A CLASS OF EXPERIMENTAL DESIGNS USING BLOCKS OF TWO PLOTS[1]

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1. Introduction and Summary. Over the past fifteen or so years, a large number of classes of experimental designs have been evolved by Yates, Bose and Nair and others (see [1] for a systematic account). The aim in all cases was to evolve patterns of observation which could utilize natural groupings in the experimental material, such as for instance litters of mice or small numbers of plots perhaps contiguous to each other. By arranging the treatments to be compared in specific ways which utilize the natural grouping, it is possible to enable treatment contrasts to be estimated by comparisons of observations on experimental units, which we shall call plots, within the natural groups. This enables the comparisons to be made usually with considerably greater accuracy than would obtain if the experimenter were forced to randomize the positions of the treatments without respect to these groupings.

In experimental work in some branches of biology, natural groups of size two are of fairly frequent occurrence, for example twins, or halves of plants, or halves of leaves. The development of experimental designs is not complete in this particular respect. The designs which have been developed for blocks of two plots or experimental units are as follows:

(1) symmetrical pairs (Yates, [2]) which require $(t - 1)$ replicates if there are t treatments.

(2) quasifactorial designs if the number of treatments is a power of 2 (see [1] in this respect).

It appears therefore that development of a class of designs using blocks of two plots is desirable, and this is the purpose of the present paper.

2. Structure of the Designs. Let the number of treatments be n and suppose r replicates of each treatment are desired. The structure of the class of designs is that treatment i $(= 1, 2, \cdots, n)$ is placed in a block with each of the treatments $i + s, i + s + 1, \cdots, i + s + r - 1$, where each of the numbers $i + s$ to $i + s + r - 1$ is to be reduced modulo n, that is, is to be replaced by the remainder after dividing it by n considering 0 to be identical to n. This structure is possible only if $2s + r - 1 = n$, that is, if $n + 1 - r$ is even.

The pattern of observations is specified therefore by n and r, and $n + r - 1$ must be even. The number r is the number of times each treatment is replicated and the total number of blocks is $rn/2$ and of plots is rn. In practice the treat-

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ments would be arranged in random order before making up the pairs of treatments which are to lie in a block together and these pairs would be assigned to the blocks at random, and the individuals in the pair assigned to the plot in the block at random.

3. The Analysis of the Designs. On the basis of the usual assumption of additivity of treatment effects, and by virtue of randomization, we may apply the method of least squares to obtain estimates of treatment comparisons. These normal equations may be obtained from the standard reduced normal equations for the two-way classification with unequal numbers, namely

$$\left\{N_{.j} - \sum_i \frac{n_{ij}^2}{N_{i.}}\right\} \hat{\tau}_j - \sum_{j' \neq j} \left\{\sum_i \frac{n_{ij} n_{ij'}}{N_{i.}}\right\} \hat{\tau}_{j'} = Q_j, \qquad j = 1, 2, \cdots, n \tag{1}$$

where y_{ij} is the yield of jth treatment in ith block, n_{ij} is the number of times jth treatment is represented in the ith block

$$N_{.j} = \sum_i n_{ij}, \qquad N_{i.} = \sum_j n_{ij},$$

$$Q_j = Y_{.j} - \sum_i \frac{n_{ij}}{N_{i.}} Y_{i.}$$

where $Y_{.j} = \sum_i y_{ij}$, $Y_{i.} = \sum_j y_{ij}$. In this instance $N_{.j}$ is r for all j, $N_{i.}$ is 2 for all i, and we get

$$\frac{r}{2} \hat{\tau}_j - \sum_{j' \neq j} \tfrac{1}{2} \lambda_{jj'} \hat{\tau}_{j'} = Q_j \tag{2}$$

where $\lambda_{jj}' = 1$ if j' and j occur in a block together and is zero otherwise. These equations may be written as $A\hat{\tau} = Q$.

The matrix of coefficients A in (2) is a special type of matrix, known as a circulant (see for example Ferrar [4]). The first row consists of $r/2$ followed by $(s - 1)$ zero's followed by r terms equal to $-\frac{1}{2}$, the remaining terms being zero. The second row is obtained from the first by moving the elements along one step, putting the last element of row one as the first element of row two and so on. In the present instance the circulant matrices we are concerned with are also symmetrical so that the characteristic roots are real. The roots are also nonnegative.

We shall now review briefly some properties of circulant matrices. The circulant matrix will be denoted by $[a_1 a_2 \cdots a_n]$, that is, by its first row. The properties we need are as follows.

(i) The determinant of the circulant matrix $[a_1 a_2 \cdots a_n]$ is equal to

$$\prod_{i=1}^{n} [a_1 + a_2 w_i + a_3 w_i^2 + \cdots + a_n w_i^{n-1}],$$

where w_i, $i = 1, 2, \cdots, n$, are the nth roots of unity.

(ii) The characteristic matrix is also a circulant and therefore has a determinant equal to

$$\prod_{i=1}^{n} [a_1 - \lambda + a_2 w_i + a_3 w_i^2 + \cdots + a_n w_i^{n-1}],$$

so that the latent roots of the matrix are

$$\lambda_i = a_1 + a_2 w_i + a_3 w_i^2 + \cdots a_n w_i^{n-1}, \qquad i = 1, 2, \cdots, n.$$

The matrix A is singular and the equations therefore do not have a unique solution, corresponding to the fact that the τ_j's are not estimable (see for instance [1], p. 77). However, it is easily seen that any comparison of the τ_j's, say $\sum \lambda_j \tau_j$, with $\sum \lambda_j$ equal to zero is estimable. It is also known that we may impose any condition on the normal equations, $A\hat{\tau} = Q$, so that we obtain a unique solution say $\hat{\tau}_0$ and that the best linear unbiased estimate of $\lambda'\tau = \sum \lambda_j \tau_j$, $\sum \lambda_j = 0$, is equal to $\lambda'\hat{\tau}_0$. The simplest condition which may be imposed is that $\sum \hat{\tau}_j = 0$ and the solution may be obtained by the device of writing the normal equations in the form

$$\begin{pmatrix} A & \vartheta \\ \vartheta' & 0 \end{pmatrix} \begin{pmatrix} \hat{\tau} \\ Z \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix} \tag{3}$$

where ϑ denotes an $n \times 1$ matrix whose elements are unity. We shall denote the matrix

$$\begin{pmatrix} A & \vartheta \\ \vartheta' & 0 \end{pmatrix}$$

by A^* and it is seen that A^* is nonsingular, so that it has an inverse.

Now consider the inverse of A^*. In full, we have to find the matrix C with element c_{ij}, such that

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n & 1 \\ a_n & a_1 & \cdots & a_{n-1} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_2 & a_3 & \cdots & a_1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} & c_{1,n+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} & c_{n,n+1} \\ c_{n+1,1} & c_{n+1,2} & \cdots & c_{n+1,n} & c_{n+1,n+1} \end{pmatrix} = I_{n+1},$$

where I_{n+1} is the $(n + 1) \times (n + 1)$ identity matrix. It can be shown easily that

$$c_{j,n+1} = \frac{1}{n}, \qquad j = 1, 2, \cdots n; \qquad c_{n+1,n+1} = 0,$$

and that the matrix A^* is symmetrical, so that C is also symmetrical.

Now let $w_1, w_2, \cdots, w_n$ be the nth roots of unity, where $w_1 = 1$, $w_2 = w = \cos 2\pi/n + i \sin 2\pi/n$ and $w_j = w^{j-1}$, and consider the first column of C. It is given by the equations

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n & 1 \\ a_n & a_1 & \cdots & a_{n-1} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_2 & a_3 & \cdots & a_1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ \cdot \\ \cdot \\ \cdot \\ c_{1n} \\ c_{1,n+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix}$$

Take any root of unity, say w_i, and form the sum of row 1 plus w_i times row 2 plus w_i^2 times row 3 and so on, to w_i^{n-1} times row n, and we get

$$(c_{11} + w_i^{n-1} c_{12} + w_i^{n-2} c_{13} + \cdots + w_i c_{1n})\lambda_i = 1,$$

where λ_i is as defined above. Doing this for all the nth roots of unity we obtain the set of equations in the c_{ij}'s.

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_2^{n-1} & w_2^{n-2} & \cdots & w_2 \\ 1 & w_3^{n-1} & w_3^{n-2} & \cdots & w_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & w_n^{n-1} & w_n^{n-2} & \cdots & w_n \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ \cdot \\ \cdot \\ c_{1n} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\lambda_2} \\ \frac{1}{\lambda_3} \\ \cdot \\ \cdot \\ \frac{1}{\lambda_n} \end{pmatrix}$$

But

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & w_2 & \cdots & w_2^{n-1} \\ 1 & w_3 & \cdots & w_3^{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & w_n & \cdots & w_n^{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & w_2^{n-1} & \cdots & w_2 \\ 1 & w_3^{n-1} & \cdots & w_3 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & w_n^{n-1} & \cdots & w_n \end{pmatrix} = nI_n;$$

so that

(4)
$$\begin{pmatrix} c_{11} \\ c_{12} \\ \cdot \\ \cdot \\ \cdot \\ c_{1n} \end{pmatrix} = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_2 & w_2^2 & \cdots & w_2^{n-1} \\ 1 & w_3 & w_3^2 & \cdots & w_3^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & w_n & w_n^2 & \cdots & w_n^{n-1} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\lambda_2} \\ \frac{1}{\lambda_3} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{\lambda_n} \end{pmatrix}.$$

By the same process one can show that the matrix C is a circulant after the last row and column, which we know, are blocked out. We have therefore obtained the matrix C.

Utilizing the fact that the λ's are real, so that the imaginary parts must be identically zero, we have therefore

$$\lambda_{j+1} = a_1 + a_2 \cos j\theta + a_3 \cos 2j\theta + \cdots + a_n \cos (n - 1)j\theta,$$

where $\theta = 2\pi/n$, and

(5)
$$c_{11} = \frac{1}{n} \sum_{j=2}^{n} \frac{1}{\lambda_j}, \; c_{1j} = \frac{1}{n} \sum_{k=2}^{n} \frac{\cos (k - 1)(j - 1)\theta}{\lambda_k}.$$

Since $a_1 = r/2$ and the remaining a's are either $-\frac{1}{2}$ or zero, we find that

(6)
$$\lambda_{j+1} = \frac{1}{2}\left[r + \frac{\sin (n - r)j\theta/2}{\sin j\theta/2}\right].$$

Finally we have the fact that the intrablock estimates are obtainable from

(7)
$$\begin{pmatrix} \hat{\tau}_0 \\ 0 \end{pmatrix} = C \begin{pmatrix} Q \\ 0 \end{pmatrix},$$

and the variance-covariance matrix of the $\hat{\tau}_{0j}$'s is $\sigma^2 C$.

4. The Efficiency of the Designs. The standard measure of the efficiency of an incomplete block design is the mean variance of treatment differences in terms of σ^2, the error variance, divided into $2\sigma^2/r$, where r is the number of times each treatment is replicated.

In the present case

$$\begin{aligned} \text{Var} (\hat{\tau}_j - \hat{\tau}_{j'}) &= \sigma^2(c_{jj} + c_{j'j'} - 2c_{jj'}) \\ &= 2\sigma^2(c_{jj} - c_{jj'}). \end{aligned}$$

Summing these variances over all $n(n-1)/2$ possible differences, and utilizing the fact that the relevant part of C is a circulant, we get as the mean variance of a treatment difference:

$$2\sigma^2\left[c_{11} - \frac{2}{n(n-1)}\{(n-1)c_{12} + (n-2)c_{13} + \cdots + c_{1n}\}\right]$$

$$= 2\sigma^2\left[c_{11} - \frac{2}{n(n-1)}\{nc_{12} + nc_{13} + \cdots + nc_{1,(n+1)/2}\}\right],$$

if n is odd

$$= 2\sigma^2\left[c_{11} - \frac{2}{n(n-1)}\left\{nc_{12} + nc_{13} + nc_{1,(n/2)} + \frac{n}{2}c_{1,(n/2)+1}\right\}\right],$$

if n is even. In either case it can be seen that the mean variance reduces to $2\sigma^2 c_{11}(n/n-1)$. We may note that c_{11} is in fact equal to one-nth of the sum of the reciprocals of the non-zero latent roots of A. A simple mathematical expression for this sum has not been obtained. Finally the efficiency factor of the designs is equal to $(n-1)/nrc_{11}$. Since the design is completely specified by n and r or s we have computed Table 1, which gives the efficiency factors for a range of values of n and r.

5. The Utilization of Inter-block Information. As with all classes of incomplete block design, the information contained on treatment comparisons in block comparisons must be considered. Whether it is actually worth incorporating with the intrablock information depends on the extent to which the grouping of the experimental units into blocks achieves a marked reduction in the error mean square.

The usual basis for the utilization of the interblock information will be used, namely that a block total, say, B_i has an expected value

$$\sum_{j=1}^{n} \delta_{ij}\tau_j,$$

where $\delta_{ij} = 1$ if treatment j is in block i and equals 0 otherwise, and is distributed around this expected value with a variance of σ_1^2, say. On the basis of the model we are led to the reduced normal equations

$$\sum_{j'}\left(\sum_i \delta_{ij}\delta_{ij'}\right)\hat{\tau}_{j'} = \sum_i \delta_{ij} Y_{i.} - r\frac{Y_{..}}{b}, \tag{8}$$

where b is the number of blocks.

Letting $W = 1/\sigma^2$ and $W' = 1/\sigma_1^2$ and combining these equations with equations (2) we get as the estimating equations in our particular case

$$r\frac{(W+W')}{2}\hat{\tau}_j - \frac{(W-W')}{2}\sum_{j'\neq j}\lambda_{jj'}\hat{\tau}_{j'} = WQ_j + \frac{W'}{2}\left(T_j - r\frac{Y_{..}}{b}\right), \tag{9}$$

$$j = 1, 2, \cdots, n,$$

where T_j is the total of the blocks containing treatment j. (See [2], p. 545, equation 15, for example).

TABLE 1

Efficiency Factor of Designs

Number of treatments	Number of Replicates								
	2	3	4	5	6	7	8	9	10
6	—	.532	—	.600	—	—	—	—	—
7	.375	—	.542	—	.583	—	—	—	—
8	—	.487	—	.543	—	.571	—	—	—
9	.300	—	.510	—	.542	—	.562	—	—
10	—	.435	—	.517	—	.540	—	.556	—
11	.250	—	.486	—	.521	—	.537	—	.550
12	—	.395	—	.501	—	.522	—	.536	—
13	.214	—	.458	—	.508	—	.524	—	.535
14	—	.360	—	.487	—	.512	—	NC	—
15	.187	—	.433	—	.498	—	NC	—	NC
20	—	.284	—	.437	—	.489	—	NC	—
30	—	.210	—	.372	—	.447	—	NC	—

— means design not possible, NC means not computed.

The matrix of the coefficients is again a circulant so that the solution can be written out from the previous sections. The nonzero roots of the matrix are

$$\lambda^*_{j+1} = r\frac{(W + W')}{2} + \frac{(W - W')}{2}\frac{\sin (n - r)j\theta/2}{\sin j\theta/2}$$
$$= (W - W')\lambda_{j+1} + rW'.$$

The solution of the normal equations is therefore given by

(10) $$\hat{\tau} = C^*R,$$

where C^* is the circulant $[c^*_{11}, c^*_{12}, \cdots, c^*_{1n}]$, and

$$c^*_{1j} = \frac{1}{n}\sum_{k=2}^{n}\frac{\cos (k-1)(j-1)\theta}{\lambda^*_k}.$$

In just the same way as before we find that the variance of the estimated difference between treatments i and j is

(11) $$2(c^*_{ii} - c^*_{ij})$$

which is equal to

$$2(c^*_{11} - c^*_{1,\{j-i\}+1})$$

where $\{j - i\} = (j - i) \bmod n$. Similarly the mean variance of all treatment differences is $(2n/n - 1)c_{11}^*$.

6. The Estimation of the Weights, W and W′. In order to utilize the inter-block information it is necessary to follow the usual device of estimating the weights W and W' for use in the estimating equations.

The analysis of variance which must be computed to estimate W' is given in Table 2,

TABLE 2

Analyses of Variance of Design

	df	S.Sqs.	S.Sqs.	*df*	
Blocks ignoring treatments.	$(nr/2) - 1$	B	T	$n - 1$	Treatments ignoring blocks
Treatments eliminating blocks..	$n - 1$	T'	B'	$(nr/2) - 1$	Blocks eliminating treatments
Error..........	$n(r/2 - 1) + 1$	S_E	S_E	$n(r/2 - 1) + 1$	Error
Total........	$nr - 1$	S	S	$nr - 1$	

where B is one half the sum of squares of block totals minus correction,

$$T' = \sum \hat{\tau}_j Q_j,$$

the $\hat{\tau}$'s being given by equation (7), T is $1/r$ times the sum of squares of treatment totals minus correction, $S = \sum y_{ij}^2$ minus correction, correction $= \sum y_{ij}^2/nr$ and S_E and B' are obtained by subtraction.

If we write $\sigma_1^2 = \sigma^2 + 2\sigma_b^2$, it may be verified that

$$E(S_E) = \left[n\left(\frac{r}{2} - 1\right) + 1\right]\sigma^2,$$

and that

$$E(B') = \left(n\frac{r}{2} - 1\right)\sigma^2 + n(r - 1)\sigma_b^2,$$

so that

$$E\left\{2B' - \frac{2(n - 2)S_E}{nr - 2n + 2}\right\} = n(r - 1)(\sigma^2 + 2\sigma_b^2).$$

We may therefore estimate W and W' by

$$w = \left[n \left(\frac{r}{2} - 1 \right) + 1 \right] \Big/ S_E$$

$$w' = \frac{n(r-1)}{2B' - \dfrac{2S_E(n-2)}{nr - 2n + 2}}.$$

These values are then used in the estimating equations (10).

7. Relation of Designs to Partially Balanced Incomplete Block Designs. The class of partially balanced incomplete block designs evolved by Bose and Nair [3] and later extended by Nair and Rao [5] has been found to contain many designs developed other than from the specifications of the class. It is clear (from for example [1], pp. 546–548) that this class of designs corresponds to a particular form of the reduced normal equations for the treatment constants (τ). If corresponding to any one treatment j the remaining treatments (j') can be divided into classes, say $S_{j1}S_{j2}, \cdots S_{jm}$ within which $\lambda_{jj'}$ is constant, then the following must hold. Let G_{jk} be the sum of the τ's for the treatments in S_{jk}; then the sum of the normal equations for the treatments in S_{jk} must give an equation in the G_{jk}'s, the coefficients of which do not depend on the particular treatment j originally taken.

From this point of view it is easily seen that the class of designs given in this paper belong to the class of partially balanced incomplete block designs. The classes S_{1i} are as follows: S_{11} consists of treatments 2 and n, S_{12} of treatments 3 and $n - 1$ and so on. If n is even, there are $n/2 + 1$ classes, the class $S_{1,n/2+1}$ consisting of treatment $(n/2 + 1)$. If n is odd there are $(n - 1)/2$ classes each containing 2 treatments. The associate classes for the other treatments are defined in a circular way, S_{21} consisting for example of treatments 3 and 1, S_{22} of 4 and n, and so on. The representation of the class of designs given herein as partially balanced incomplete block designs is, however, of little value, because the number of associate classes depends on the number of treatments and may be large, and because all analyses of partially balanced incomplete block designs have been worked out in terms of the number of associate classes. The accuracies of comparisons between treatments in the same or different associate classes is given by (11).

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