The function $K(x, y)$ is an increasing function of x and an increasing function of y, provided $x + y < 1$. Conditions (1) and (2) imply that $1 - L(\theta_0) \leqq \alpha$, $L(\theta_1) \leqq \beta$. Hence if $\alpha + \beta < 1$, we have

$$K(1 - L(\theta_0), L(\theta_1)) \leqq K(\alpha, \beta). \tag{15}$$

Inequality (4) now follows from relations (12) to (15).

Concerning the conditions for equality, it suffices to observe that in (10) the sign of equality holds if and only if there exist constants C_0 and C_1 such that

$$P_\theta\left\{\prod_{j=1}^{n} \frac{f(X_j, \theta)}{f(X_j, \theta')} = C_i \mid S \text{ accepts } H_i\right\} = 1, \qquad i = 0, 1,$$

where the usual notation for conditional probabilities is used. This can be verified from Wald's proof. The conditions for equality in (12), (13), (15) are obvious.

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SOME INEQUALITIES ON MILL'S RATIO AND RELATED FUNCTIONS

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1. Introduction. Mill's ratio is defined as

$$R_x = e^{\frac{1}{2}x^2} \int_x^\infty e^{-\frac{1}{2}u^2}\, du. \tag{1}$$

Gordon [1] and Birnbaum [2] have given, respectively, upper and lower limits for R_x as

$$\tfrac{1}{2}\{\sqrt{4 + x^2} - x\} < R_x < 1/x, \qquad x > 0. \tag{2}$$

Murty [3] has shown how limits to R_x of any required degree of accuracy can be derived for $x > 0$ by the use of successive convergents of Laplace's expression for the normal integral as a continued fraction. No limits have, as yet, been published for $x < 0$.

If the functions $\nu(x)$ and $\lambda(x)$ are defined by $\nu(x) = 1/R_x$, $\lambda(x) = \nu'(x) = \nu(\nu - x)$, the inequalities

$$0 < \lambda < 1, \tag{3}$$

$$\lambda' = \nu\{(\nu - x)(2\nu - x) - 1\} > 0 \tag{4}$$

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are of importance in the theory of the analysis of response-times and of truncated normal data generally. The result (4) was conjectured as true for positive x by Birnbaum [4] and was proved for sufficiently large positive x by Murty [3]. In this paper it is proved for all finite x, and is used to provide an upper limit for R_x valid over the range $x > -1$. The result (3) is also proved for all finite x. The upper limit is equivalent to the lower limit in (2), which is thus valid for $x < 0$ as well as for positive x.

2. Proof of the inequality on λ. The function

$$e^{-\frac{1}{2}u^2} \Big/ \int_x^\infty e^{-\frac{1}{2}u^2}\,du$$

is a p.d.f. over the range $x \leqq u < \infty$, and its variance is easily shown to be $1 - \nu(\nu - x)$. Since this must be positive for finite x, the upper limit in (3) follows. Also $\nu > 0$ by definition. Hence $(\nu - x) > 0$ for $x \leqq 0$ and, by (2), for $x > 0$, and the lower limit follows.

3. Proof of the inequality on λ'. The result (4) is equivalent to

$$\varphi = (\nu - x)(2\nu - x) > 1. \tag{5}$$

An expansion by parts for $x > 0$ gives

$$\int_x^\infty e^{-\frac{1}{2}u^2}\,du = \left\{1 - \frac{1}{x^2} + R\right\}\frac{e^{-\frac{1}{2}x^2}}{x},$$

where $R = 0(1/x^2)$; whence

$$\varphi = \frac{1 + x^2 0(1/x^2)}{\left\{1 - \frac{1}{x^2} + R\right\}^2} \to 1 \quad \text{as} \quad x \to \infty. \tag{6}$$

Also, as $x \to -\infty$, $\nu \to 0$ and

$$\varphi \to x^2 \to \infty. \tag{7}$$

Now suppose there exists a finite x_1 such that

$$\varphi(x_1) = 1. \tag{8}$$

Then, since φ is continuous and differentiable, (6), (7), and (8) imply the existence of a finite point x_2 for which

$$\begin{aligned} \varphi(x_2) &\leqq 1, \\ \varphi'(x_2) &= 0. \end{aligned} \tag{9}$$

But $\varphi' = (\lambda - 1)(2\nu - x) + (\nu - x)(2\lambda - 1) = \nu(\varphi - 1) + 2(\nu - x)(\lambda - 1)$, whence, for finite x, $\varphi' < \nu(\varphi - 1)$, so that conditions (9) cannot be satisfied simultaneously, and the result (4) follows.

The quadratic $\varphi = 1$ has solutions

$$\nu = \frac{3x \pm \sqrt{x^2 + 8}}{4}. \tag{10}$$

As ν is known to be greater than x, only the positive sign in (10) need be considered. The result so obtained is everywhere greater than x, and positive for all $x > -1$, giving the result

$$R_x < 4/\{3x + \sqrt{8 + x^2}\}, \qquad x > -1.$$

4. A corollary on the weight function in probit analysis. The function

$$\psi(x) = e^{-x^2} \Big/ \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du \int_{x}^{\infty} e^{-\frac{1}{2}u^2}\, du$$

is well known as the weight function in probit analysis. From tables it is obvious that ψ is a decreasing function of x^2. Hammersley [5] has given a rather complicated proof of this result, and has remarked on the apparent lack of a simple proof. In fact

$$\begin{aligned}\psi'(x) &= \psi(x)\{\nu(x) - \nu(-x) - 2x\} \\ &= 2x\psi(x)\{\lambda(x') - 1\}, \quad \text{where } -|x| \leqq x' \leqq |x|,\end{aligned}$$

by the Mean Value Theorem, and, since ψ is positive by definition, the result follows immediately from (3) above.

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ON A DOUBLE INEQUALITY OF THE NORMAL DISTRIBUTION[1]

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In this note we shall extend certain results of R. D. Gordon and Z. W. Birnbaum concerning bounds for the normal distribution function.

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