4. Numerical results. Numerical values of the coefficient in (3.4) are tabulated in Table 1, together with the corresponding values $n^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}(n-1)}$ for normal and $(2\sqrt{3})^{-(n-1)}$ for rectangular population. All these values approach 1 when $n \to 1$, as might be expected from the fact that for any distribution $\Omega_1 = 1$. It is to be noted that the curve for the lower bound would be fairly parallel in logarithmic scale to the curve for rectangular population. In fact it is easily shown that when n becomes large the former is given by

$$(4.1)\quad \frac{1}{(2\sqrt{3})^{n-1}} \frac{2}{3} \exp\left[-\frac{1}{3} + \psi\left(\frac{1}{2}\right) - \psi(0)\right]\left(1 + O\left(\frac{1}{n}\right)\right) = \frac{1}{(2\sqrt{3})^{n-1}} \frac{e^{5/3}}{6}\left(1 + O\left(\frac{1}{n}\right)\right) = \frac{0.8824}{(2\sqrt{3})^{n-1}}\left(1 + O\left(\frac{1}{n}\right)\right),$$

where $\psi(x)$ denotes the digamma function $\Gamma'(x + 1)/\Gamma(x + 1)$. The first term happens to be close to the true value even for small n as we see in Table 1.

5. Acknowledgement. The author wishes to express his thanks to Professor Harold Hotelling for his kind supervision of the work.

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UNIFORMITY FIELD TRIALS WHEN DIFFERENCES IN FERTILITY LEVELS OF SUBPLOTS ARE NOT INCLUDED IN EXPERIMENTAL ERROR

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1. Introduction. The present note is confined to the consideration of two randomized blocks with two subplots each. The usual mathematical model for the analysis of variance of such an experiment assumes that

$$(1.1)\quad v_{ij} = g + b_i + t_j + \epsilon_{ij}, \qquad i = 1, 2; j = 1, 2,$$

where v_{ij} is the yield of the jth variety in the ith block, and the block effect b_i is the average for the subplots of the ith block. Any difference between b_i and the yield of subplots due to differences in fertility is one component of the random parts, ϵ_{ij}. The random parts, ϵ_{ij}'s, are then assumed to be normally and independently distributed with *zero* means and uniform variance. That these assumptions may break down in many cases because of the magnitude and nonrandomness of the differences between subplots has been indicated in a recent paper [1]. It should be understood that it is practically impossible with our present knowledge to determine the relative or absolute fertility levels of any set of plots, so that the present discussion will add only to the background knowledge and general understanding of the behavior of field trials. It is possible to discuss the present simple case in some detail while the situation becomes more complex with more degrees of freedom. (See [1], pages 64 and 65.) The effect of randomization is considered, and it is found to have a "beneficial" effect in some cases, and no effect in others.

2. Theoretical development. The following development follows closely suggestions made by a referee, especially with respect to randomization. Let us consider the following setup:

Block 1	Block 2
v_{11}	v_{21}
v_{12}	v_{22}

The v_{ij}'s refer to observed yields of a uniformity trial. The subscript i refers to the block number and j to a dummy variety. Since this is a uniformity trial the t_j's of equation (1.1) are zero. The plots we shall consider as being assigned at random to the dummy varieties.

Let σ_0^2 be the assumed uniform error variance and $\xi_{ih}(i = 1, 2; h = 1, 2)$ be the "true" unknown fertility level in the hth subplot of the ith block. Let

$$v_{ij} = x_{ij} + \xi'_{ij}. \tag{2.1}$$

Then it is assumed that the x_{ij}'s are distributed as $N(0, \sigma_0)$. The jth "variety" has equal chance of being assigned either to the first or to the second plot within the block. Thus ξ'_{ij} itself is a stochastic variate, with probability 1/2 of taking the values ξ_{i1}, ξ_{i2}. Put

$$b_i = \frac{\xi_{i1} + \xi_{i2}}{2}, \; d_i = \frac{\xi_{i1} - \xi_{i2}}{2} \qquad (i = 1, 2). \tag{2.2}$$

Since if variety 1 is assigned to a given plot in the ith block then variety 2 must be assigned to the second plot we have

$$\begin{aligned} v_{11} &= b_1 + x_{11} + a_1, & v_{21} &= b_2 + x_{21} + a_2, \\ v_{12} &= b_1 + x_{12} - a_1, & v_{22} &= b_2 + x_{22} - a_2, \end{aligned} \tag{2.3}$$

where a_i is a stochastic variate which takes the values $\pm d_i$ with equal probabilities 1/2.

If we apply the conventional analysis of variance we obtain

$$S_v^2 = \tfrac{1}{4}(v_{11} + v_{21} - v_{12} - v_{22})^2, \tag{2.4}$$

$$S_e^2 = \tfrac{1}{4}(v_{11} - v_{21} - v_{12} + v_{22})^2, \tag{2.5}$$

where S_v^2 is the variety sum of squares and S_e^2 is the error sum of squares, each with one degree of freedom. These expressions (2.4) and (2.5) in terms of x_{ij}'s and a_i's are

$$S_v^2 = \tfrac{1}{4}(x_{11} + x_{21} - x_{12} - x_{22} + 2a_1 + 2a_2)^2, \tag{2.6}$$

$$S_e^2 = \tfrac{1}{4}(x_{11} - x_{21} - x_{12} + x_{22} + 2a_1 - 2a_2)^2. \tag{2.7}$$

Put

$$\begin{aligned} v &= x_{11} + x_{21} - x_{12} - x_{22}, \\ u &= x_{11} - x_{21} - x_{12} + x_{22}, \\ m_2' &= 2a_1 + 2a_2, \\ m_1' &= 2a_1 - 2a_2, \end{aligned} \tag{2.8}$$

and we get

$$F_r = \left(\frac{v + m_2'}{u + m_1'}\right)^2, \tag{2.9}$$

where u and v are independent variates distributed $N(0, 4\sigma_0^2)$. Pur

$$\begin{aligned} m_1 &= \xi_{11} - \xi_{12} - \xi_{21} + \xi_{22}, \\ m_2 &= \xi_{11} - \xi_{12} + \xi_{21} - \xi_{22}, \end{aligned} \tag{2.10}$$

and then the pair (m_2', m_1') has the four possible values (m_2, m_1), (m_1, m_2), $(-m_2, -m_1)$, $(-m_1, -m_2)$, each with probability 1/4.

If we used a systematic arrangement with the variety number the same as subplot number instead of a randomized arrangement we would have

$$F = \left(\frac{v + m_2}{u + m_1}\right)^2 \tag{2.11}$$

instead of (2.9). If we apply the result given in [2], especially equation (17) page 5, and transform to a new variable, we obtain the distribution of F as

$$\begin{aligned} f(F, m_2, m_1) &= \pi^{-1}(1 + F)^{-1}F^{-\frac{1}{2}}e^{-\frac{1}{2}(m^2{}_1+m_2{}^2)/\sigma^2} \\ &+ a(2\pi)^{-\frac{1}{2}}(1 + F)^{-1}F^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(m_1F^{\frac{1}{2}} - m_2)^2/(\sigma^2(1 + F))\right] \int_0^a N(x)\, dx \\ &+ b(2\pi^{-\frac{1}{2}})(1 + F)^{-1}F^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(m_2 + m_1F^{\frac{1}{2}})^2/(\sigma^2(1 + F))\right] \int_0^b N(x)\, dx, \end{aligned} \tag{2.12}$$

$0 \leqq F \leqq \infty$, where

$$\begin{aligned} a &= (m_2F^{\frac{1}{2}} + m_1)/(\sigma(1 + F)^{\frac{1}{2}}), \\ b &= (m_1 - m_2F^{\frac{1}{2}})/(\sigma(1 + F)^{\frac{1}{2}}), \\ N(x) &= (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}. \end{aligned}$$

We note that if both m_1 and m_2 are zero (2.12) reduces to the tabled F distribution which is used almost universally in testing the significance of "variety" difference with 1 and 1 degrees of freedom.

TABLE 1

Distributions (2.12) *and* (2.13) *for seven pairs of values of the parameters*

	DESIGNATING NUMBER OF PARAMETRIC PAIR								
F	1[c]	2[a]	3[a]	$\frac{1}{2}(2+3)$[b]	4[c]	5[a]	6[a]	$\frac{1}{2}(5+6)$[b]	7[c]
0.0001	31.828	19.307	46.529	32.918	28.229	4.309	80.439	42.374	10.893
0.005	4.479	2.724	6.524	4.624	3.992	0.618	11.185	5.902	1.620
0.010	3.152	1.948	4.572	3.260	2.822	0.444	7.774	4.109	1.187
0.025	1.964	1.221	2.817	2.019	1.783	0.293	4.674	2.483	0.862
0.05	1.356	0.862	1.909	1.385	1.257	0.221	3.045	1.633	0.712
0.10	0.915	0.607	1.244	0.926	0.879	0.175	1.844	1.010	0.650
0.20	0.593	0.423	0.758	0.591	0.613	0.147	0.987	0.567	0.604
0.40	0.360	0.286	0.417	0.352	0.388	0.131	0.440	0.285	0.518
0.60	0.267	0.238	0.276	0.257	0.282	0.126	0.249	0.188	0.427
0.80	0.198	0.183	0.200	0.192	0.220	0.116	0.158	0.137	0.348
1.00	0.159	0.154	0.154	0.154	0.177	0.109	0.109	0.109	0.284
1.20	0.132	0.132	0.123	0.127	0.147	0.103	0.080	0.091	0.233
1.40	0.112	0.116	0.101	0.108	0.124	0.097	0.060	0.078	0.194
1.60	0.096	0.103	0.084	0.094	0.107	0.091	0.048	0.069	0.162
1.80	0.085	0.092	0.072	0.082	0.093	0.086	0.038	0.062	0.138
2.00	0.075	0.084	0.062	0.073	0.082	0.081	0.032	0.056	0.118
2.20	0.067	0.076	0.055	0.065	0.073	0.076	0.027	0.052	0.102
2.40	0.060	0.070	0.049	0.059	0.065	0.072	0.023	0.048	0.089
2.60	0.055	0.064	0.043	0.054	0.059	0.068	0.020	0.044	0.078
2.80	0.050	0.059	0.039	0.049	0.053	0.065	0.017	0.041	0.068
3.00	0.046	0.055	0.035	0.045	0.048	0.062	0.015	0.038	0.061
3.20	0.042	0.051	0.032	0.042	0.043	0.059	0.013	0.036	0.054
3.40	0.039	0.048	0.030	0.039	0.041	0.056	0.012	0.034	0.049
3.60	0.036	0.045	0.027	0.036	0.038	0.053	0.011	0.032	0.044
3.80	0.034	0.042	0.025	0.034	0.035	0.051	0.010	0.030	0.040
4.00	0.032	0.040	0.024	0.032	0.033	0.049	0.009	0.029	0.036
4.20	0.030	0.037	0.022	0.030	0.031	0.046	0.008	0.027	0.033
4.40	0.028	0.035	0.020	0.028	0.029	0.045	0.007	0.026	0.031
4.60	0.026	0.034	0.019	0.026	0.027	0.043	0.007	0.025	0.028
4.80	0.025	0.032	0.018	0.025	0.025	0.041	0.006	0.024	0.026
5.00	0.024	0.030	0.017	0.024	0.024	0.040	0.006	0.023	0.024
100	0.(2)03[d]	0.(2)05	0.(3)19	0.(3)34	0.(3)14	0.(3)77	0.(3)04	0.(2)04	0.(3)11
10,000	0.(6)32	0.(6)46	0.(6)19	0.(6)33	0.(6)28	0.(6)84	0.(7)43	0.(6)44	0.(6)11
Limiting ratio[e]	1.	1.462	0.606	1.034	0.887	1.332	0.135	0.733	0.180

[a] Distributions headed 2, 3, 5, and 6 are not randomized (2.12).

[b] Distributions $(2 + 3)/2$, $(5 + 6)/2$ are randomized (2.13).

[c] Distributions 1, 4, and 7 are identical under randomization and nonrandomization.

[d] Number in parenthesis indicates the number of omitted zeros, thus 0.(2)03 means 0.0003.

[e] Limiting ratio of the ordinates of the indicated distributions to the ordinates of the conventional F distribution (1) as F approaches zero and as F approaches infinity.

The distribution of F when randomization is allowed is

$$\frac{1}{2}[f(F, m_2, m_1) + f(F, m_1, m_2)], \tag{2.13}$$

since $f(F, m_2, m_1) = f(F, -m_2, -m_1)$.

3. Discussion. To show how (2.12) and (2.13) may differ from the usually assumed distribution we have considered the following seven pairs of values of m_1/σ and m_2/σ:

DESIGNATING NUMBER	$\frac{m_1}{\sigma}$	$\frac{m_2}{\sigma}$
1	0	0
2	0	1
3	1	0
4	1	1
5	0	2
6	2	0
7	2	2

Selected ordinates for systematic and randomized procedures for these 7 pairs of values are presented and compared in Table 1. It is seen that the tails of some of the curves are much heavier than for case 1 ($m_1 = m_2 = 0$), indicating that much larger values of F are required for significance. On the other hand, some of the tails are lighter than for case 1 so that smaller F-values are indicative of significance at the usual levels. Randomization is effective in some cases in giving a distribution that is closer to the conventional F distribution than is the F distribution for a systematic procedure.

It is easy to find the limiting values of the ratios of the ordinates of (2.12) and (2.13) to the ordinates of the conventional F distribution as F approaches 0 and ∞ (same). These limiting values are also indicated in Table 1.

When (2.13) is a greatly curtailed distribution making errors of the first kind less probable than expected then the probability of errors of the second kind may be greatly enhanced.

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A GENERALIZATION OF A THEOREM DUE TO MacNEISH[1]

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1. Summary and introduction. In 1922 MacNeish [1] considered the problem of orthogonal Latin squares and showed that if the number s is written in standard form:

$$s = p_0^{n_0} p_1^{n_1} \cdots p_k^{n_k},$$

[1] This note is a revision of one section of the author's doctoral dissertation submitted to the University of North Carolina at Chapel Hill.