')$, where k' = total number of minus signs in B^*. Hence $\sigma(r_p, k) = \sigma(s_p, k) + \sigma(s_p, k') = \sigma(s_p, k + k') = \sigma(s_p, n - 1) = 0$, since $n - 1$ is a constant.

Although k is not independent of $r_1, \cdots, r_p, r'_{p+1}$ under H_0 it is uncorrelated with them, and since k and the r's have a joint normal distribution in the limit, it follows that $Q(k)$ and $Q(r_1, \cdots, r_p, r'_{p+1})$ are independently distributed in the limit as χ_1^2 and χ_{p+1}^2, respectively. Thus, for example, the λ statistic depending only on k and r is

$$\frac{(k - E(k))^2}{\sigma^2(k)} + \frac{(r - E(r))^2}{\sigma^2(r)} = \chi_2^2. \tag{4.6}$$

This statistic is very easy to compute and use. A rough idea of the type of departure from randomness may be obtained from the relative size of the two components, since it can be shown, for example, that the test based on k is more powerful for linear trends and less powerful for certain cyclical trends than is the test based on r.

5. The asymptotic power function. Under H_0 the exact distribution of u-run statistics is extremely cumbersome and impractical. For any alternative the exact distribution would be still more complicated, if, indeed, it could be obtained at all. Since we are thus constrained to use the asymptotic theory in any case, we may as well take advantage of this to introduce certain simplifications. Let F represent an infinite sequence $\{F^{(n)}\}$ such that, for $k < m$, $F^{(k)}(x_1, \cdots, x_k) = F^{(m)}(x_1, \cdots, x_k, \infty, \cdots, \infty)$. If u and v are any two u-run statistics, then for H_0 or for a number of important alternatives, for example, a linear trend, a cyclic alternative, or a stable stochastic difference equation, we have

$$\begin{aligned} E(u) &= na_1 + a_2, \\ \sigma^2(u) &= na_3 + a_4, \\ \sigma(u, v) &= na_5 + a_6, \end{aligned} \tag{5.1}$$

where the a_i are constants depending on F. Let

$$\begin{aligned} E'(u) &= \lim_{n\to\infty} \frac{1}{n} E(u) = a_1, \\ \sigma'^2(u) &= \lim_{n\to\infty} \frac{1}{n} \sigma^2(u) = a_3, \\ \sigma'(u, v) &= \lim_{n\to\infty} \frac{1}{n} \sigma(u, v) = a_5. \end{aligned} \tag{5.2}$$

Then for large n, $E(u) \sim nE'(u)$, $\sigma(u) \sim \sqrt{n}\, \sigma'(u)$, and $\sigma(u, v) \sim n\, \sigma'(u, v)$, where the symbol $\sim$ means "is asymptotically equal to."

Furthermore, if F is such that the u-run statistics are normally distributed in the limit (see Section 3) the limits $E'(u)$ and $\sigma'(u, v)$ will usually exist; and the asymptotic distribution of the u-run statistics is completely determined as soon as we have the $E'(u)$'s and the $\sigma'(u, v)$ matrix. For the remainder of this paper we consider only F of this type.

Now suppose we consider the hypothesis H_0 and a definite alternative hypothesis $H_1: F = F_1$. Then let $E'_0(u)$ and $\sigma'_0(u, v)$ be the expected values and covariances under H_0, and $E'_1(u)$ and $\sigma'_1(u, v)$ be the corresponding values under H_1. We can then compute the power of the test.

For concreteness suppose we have a linear trend:

$$F^{(n)} = \prod F^{(1)}(x_i - i\theta). \tag{5.3}$$

We first consider the test based on total runs r. Then $E_0(r) > E_1(r)$. Suppose we use the lower tail of r as critical region. For size α the test will have power at least $1 - \beta$ if

$$E_0(r) - \lambda_\alpha \sigma_0(r) \geq E_1(r) + \lambda_\beta \sigma_1(r), \tag{5.4}$$

where

$$\frac{1}{\sqrt{2\pi}} \int_{\lambda_\alpha}^{\infty} e^{-\frac{1}{2}t^2}\, dt = \alpha, \tag{5.5}$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{\lambda_\beta}^{\infty} e^{-\frac{1}{2}t^2}\, dt = \beta. \tag{5.6}$$

(5.4) may be written

$$E_0(r) - E_1(r) > \lambda_\alpha \sigma_0(r) + \lambda_\beta \sigma_1(r),$$

or, using the approximate values,

$$\sqrt{n}[E'_0(r) - E'_1(r)] > [\lambda_\alpha \sigma'_0(r) + \lambda_\beta \sigma'_1(r)]. \tag{5.7}$$

Since the terms in brackets depend on θ but not on n, the inequality will hold for large enough n for any fixed $\theta > 0$, and any λ_α and λ_β. In order to have a situation of statistical interest, it is necessary to let $\theta \to 0$ as $n \to \infty$ in such a way that $\sqrt{n}[E'_0(r) - E'_1(r)]$ remains constant. Under these conditions $\sigma'_0(r) \to \sigma'_1(r)$, and consequently we may write (5.7) as

$$\sqrt{n}[E'_0(r) - E'_1(r)] > (\lambda_\alpha + \lambda_\beta)\sigma'_0(r). \tag{5.8}$$

Thus for large n the power depends only on

$$\Delta(r) = \frac{E'_0(r) - E'_1(r)}{\sigma'_0(r)}.$$

Similarly, if a two-tail test were used, we should find that the power of the test depended only on

$$\Delta^2(r) = \frac{[E'_0(r) - E'_1(r)]^2}{[\sigma'_0(r)]^2}. \tag{5.9}$$

We shall call $\Delta^2(r)$, which is a monotonic function of the asymptotic power of the test, the *asymptotic power index*.

Now let $u_1, \cdots, u_p$ be a set of linearly independent u-run statistics with covariance matrix $\| \sigma_{ij} \|$. Let $\| \sigma^{ij} \| = \| \sigma_{ij} \|^{-1}$. Then the critical region is

$$Q = \sum_{ij} \sigma_0^{ij}[u_i - E_0(u_i)][u_j - E_0(u_j)] \geq C. \tag{5.10}$$

Again

$$Q \sim n \sum_{ij} (\sigma^{ij})'_0 \left[\frac{u_i}{n} - E'_0(u_i)\right]\left[\frac{u_j}{n} - E'_0(u_j)\right]. \tag{5.11}$$

In determining the distribution of Q under H_1 when n is large and for the cases of interest the matrix $\| (\sigma_{ij})'_1 \|$ can be replaced by the null covariance matrix $\| (\sigma_{ij})'_0 \|$. We then have Q distributed under H_0 as χ_p^2 and under H_1 as a sum of noncentral squares $\chi_p'^2$ with parameter

$$n\Delta^2 = n \sum_{ij} (\sigma^{ij})'_0[E'_0(u_i) - E'_1(u_i)][E'_0(u_j) - E'_1(u_j)]. \tag{5.12}$$

(See Tang [11] for the χ'^2 distribution. Tang uses the parameters $\lambda = n\,\Delta^2/2$ and $\varphi = \Delta\sqrt{n/(p+1)}$.) We will call $\Delta^2 = \Delta^2(u_1, \cdots, u_p)$ the *multivariate asymptotic power index*. It is easy to see that $\Delta^2(r)$ defined above is a special case of this.

The above reasoning will hold for a number of other types of alternative as well as it does for the linear trend. We thus see that an investigation of the asymptotic power of u-run statistics in these cases requires only the finding of the $E'_1(u)$, and we will consider ways of doing this in the next three sections. This situation is very fortunate for three reasons. First, it is much more labor to find the covariances than the expected values. Second, when the $E'_1(u_i)$ differ from the $E'_0(u_i)$ and the $(\sigma_{ij})'_1$ differ from the $(\sigma_{ij})'_0$, the asymptotic distribution of Q becomes the distribution of an arbitrary quadratic form in normal variates, and is extremely difficult to handle. Third, we can now show that Q, recommended on intuitive grounds in the last section as the likelihood ratio statistic, has optimum properties. In the space of the u-run statistics $(u_1, \cdots, u_p)$, say, we are essentially testing the simple hypothesis that the variables $(u_1, \cdots, u_p)$, normally distributed with the covariance matrix $\| \sigma_{ij} \|$, have means $(\mu_1^0, \cdots, \mu_p^0)$, against the alternative that the means are $(\mu_1, \cdots, \mu_p)$, with $\max | \mu_i^0 - \mu_i | = O(n^{-\frac{1}{2}})$. For this hypothesis Wald [11] has shown that $W_n(\alpha): Q > C$ has optimum properties.

6. Expected values in general. So far we have only assumed that $F^{(n)}$ was continuous. To obtain the expected values, we assume that the probability density function, $f^{(n)}(x_1, \cdots, x_n)$, exists. Then

$$E(s) \sim \sum_{i=1}^{n-2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdot \left(\int_{-\infty}^{x_{i+2}} \left[\int_{x_{i+1}}^{\infty} f^{(n)}(x_1, \cdots, x_n)\, dx_i\right] dx_{i+1}\right) \prod_{j \neq i, i+1} dx_j, \tag{6.1}$$

where in each term of the sum the integration is to be from $-\infty$ to $+\infty$ for every x_j ($j = 1, \cdots, n$) except x_i and x_{i+1}. For the remainder of this paper we will further assume that $f^{(n)} = \prod f_i^{(1)}(x_i)$, and will omit the superscript indicating dimensionality, writing the joint density function as

$$f(x_1, \cdots, x_n) = \prod_{i=1}^{n} f_i(x_i). \tag{6.2}$$

Equation (6.1) then becomes

$$E(s) \sim \sum_{i=1}^{n-2} \int_{-\infty}^{\infty} f_{i+2}(x_{i+2}) \int_{-\infty}^{x_{i+2}} f_{i+1}(x_{i+1}) \int_{x_{i+1}}^{\infty} f_i(x_i)\, dx_i\, dx_{i+1}\, dx_{i+2}. \tag{6.3}$$

Similarly we have

$$E(k) \sim \sum_{i=1}^{n-1} \int_{-\infty}^{\infty} f_{i+1}(x_{i+1}) \int_{-\infty}^{x_{i+1}} f_i(x_i)\, dx_i\, dx_{i+1} \tag{6.4}$$

and

$$E(s'_p) = \sum_{i=1}^{n-p-1} \int_{-\infty}^{\infty} f_{i+p+1}(x_{i+p+1}) \int_{-\infty}^{x_{i+p+1}} f_{i+p}(x_{i+p}) \cdots \int_{-\infty}^{x_{i+2}} f_{i+1}(x_{i+1}) \int_{x_{i+1}}^{\infty} f_i(x_i) \times dx_i \cdots dx_{i+p+1}. \tag{6.5}$$

For the linear trend, $f_i(x_i) = f(x_i - i\theta)$, all terms in the sum (6.3) are equal and we have

$$E'(s) = \int_{-\infty}^{\infty} f(x_3 - 3\theta) \int_{-\infty}^{x_3} f(x_2 - 2\theta) \int_{x_2}^{\infty} f(x_1 - \theta)\, dx_1\, dx_2\, dx_3, \tag{6.6}$$

while for a cyclic alternative of period T we have

$$E'(s) = \frac{1}{T} \sum_{i=1}^{T} \int_{-\infty}^{\infty} f_{i+2}(x_{i+2}) \int_{-\infty}^{x_{i+2}} f_{i+1}(x_{i+1}) \int_{x_{i+1}}^{\infty} f_i(x_i)\, dx_i\, dx_{i+1}\, dx_{i+2}, \tag{6.7}$$

where $E'(s)$ is defined by (5.2) as $\lim \frac{1}{n} E(s)$. These simplifications hold for every u-run statistic.

We will deal only with $E'(s'_p)$ and $E'(t'_p)$, since $s_p = s'_p - s'_{p+1}$, etc. We also note that $E'(s) = E'(t) = \frac{1}{2} E'(r)$, since $| s - t | \leq 1$, and that the distribution of t'_p for a sequence $\{X_i\}$ is the distribution of s'_p for $\{-X_i\}$.

Even in the simplest possible case, a linear trend with θ given, the value of $E'(s)$, etc., depends on the underlying distribution $f(x)$, which must be specified before we can integrate. We will obtain expected values for $f(x)$ rectangular, and for the most important case, $f(x)$ normal.

7. Expected values for rectangular populations. Let

$$f_i(x_i) = \begin{cases} 1, & i\theta \leq x_i \leq i\theta + 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Then the integrations indicated in Section 6 can be performed explicitly. The only complication is the usual one associated with rectangular distributions: the integral must be broken up into a number of parts, each with a different integrand; and the enumeration of the possibilities rapidly becomes tedious as the number of integrations increases. We list a number of the simpler results.

$$(7.1)\quad E'(k) = \begin{cases} \frac{1}{2}[1 + \theta(2 - |\theta|)], & -1 \leq \theta \leq 1, \\ 0, & \theta < -1, \\ 1, & \theta > 1, \end{cases}$$

$$(7.2)\quad E'(s) = \begin{cases} \frac{1}{6}(2 - 9\theta^2 + 8|\theta|^3), & |\theta| \leq \frac{1}{2}, \\ (1 - |\theta|)^2, & \frac{1}{2} \leq |\theta| \leq 1, \\ 0, & |\theta| \geq 1, \end{cases}$$

$$(7.3)\quad E'(s_2') = \tfrac{1}{24}(3 + 12\theta - 18\theta^2 - 52\theta^3 + 75\theta^4), \qquad 0 \leq \theta \leq \tfrac{1}{3},$$

$$(7.4)\quad E'(t_2') = \tfrac{1}{24}(3 - 12\theta - 6\theta^2 + 76\theta^3 - 81\theta^4), \qquad 0 \leq \theta \leq \tfrac{1}{3},$$

$$(7.5)\quad E'(r_2') = \tfrac{1}{4}(1 - 4\theta^2 + 4\theta^3 - \theta^4), \qquad 0 \leq \theta \leq \tfrac{1}{3}.$$

It will be noted that for θ close to zero, which is the most interesting case, the test based on s_2' (or t_2') is much more powerful than the test based on r_2', since the asymptotic power indexes (see 5.9) are of order θ^2 and θ^4 respectively.

For one special case a simple general formula is possible, namely

$$E'(t_p') = \begin{cases} \dfrac{1}{(p+1)!}(1 - p\theta)^{p+1} - \dfrac{1}{(p+2)!}[1 - (p+1)\theta]^{p+2}, & 0 \leq \theta \leq \dfrac{1}{p+1}, \\ \dfrac{1}{(p+1)!}(1 - p\theta)^{p+1}, & \dfrac{1}{p+1} \leq \theta \leq \dfrac{1}{p}, \\ 0, & \dfrac{1}{p} \leq \theta. \end{cases}$$

8. Expected value of k and s for normal populations. Let

$$(8.1)\qquad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

and

$$(8.2)\qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt.$$

We will consider in this section the alternative of a normal population with change of position, that is,

$$(8.3)\qquad f_i(x_i) = \phi(x_i - \mu_i)$$

for some set of parameters $\mu_i (i = 1, \cdots, n)$. We can suppose without loss of generality that $\mu_1 = 0$. It will be enough to show how the first term of the sums in (6.3) and (6.4) can be evaluated. Now

$$\int_{-\infty}^{\infty} \phi(x_2 - \mu_2) \int_{-\infty}^{x_2} \phi(x_1)\, dx_1 \ dx_2 \tag{8.4}$$

is the integral of the circular normal distribution with center at $(0, \mu_2)$ over the half plane above and to the left of the line $x_1 = x_2$. Evidently the distance from $(0, \mu_2)$ to this line is $\mu_2/\sqrt{2}$ and the integral (8.4) is equal to $\Phi(\mu_2/\sqrt{2})$. Thus

$$E(k) \sim \Sigma\Phi[(\mu_{i+1} - \mu_i)/\sqrt{2}], \tag{8.5}$$

and for the linear trend, $\mu_i = (i - 1)\theta$, we have

$$E'(k) = \Phi(\theta/\sqrt{2}). \tag{8.6}$$

Here, essentially, we rotated axes so that one variate was independent of the other and then integrated it out. This elimination of one variable can be done in the general case of the integration in (6.5). In particular, the evaluation of $E'(s)$ reduces to evaluating the circular normal distribution over a region bounded by two half lines meeting in an obtuse angle. By a further linear transformation we obtain the relation

$$\int_{-\infty}^{\infty} \phi(x_3 - \mu_3) \left(\int_{-\infty}^{x_3} \phi(x_2 - \mu_2) \left[\int_{x_2}^{\infty} \phi(x_1)\, dx_1 \right] dx_2 \right) dx_3 = K \int_a^{\infty} \int_b^{\infty} e^{-[1/2(1-\rho^2)](y_1^2 - 2\rho y_1 y_2 + y_2^2)}\, dy_1\, dy_2, \tag{8.7}$$

where $\rho = \frac{1}{2}$, $a = \mu_3\sqrt{\frac{1}{2}} - \mu_2\sqrt{\frac{1}{2}}$, and $b = -\mu_2\sqrt{\frac{1}{2}}$. The right member of (8.7) is given in Table VIII, Vol. 2 of Pearson's Tables [12].

For a linear trend, $\mu_i = (i - 1)\theta$, $E'(s)$ is given by the right member of (8.7), with $a = \theta\sqrt{\frac{1}{2}}$, $b = -\theta\sqrt{\frac{1}{2}}$.

Table 1 gives values of $1 - E'(k)$, $E'(s)$ and $\sigma'^2(k)$ (see Section 10) for a linear trend with various values of θ. Pearson's table goes only to $\theta\sqrt{\frac{1}{2}} = 2.6$, $(\theta = 3.676955)$; however, it will be noted that for $\theta > 2.8$, $E'(s) = 1 - E'(k)$ correct to five decimal places, and hence we can obtain $E'(s)$ for $\theta > 2.8$ by computing $1 - E(k)$. The reason for this is that as $\theta \to \infty$ the number of minus signs becomes small, and nearly every run down is of length one. For other values of θ, $E'(k)$ can be obtained from a table of the normal integral, while $E'(s)$ and $\sigma'^2(k)$ can be obtained by interpolation in Table 1, using four-point formulas for four decimal places or six-point formulas for full accuracy.

In Fig. 1, $E'(s)$ is plotted against θ, the full line for $f_i(x_i)$ normal and the broken line for $f_i(x_i)$ rectangular. In order to make these comparable, the rectangular distribution has been taken with unit variance (i.e., range $= \sqrt{12}$). It will be noted that the graphs are surprisingly close, suggesting that $E'(s)$ is not very sensitive to changes in the form of $f_i(x_i)$ for fixed mean and variance.

9. Expected value of s'_p for normal populations. For simplicity we will confine our attention to a normal population with linear trend

$$f_i(x_i) = \phi(x_i - \mu_i) \qquad (\mu_i = i\theta). \tag{9.1}$$

At the end of this section we will extend the method to the general case.

TABLE 1

Limiting values for a normal population with unit variance and linear trend $\mu_i = i\theta$

$\theta\sqrt{\frac{1}{2}}$	θ	$E'(s)$	$1 - E'(k)$	$\sigma'^2(k)$	$\sigma'(k)$
0	.000000	.333333	.50000	.08333	.289
.1	.141421	.330590	.46017	.08405	.290
.2	.282843	.322524	.42074	.08611	.293
.3	.424264	.309601	.38209	.08910	.298
.4	.565685	.292542	.34458	.09244	.304
.5	.707107	.272240	.30854	.09554	.309
.6	.848528	.249673	.27425	.09777	.313
.7	.989949	.225818	.24196	.09862	.314
.8	1.131371	.201577	.21186	.09779	.313
.9	1.272792	.177722	.18406	.09510	.308
1.0	1.414214	.154873	.15866	.09072	.301
1.1	1.555635	.133483	.13567	.08484	.291
1.2	1.697056	.113851	.11507	.07779	.279
1.3	1.838478	.096143	.09680	.07000	.265
1.4	1.979899	.080415	.08076	.06189	.249
1.5	2.121320	.066635	.06681	.05378	.232
1.6	2.262742	.054716	.05480	.04597	.214
1.7	2.404163	.044526	.04457	.03869	.197
1.8	2.545584	.035913	.03593	.03209	.179
1.9	2.687006	.028708	.02872	.02628	.162
2.0	2.828427	.022747	.02275	.02120	.146
2.1	2.969848	.017863	.01786	.01689	.130
2.2	3.111270	.013903	.01390	.01332	.115
2.3	3.252691	.010724	.01072	.01038	.102
2.4	3.394113	.008197	.00820	.00800	.089
2.5	3.535534	.006210	.00621	.00609	.079
2.6	3.676955	.004661	.00466	.00463	.068
2.7	3.818377		.00347	.00344	.059
2.8	3.959798		.00256	.00253	.050
2.9	4.101219		.00187	.00187	.043
3.0	4.242641		.00135	.00135	.037
3.2	4.525483		.00069	.00069	.026
3.4	4.808326		.00034	.00034	.018
3.6	5.091169		.00016	.00016	.013
3.8	5.374012		.00007	.00007	.008
4.0	5.656854		.00003	.00003	.005

We have

$$E'(s'_p) = \int_{-\infty}^{\infty} \phi[x_{p+2} - (p+1)\theta] \cdot \int_{-\infty}^{x_{p+2}} \phi(x_{p+1} - p\theta) \cdots \int_{-\infty}^{x_3} \phi(x_2 - \theta) \int_{x_2}^{\infty} \phi(x_1)\, dx_1 \cdots dx_{p+2}. \tag{9.2}$$

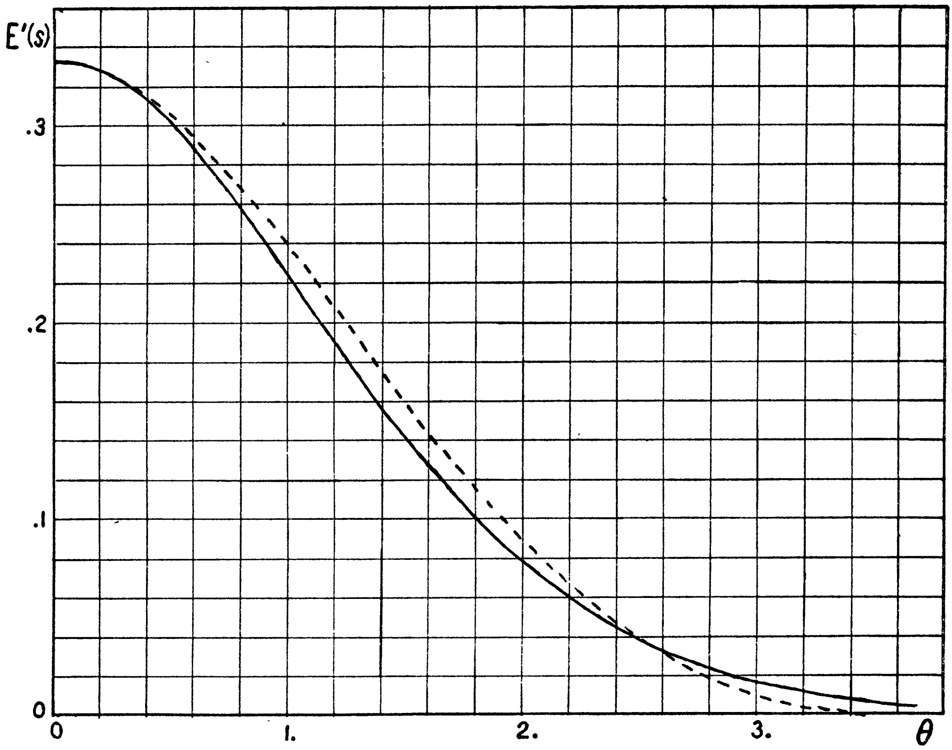

FIG. 1. Value of $E'(s)$ for linear trend: X_i independent with unit variances and mean $i\theta$. Solid line denotes normal population; broken line denotes rectangular population.

This could be reduced to a p-tuple integral geometrically as was done for $E'(s)$ in Section 8, but it is easier to use a method due to Kendall [13]. Let

$$y_i = \frac{x_{i+1} - x_i}{\sqrt{2}} \qquad (i = 1, \cdots, n-1). \tag{9.3}$$

Then

$$E'(s'_p) = K \int_{-\infty}^{(\mu_1-\mu_2)/\sqrt{2}} \int_{(\mu_2-\mu_3)/\sqrt{2}}^{\infty} \cdots \int_{(\mu_{p+1}-\mu_{p+2})/\sqrt{2}}^{\infty} e^{-\frac{1}{2}\Sigma\sigma^{ij}y_iy_j}\, dy_{p+1} \cdots dy_1, \tag{9.4}$$

where

$$(9.5)\qquad \| \sigma^{ij} \|^{-1} = \| \sigma_{ij} \| = \left\| \begin{matrix} 1 & -\frac{1}{2} & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & 0 \\ \cdot & \cdot & & & & & & \\ \cdot & \cdot & & & & & & \\ \cdot & \cdot & & & & & & \\ 0 & \cdot & & & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & -\frac{1}{2} & 1 \end{matrix} \right\|,$$

and for a linear trend $\mu_i - \mu_{i+1} = -\theta$ for all $i < n$: Kendall [13] was not interested in power function considerations, but was investigating runs for a different purpose. He did not consider a linear trend, but the case where the X_i satisfy the stochastic difference equation

$$(9.6)\qquad X_{i+2} + aX_{i+1} + bX_i = U_{i+2},$$

where the U are independently normally distributed with zero mean and unit variances. In his case all the $\mu_i = 0$, but the matrix $\| \sigma_{ij} \|$ has no zero terms. Kendall gave $1/E'(s)$ for certain a and b, and suggested evaluating the general integral (9.4) by the generalized tetrachoric series expansion given in Kendall [14]. For a general multivariate normal distribution the evaluation of this series is extremely laborious. For the linear trend, the many zero terms in the covariance matrix reduce the number of terms in the expansion, but the labor is still very great, and increases geometrically with p.

An alternative way of evaluating the integral in (9.4) would be by numerical quadrature. This would involve computing and adding N terms, with N of the order of α^{p+1} and α between 10 and 30 if reasonable accuracy were to be obtained. This would be very laborious. It is much easier to work with the integral in (9.2) and to evaluate it by repeated numerical quadrature. This will involve only $(p + 2)\alpha$ operations, and is the method we will actually use.

There are many methods of numerical quadrature, from simple ones such as the trapezoidal formula and Simpson's formula to relatively complicated ones involving many ordinates with different weights, and even ordinates spaced at irrational intervals. Any of these methods will give any desired degree of accuracy if the function to be integrated is well behaved and ordinates are taken sufficiently close together, but the methods differ in the number of ordinates required for specified accuracy. In a specific situation the formula requiring the least amount of labor should be used.

For our problem, the easiest method is the most elementary one, namely, the tangent formula. For α, ξ integers, $a = \frac{1}{2}(2\alpha h - 1)$, $x = \frac{1}{2}(2\xi h + 1)$, the tangent formula with remainder is

$$\int_a^x f(t)\,dt = h\sum_{j=\alpha}^{\xi} f(jh) + \sum_{j=\alpha}^{\xi} \frac{h^3}{24} f''(\zeta_j), \tag{9.7}$$

where $\frac{1}{2}(2jh - 1) \leq \zeta_j \leq \frac{1}{2}(2jh + 1)$ (see Steffensen [15] p. 159). Thus if $f''(x)$ exists and is continuous on the interval of integration, the error in using the tangent formula is of order h^2, where h is the distance between ordinates. The advantage of the tangent formula for our purposes is that it gives the indefinite integral with no extra labor; for example, if we start with values of $f(x)$ at $x = 0, 1, 2, \cdots$, we obtain approximations to $\int_{-\frac{1}{2}}^{x} f(t)\,dt$ at $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$. These values are then used for the second integration, and so on.

To illustrate the method used, consider the following expressions, where the variables of summation vary in steps of h:

$$G_1(y) = h\sum_{z=y-h/2}^{z\doteq 6} \phi(z-\theta)\left[\int_{-\infty}^{z} \phi(t)\,dt\right] \doteq \int_y^{\infty} \phi(z-\theta)\left[\int_{-\infty}^{z} \phi(t)\,dt\right] dz, \tag{9.8}$$

where the symbol $\doteq$ means "approximately equal to", and

$$\begin{aligned} G_2(x) &= h\sum_{y\doteq -6}^{x-h/2} \phi(y-2\theta)[G_1(y)] \\ &\doteq \int_{-\infty}^{x} \phi(y-2\theta)\left(\int_y^{\infty} \phi(z-\theta)\left[\int_{-\infty}^{t} \phi(t)\,dt\right] dz\right) dy. \end{aligned} \tag{9.9}$$

Then $G_1(-\infty) \doteq E'(k)$, and $G_2(\infty) \doteq E'(t)$.

We will see later in Table 2 that these two formulas give values close to the exact ones even when fairly few points are used. However, for the longer runs the exact values are not available for comparison, so something needs to be said about the errors in this method. We confine our remarks here to the normal case.

The first approximation made is the use of finite sums to represent infinite integrals. We have

$$\int_{-\infty}^{\infty} \phi(x-\mu)\,dx - \int_{-5+\mu}^{5+\mu} \phi(x-\mu)\,dx = .00000 \tag{9.10}$$

correct to five decimals. Since all our integrands are of the form $\phi(x - \mu)\cdot\psi(x)$ with $0 \leq \psi(x) \leq 1$, the error committed by using a finite range is always less than .00001, and the only question is whether our finite sums are sufficiently close to the corresponding finite integrals. We will now consider this question.

There is one source of error in $G_1(x)$, due to summing instead of integrating. On the other hand, $G_2(x)$ has two sources of error: first, we sum instead of integrating; second, the ordinates are themselves in error, because of errors in $G_1(x)$. It thus seems at first sight that the errors accumulate and that only a few iterations can be performed safely. Fortunately this is not so. The author has shown in his dissertation [16] that the error after m numerical integrations due to the accumulated error in the ordinates used for the last summation is less than the error due to replacing integration by summation at the last step; that is, no great improvement in accuracy would result if the approximate ordinates used at the last step were replaced by the corresponding exact values obtained by integration. Since this is so, it is only necessary to consider the error caused by a single numerical integration.

Let $G_j(x)$ stand for the result of the jth summation, let $g_j(x) = \phi(x - j\theta)G_{j-1}(x)$, and let $\epsilon_j(x)$ be the error introduced at the jth summation. If we knew $g_j''(x)$, we could obtain a bound on $\epsilon_j(x)$ from (9.7). Since the $g_j''(x)$ are no easier to obtain than the expected values we are looking for, we use the approximation

$$g_j''(x) = \frac{1}{h^2}\Delta^2 g_j(x), \tag{9.11}$$

where $\Delta f(x) = f(x + h) - f(x)$, and $\Delta^i f(x) = \Delta[\Delta^{i-1}f(x)]$. For a general analytic $f(x)$, the fact that $f''(x)$ was small on a tabulated set of points would not prevent it from being uncomfortably large at some intermediate point. However, we know that $g_j(x)$ represents a multiple normal integral and that neither the function nor its derivatives have any sudden changes. Accordingly, if h is so small that $g_j''(x)$ changes smoothly, we can be sure that the maximum second difference is close to the maximum of $g_j''(x)$, and that it is safe to write

$$\epsilon_j(x) \doteq \sum_{i=\alpha}^{\xi} \frac{h}{24}\Delta^2 g_j(ih). \tag{9.12}$$

This suggests the use of Gauss's first summation formula (Steffensen [15], p. 104),

$$\int_a^x f(t)\,dt = h\left[\sum_{j=\alpha}^{\xi} f(jh) + \sum_{\nu=1}^{r-1} K_{2\nu}[\delta^{2\nu-1}f(x)]_{a-h/2}^{x+h/2}\right] + h^{2r}K_{2r}\sum_{j=\alpha}^{\xi} f^{(2r)}(\zeta_j), \tag{9.13}$$

where

$$K_2 = \frac{1}{24}, \qquad K_4 = -\frac{17}{5760}, \qquad K_6 = \frac{367}{967680}, \qquad K_8 = -\frac{27859}{464486400}, \cdots$$

and $\delta^{2\nu-1}f(x)$ is the $(2\nu - 1)$th central difference of $f(x)$. By the same argument as above, the remainder term can be approximated by the first correction term omitted, provided the differences of the requisite order are changing smoothly. For evaluating $E'(s_p')$ with $\theta = \frac{1}{2}$, $h = 1$ is too large for smooth differences, and successive orders of differences become large. However, with $h = .2$ the differences change smoothly, and successive orders of differences rapidly become small. Consider for example $G_1(.5)$ for $\theta = \frac{1}{2}$. The uncorrected sum is .296610 and the successive correction terms from 9.13 are −.000,233, −.000,002, −.000,000,1. The value $x = .5$ was chosen because the second correction term here assumes its extreme value. Evidently all correction terms except the first can be ignored, and the error of $G_1(.5)$ is only 0.1%. Similarly, the maximum error of $G_2(x)$ is 0.1% at $x = 2$, while the error of $G_2(\infty)$ is only 0.03%.

Table 2 gives values of $E'(k)$ and $E'(s_p')$ for $\theta = \frac{1}{2}$ and $-\frac{1}{2}$ obtained by setting h equal to 1, .5, and .2 and also some values obtained by using the first correction term of (9.13). For $h = .2$ and $\theta = \frac{1}{2}$ the corrected values are accurate to five decimal places and the uncorrected values to four. Furthermore, the error

TABLE 2

Limiting expected values obtained by various methods for a normal population with unit variance and linear trend $\mu_i = i\theta$

Method	θ	$E'(k)$	$E'(s_1')$	$E'(s_2')$	$E'(s_3')$	$E'(s_4')$	$E'(s_5')$	$E'(s_6')$	$E'(s_7')$
Summing, $h = 1$	$\frac{1}{2}$	.63816	.30333	.17759	.08893	.04101	.01785	.00749	.00306
Summing, $h = .5$	$\frac{1}{2}$	.63817	.30149						
Summing, $h = .2$	$\frac{1}{2}$	.63816	.30102	.17525	.08812	.04117	.01848		
Summing, $h = .2$ (with correction term)	$\frac{1}{2}$	.63816	.30092	.17516	.08809	.04118	.01850		
Exact	$\frac{1}{2}$	.63816	.300926						
Exact	0	.500000	.333333	.125000	.033333	.006694	.001190	.000174	.000022
Exact	$-\frac{1}{2}$	.36184	.300926						
Summing, $h = .2$	$-\frac{1}{2}$	.36184	.30091	.05561	.00488	.00021	.000004		
Summing, $h = .5$	$-\frac{1}{2}$	.36184	.30080	.05488					
Summing, $h = 1$	$-\frac{1}{2}$	.36184	.30045	.05203	.00371	.00009	.00000		

actually decreases with repeated iteration (i.e., large p), although the percentage error increases. The uncorrected values for $h = .2$ and $\theta = -\frac{1}{2}$ are probably slightly less accurate, since the second derivatives are somewhat larger. Nevertheless, for the purpose of investigating the power functions it appears that uncorrected summation will give ample accuracy with $h = .2$. It should be noted that even with $h = 1$, where the errors in the $G_i(x)$ are large for intermediate x, the final values are surprisingly good.

The method of repeated summation is of very general applicability. It can be used freely when the X_i are independently normally distributed with variances close to 1 and $| \mu_i - \mu_{i-1} | \leq \frac{1}{2}$. For more extreme variation it may be necessary to use the correction term or take $h < .2$. It seems to be easier to take more ordinates than to compute the differences and apply the correction term, but the latter course should be taken occasionally to obtain an idea of the degree of accuracy attained. The method may also be used when the X_i are independent but not normal; however, in such a case the error would have to be investigated in the same way as we have done it here.

10. Variance. We have seen in Section 5 that it is not essential to know the variances under the alternative hypothesis. However, if they should be desired they can be obtained by the same methods used in Levene and Wolfowitz [1]. The only difference is that whereas under H_0 such probabilities as Prob $\{- +^p - +\}$ could be obtained explicitly as rational functions of p, they must now be obtained numerically for fixed p by the methods of the three preceding sections. In general this requires excessive additional work; however, there are two variances which can be obtained as byproducts of the expected values. These are

$$\sigma'^2(k) = 3E'(k) - 3[E'(k)]^2 - 2E'(s), \tag{10.1}$$

and

$$\sigma'^2(s) = 2E'(s) - 5[E'(s)]^2 - E'(s_3') - E'(t_3'). \tag{10.2}$$

Since both $E'(k)$ and $E'(s)$ are tabulated in Table 1, Section 8, for a normal population with linear trend, it was possible to give $\sigma'^2(k)$ in the same table. The surprising fact will be noted that $\sigma'^2(k)$ is a maximum at $\theta \doteq 1$ rather than at $\theta = 0$. For $\theta = 0$, the signs of adjacent differences have a negative correlation, and apparently a moderate trend tends to make the differences more nearly independent, thus increasing the variance of the sum, k, even though the variance of an individual difference is greatest at $\theta = 0$. A similar condition holds for $\sigma'^2(s)$.

For the special case $\theta = \frac{1}{2}$, we have $\sigma'^2(k) = .09088$ and $\sigma'^2(s) = .05610$, compared with $\sigma'^2(k) = .08333$ and $\sigma'^2(s) = .04444$ for $\theta = 0$. We can then compute the exact asymptotic power of the tests. For the test of H_0 against the one sided alternative $\theta > 0$, the asymptotically most powerful tests based on k and s respectively are

$$\frac{k - n/2}{\sqrt{n/12}} \geq \lambda_\alpha \tag{10.3}$$

and

$$\frac{n/3 - s}{\sqrt{n/90}} \geq \lambda_\alpha. \tag{10.4}$$

For the level of significance $\alpha = .05$ and for power $1 - \beta = .95$ we will then require approximately $n = 50$ observations for the k-test and approximately $n = 517$ observations for the s-test. Thus, for this alternative, the test based on k is about ten times as good in a certain sense as the s-test.

For the sake of comparison, we find that the asymptotic power index defined in (5.12) is $\Delta^2(s) = .02362$. For $\alpha = \beta = .05$ and a one-sided test we must have $n\,\Delta^2 = 10.822$, leading to $n = 459$, compared to the correct value, 517. Thus we see that even for this considerable departure from H_0, the asymptotic power index gives us a correct general idea of the power of the test.

The power of these tests will be compared with the power of tests based on runs above and below the median in a forthcoming paper, where cyclic alternatives will also be considered.

Appendix. Covariance matrix of u-run statistics under H_0. When the sequence $(X_1, \cdots, X_n)$ is random, the expected values are

$$E(s_p) = E(t_p) = \tfrac{1}{2}E(r_p) = n\,\frac{p^2 + 3p + 1}{(p+3)!} - \frac{p^3 + 3p^2 - p - 4}{(p+3)!},$$

$$E(s'_p) = E(t'_p) = \tfrac{1}{2}E(r'_p) = n\,\frac{p+1}{(p+2)!} - \frac{p^2 + p - 1}{(p+2)!},$$

and $E(k) = (n-1)/2$.

The exact covariances and selected numerical values are given below. Formulas not given below may be obtained by interchanging t and s; thus $\sigma^2(s_p) = \sigma^2(t_p)$ and $\sigma(s_p, t'_q) = \sigma(t_p, s'_q)$. An exception to this rule is $\sigma(k, s_p) = -\sigma(k, t_p)$ and $\sigma(k, s'_p) = -\sigma(k, t'_p)$.

$$\begin{aligned}\sigma(s_p, s_q) = n\{&-[1/(q+3)!(p+3)!]\,[p^3(q^2+3q+1) + p^2q(q^2+7q+11)\\ &+ p(3q^3+11q^2+3q-10) + (q^3-10q-7)]\\ &- [2/(p+q+5)!]\,[(p+q)^3 + 9(p+q)^2 + 23(p+q) + 14]\\ &+ [\delta_{pq}/(p+3)!][p^2+3p+1]\} + \{[1/(q+3)!(p+3)!][p^4(q^2\\ &+3q+1)\\ &+ p^3q(q^2+7q+11) + p^2(q^4+7q^3+9q^2-14q-18)\\ &+ p(3q^4+11q^3-14q^2-65q-25) + (q^4-18q^2-25q+4)]\\ &+ [2/(p+q+5)!][(p+q)^4 + 10(p+q)^3 + 29(p+q)^2 + 16(p+q)\\ &- 19] - [\delta_{pq}/(p+3)!][p^3+3p^2-p-4]\},\end{aligned}$$

where $\delta_{pq} = 1$ if $p = q$ and $= 0$ if $p \neq q$.

$$\sigma^2(s_p) = n\{-[1/(p+3)!(p+3)!][2p^5 + 13p^4 + 24p^3 + (3p+1)(p-7)]$$
$$-[2/(2p+5)!][8p^3 + 36p^2 + 46p + 14] + [1/(p+3)!][p^2 + 3p + 1]\}$$
$$+\{[1/(p+3)!(p+3)!][p^4(3p+11)(p+3 - p(28p^2 + 101p + 50) + 4]$$
$$+[2/(2p+5)!][16p^4 + 80p^3 + 116p^2 + 32p - 19]$$
$$-[1/(p+3)!][p^3 + 3p^2 - p - 4]\}.$$

$$\sigma(s'_p, s'_q) = n\{-[1/(q+2)!(p+2)!][p^2(q+1) + p(q^2 + 3q + 1)$$
$$+(q^2 + q - 1)] - [2/(p+q+3)!][p+q+2]$$
$$+[1/(G+2)!][G+1]\} + \{[1/(q+2)!(p+2)!]$$
$$[p^3(q+1) + p^2(q^2 + 3q + 1) + p(q^3 + 3q^2 - q - 4)$$
$$+(q^3 + q^2 - 4q - 3)] + [2/(p+q+3)!][(p+q)^2 + 3(p+q) + 1]$$
$$-[1/(G+2)!][G^2 + G - 1]\},$$

where $G = \text{Max}(p, q)$.

$$\sigma^2(s'_p) = n\{-[1/(p+2)!(p+2)!][(p+1)(2p^2 + 3p - 1)]$$
$$-[4/(2p+3)!][p+1] + [1/(p+2)!][p+1]\} + \{[1/(p+2)!(p+2)!]$$
$$[3p^4 + 8p^3 + p^2 - 8p - 3] + [2/(2p+3)!][4p^2 + 6p + 1]$$
$$-[1/(p+2)!][p^2 + p - 1]\}.$$

$$\sigma(s_p, t_q) = n\{-[1/(q+3)!(p+3)!][p^3(q^2 + 3q + 1)$$
$$+p^2(q^3 + 11q^2 + 27q + 12) + p(3q^3 + 29q^2 + 77q + 48)$$
$$+(q^3 + 18q^2 + 68q + 59)] - [2/(p+q+3)(q+2)!(p+1)!]$$
$$[p - q - 1] - [2/(p+q+5)(q+3)!(p+1)!]$$
$$+[2/(p+q+1)q!p!]\} + \{[1/(q+3)!(p+3)!][p^4(q^2 + 3q + 1)$$
$$+p^3(q^3 + 11q^2 + 27q + 12) + p^2(q^4 + 11q^3 + 47q^2 + 84q + 40)$$
$$+p(3q^4 + 29q^3 + 94q^2 + 123q + 45) + (q^4 + 18q^3 + 72q^2 + 89q + 16)]$$
$$+[2/(p+q+3)(q+2)!(p+1)!][(p+q+2)(p-q-1)]$$
$$+[2/(p+q+5)(q+3)!(p+1)!][p+q+4]$$
$$-[2/(p+q+1)q!p!][p+q]\}.$$

$$\sigma(s'_p, t'_q) = n\{-[1/(q+2)!(p+2)!][(p+q+3)(p+1)(q+1)] + [2/(p+q+1)q!p!] - [1/(p+q+3)(q+2)!(p+2)!] [(p+1)(p+2) + (q+1)(q+2)]\} + \{[1/(q+2)!(p+2)!][p^3(q+1) + p^2(q^2+5q+4) + p(q^3+5q^2+7q+2) + (q^3+4q^2+2q-3)] - [2/(p+q+1)q!p!][p+q] + [1/(p+q+3)(q+2)!(p+2)!] [(p+q+2)[(p+1)(p+2)+(q+1)(q+2)]]\}.$$

$$\sigma(s_p, t'_q) = n\{-[1/(p+3)!(q+2)!][p^3(q+1) + p^2(q^2+8q+9) + p(3q^2+17q+24) + (q^2+8q+19)] + [2/(p+q+1)p!q!] + [1/(p+q+2)(p+1)!(q+1)!][p-q] + [1/(p+q+3)(p+2)!(q+2)!][(p-q+1)(q+2)] + [1/(p+q+4)(p+3)!(q+2)!][(p+2)(p+3)+(q+1)(q+2)]\} + \{[1/(p+3)!(q+2)!][p^4(q+1) + p^3(q^2+8q+9) + p^2(q^3+8q^2+24q+29) + p(3q^3+17q^2+34q+43) + (q^3+8q^2+21q+29)] - [2/(p+q+1)p!q!][p+q] - [1/(p+q+2)(p+1)!(q+1)!][p^2+2p-q^2+2] - [1/(p+q+3)(p+2)!(q+2)!][(p+2)[(p+2)(q+3)-1] - q(q+1)(q+2)] - [1/(p+q+4)(p+3)!(q+2)!] [(p+q+3)[(p+2)(p+3)+(q+1)(q+2)]]\}.$$

$$\sigma(s_p, s'_q) = n\{-[1/(p+3)!(q+2)!][p^3(q+1) + p^2(q^2+5q+3) + p(3q^2+5q-1) + (q^2-2q-4)] - [2/(p+q+4)!] [(p+q)^2 + 5(p+q) + 5] + [\eta_{pq}/(p+3)!][p^2+3p+1]\} + \{[1/(p+3)!(q+2)!][p^4(q+1) + p^3(q^2+5q+3) + p^2(q^3+5q^2+2q-5) + p(3q^3+5q^2-15q-16) + (q^3-2q^2-11q-4)] + [2/(p+q+4)!][(p+q)^3 + 6(p+q)^2 + 8(p+q) - 1] - [\eta_{pq}(p+3)!][p^3+3p^2-p-4]\},$$

where $\eta_{pq} = 1$ if $p \geq q$ and $= 0$ if $p < q$.

$$\sigma(m, s'_p) = n[1/2(p+3)!][(p-1)(p^2+3p+1)] + [1/2(p+3)!] [-p^4 - 2p^3 + 5p^2 + 7p - 1]$$

$$\sigma(m, s_p) = n[1/2(p + 4)!][p^4 + 5p^3 + p^2 - 14p - 4] + [1/2(p + 4)!] [-p^5 - 5p^4 + 3p^3 + 34p^2 + 20p - 12].$$

Numerical Values

$$\sigma^2(s_1) = \frac{2843n - 1525}{20160}, \qquad \sigma^2(s_2) = \frac{54563n - 58747}{907200},$$

$$\sigma^2(s_1') = \sigma^2(s) = \frac{2n + 2}{45}, \qquad \sigma^2(s_2') = \frac{369n - 191}{6720},$$

$$\sigma^2(s_3') = \frac{11824n - 27551}{453600}, \qquad \sigma^2(s_1, s_2) = \frac{-989n + 319}{20160},$$

$$\sigma(s_1, s_2') = \frac{-509n + 499}{6720}, \qquad \sigma(s_1, s_3') = \frac{-269n + 589}{10080},$$

$$\sigma(s_2, s_3') = \frac{-7099n + 22016}{453600}, \qquad \sigma(s_1', s_3') = \frac{-82n + 233}{5040},$$

$$\sigma(s_2', s_3') = \frac{35n - 41}{3360}, \qquad \sigma(s_1, t_1) = \frac{1427n - 3333}{20160},$$

$$\sigma(s_1, t_2) = \frac{11n - 121}{2880}, \qquad \sigma(s_2, t_2) = \frac{-3457n - 15112}{907200},$$

$$\sigma(s_1, t_2') = \frac{-111n - 391}{20160}, \qquad \sigma(s_1, t_3') = \frac{-47n + 114}{5040},$$

$$\sigma(s_2, t_3') = \frac{-4148n + 4987}{907200}, \qquad \sigma(s_1', t_1') = \frac{8n - 37}{180},$$

$$\sigma(s_1', t_3') = \frac{-82n + 163}{5040}, \qquad \sigma(s_2', t_2') = \frac{-309n - 29}{20160},$$

$$\sigma(s_2', t_3') = \frac{-5n + 7}{720}, \qquad \sigma(s_3', t_3') = \frac{-2152n + 3833}{907200},$$

$$\sigma^2(k) = \frac{n + 1}{12}, \qquad \sigma(k, s_1) = \frac{-11n + 39}{120},$$

$$\sigma(k, s_2) = \frac{7n + 19}{180}, \qquad \sigma(k, s) = \frac{1}{3},$$

$$\sigma(k, s_2') = \frac{11n + 1}{120}, \qquad \sigma(k, s_3') = \frac{19n - 35}{360}.$$

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