CONTRIBUTIONS TO THE STATISTICAL THEORY OF COUNTER DATA[1]

By G. E. Albert and Lewis Nelson

University of Tennessee and Oak Ridge National Laboratory

Summary. A new mathematical model is proposed for the action of counters such as the Geiger-Mueller or the scintillation counters. It is assumed that after each registration the counter is inoperative for a time interval of random length. The distribution of lengths of the inoperative periods is so defined that the Type I and Type II models familiar in the literature on counters are special cases. More important, it also allows an action that is a compromise between those two types. Assuming that the sequence being counted is a Poisson process with mean rate of occurrence mT, $m > 0$, in an arbitrary interval of length T, the process generated by the counter is discussed and rules are established for obtaining confidence intervals for the parameter m from various types of counting experiments.

1. The counter selection principle; formulation of the distribution problem. A counter observes a segment of a sequence $\{f\}$ of events $f_1, f_2, f_3, \cdots$ that are randomly spaced on the positive time axis, $t > 0$. Due to an inherent resolving time, the instrument may fail to record all of the events of $\{f\}$ that occur during the interval of observation. Thus the recorded events (registrations by the counter) form a second sequence $\{g\}$ of events $g_1, g_2, g_3, \cdots$ also randomly spaced in time with a distribution law that depends upon that of $\{f\}$ and upon the mathematical model used for the action of the counter. Thus, a precise rule for the selection by the counter of the sequence $\{g\}$ from the sequence $\{f\}$ must be specified. Two such rules have received attention in the literature on counters. Briefly, they are as follows.

In a Type I counter there is a fixed resolving time $u > 0$ such that an event of $\{f\}$ is selected for $\{g\}$ if and only if no event of $\{g\}$ has taken place during the preceding time interval of length u. In a Type II counter the resolving time is random and specified by the rule that an event of $\{f\}$ is selected for $\{g\}$ if and only if no event of $\{f\}$ has taken place during the preceding time interval of length u.

In a Type II counter an event of $\{f\}$ that occurs while the counter is locked prolongs the locked period. In theory the counter can remain locked indefinitely. This is not true of a Type I counter. It has been recognized by some authors on this subject that actual counters select $\{g\}$ from $\{f\}$ in some manner that is a compromise between the types I and II rules; (Feller [2]). Such a compromise will

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be proposed and used in this paper. It includes the above rules as special cases. Briefly, it will be assumed that the Type II rule holds with the exception that an event of $\{f\}$ that occurs while the counter is locked may or may not prolong the locked period subject to chance. This will be made more precise in the following abstraction.

General rule of selection. Specify a positive number u and a number θ in the range $0 \leqq \theta \leqq 1$. A sequence $\{f\}$ of events is assumed generated on the positive time axis by some (physical) stochastic process. The successive events of $\{f\}$ will be denoted by $f_1, f_2, f_3, \cdots$. A subsequence of these, $f_{k_1}, f_{k_2}, f_{k_3}, \cdots$, will be selected to be the respective elements $g_1, g_2, g_3, \cdots$ of $\{g\}$ according to the following:

(i) The event f_1 is selected as g_1; that is, $k_1 = 1$.

(ii) Following each selection of an event f_{k_n} as g_n, there is a time interval τ_n, to be defined in (iii), during which no further selection for $\{g\}$ may be made. That event of $\{f\}$ *first* following g_n by time at least τ_n is selected as g_{n+1}. That is, k_{n+1} is the smallest integer such that the time interval between f_{k_n} and $f_{k_{n+1}}$ is of length τ_n or more.

(iii) The time τ_n is random in the range $\tau_n \geqq u > 0$. It is selected as follows. From the segment of the sequence $\{f\}$

$$(1) \qquad f_{k_n}, f_{k_n+1}, f_{k_n+2}, \cdots,$$

choose a subsequence $\{f^{(n)}\}$ whose elements are as follows. The first element f_{k_n} of (1) is the first element $f_1^{(n)}$ of $\{f^{(n)}\}$; the succeeding events of (1) are considered in succession and independently as candidates for $\{f^{(n)}\}$. Each is either selected or rejected for $\{f^{(n)}\}$ with probabilities θ and $1 - \theta$ respectively. Let $f_s^{(n)}$ be the *first* element of $\{f^{(n)}\}$ such that the time interval between $f_s^{(n)}$ and $f_{s+1}^{(n)}$ is u or more in length. The time τ_n is defined to be the sum of u and the time from $f_1^{(n)}$ to $f_s^{(n)}$. (Observe that $f_{s+1}^{(n)}$ is not necessarily the element g_{n+1} defined in (ii) above.)

Note that if $\theta = 0$, the sequence $\{f^{(n)}\}$ consists of the single element f_{k_n}. Thus, $\tau_n = u$ and the general rule reduces to the Type I rule. If $\theta = 1$, $\{f^{(n)}\}$ is the sequence (1) and the general rule becomes the Type II rule. Other values of θ lead to a compromise between those two rules.

Some notations will be needed before proceeding. Probability functions and cumulative distributions and their dependence upon parameters and conditions will be denoted as in Cramér [1]. Let $S_1, S_2, \cdots, S_n, \cdots$ denote the times of occurrence of the successive events of $\{g\}$. From the standpoint of a counter, each of the intervals $S_{n+1} - S_n$, $n \geqq 1$, is the sum of two parts, a time τ_n during which the counter is inoperative following the occurrence of g_n and a time T_n during which the counter is ready to make the next selection g_{n+1}. Thus,

$$(2) \qquad S_{n+1} - S_n = \tau_n + T_n, \qquad n \geqq 1.$$

It is implicitly assumed in the rule of selection that the counter is unlocked at time zero. To complete the above notations, let $T_0 = S_1$. Define the quantities ξ_n, η_n, ζ_n by

$$\text{(3)} \qquad \begin{cases} \xi_n = \sum_{j=0}^{n-1} T_j, \\ \eta_{n-1} = \sum_{j=1}^{n-1} (\tau_j - u), \qquad n \geqq 2, \qquad \eta_0 = 0, \\ \zeta_n = S_n - (n-1)u. \end{cases}$$

Actually a counter generates only a segment of the sequence $\{g\}$; that is, a counter observes $\{f\}$ only during some finite time interval $0 < t \leqq T$. Two types of intervals of observation are used in practice. The number T may be the random time S_N where N is some preassigned integer or T may itself be preassigned. The first case will be designated as a *fixed count* experiment. The distribution function of interest is

$$\text{(4)} \qquad H_N(t) = P(\zeta_N \leqq t), \qquad t > 0.$$

The second case will be designated as a *fixed time* experiment. Here the number $n(T)$ of events $g_1, g_2, \cdots, g_{n(T)}$ of $\{g\}$ that occur in $0 < t \leqq T$ is a random variable with a distribution function

$$\text{(5)} \qquad P[n(T) \leqq N; T], \qquad N = 0, 1, 2, \cdots,$$

which may be expressed in terms of (4). Let $P_k \equiv P[n(T) = k; T]$, $k = 0, 1, 2, \cdots$. Clearly, $P_0 = P(S_1 > T) = 1 - H_1(T)$ and for $k \geqq 1$, $P_k = P(S_k \leqq T, S_{k+1} > T; T) = H_k[T - (k-1)u] - H_{k+1}[T - ku]$. It follows that

$$\text{(6)} \qquad P[n(T) \leqq N; T] = 1 - H_{N+1}(T - Nu).$$

Thus the determination of the function (4) solves the distribution problem for either type of observation interval.

2. Calculation of the distributions for a Poisson case. Two general theories have been given in the literature for the treatment of a distribution problem such as that in hand; see Feller [2] and Malmquist [7]. Malmquist introduces certain auxiliary distributions which the present authors feel are extraneous to the counter problem. Feller presents a simple, elegant theory based upon operational calculus. His method will be followed here.

It will be assumed that the time from an arbitrary point A on the positive time axis to the first event of $\{f\}$ that follows A is a random variable with cumulative distribution

$$\text{(7)} \qquad F(t; m) \equiv 1 - \exp(-mt), \qquad \text{m} > 0 \text{ constant},$$

and that any number of such intervals which are nonoverlappling in pairs, form a set of independent variables. It follows that the 2N-1 terms on the right in

$$\zeta_N = \sum_{j=0}^{N-1} T_j + \sum_{j=1}^{N-1} (\tau_j - u) \tag{8}$$

form a set of independent random variables. The T_j in (8) all have the same distribution $P(T_j \leqq t; m) = F(t; m)$ and the $\tau_j - u$ have a common distribution $P(\tau_j - u \leqq t; m, \theta) \equiv G(t; m, \theta)$ which must be determined by application of the general rule of selection.

The Laplace-Stieltjes transform is a convenient operational tool for the derivation of the distribution of the sum of independent nonnegative random variables. Define the transforms

$$\begin{aligned} \varphi(s) &= \int_0^\infty [\exp(-st)]\, dF(t; m), \\ \psi(s) &= \int_0^\infty [\exp(-st)]\, dG(t; m, \theta), \\ \chi_N(s) &= \int_0^\infty [\exp(-st)]\, dH_N(t; m, \theta). \end{aligned} \tag{9}$$

where the dependence of H_N defined in (4) upon the parameters m and θ has been put in evidence. By well known rules

$$\chi_N(s) = [\varphi(s)]^N [\psi(s)]^{N-1}. \tag{10}$$

By (7) the transform $\varphi(s)$ is $(1 + s/m)^{-1}$ so that

$$[\varphi(s)]^N = (1 + s/m)^{-N}. \tag{11}$$

The inverse of (11) is known:

$$F_N(t, m) = \frac{m^N}{\Gamma(N)} \int_0^t [\exp(-mx)] x^{N-1}\, dx. \tag{12}$$

In order to find $\psi(s)$, the following lemma concerning part (iii) of the general rule of selection is needed.

LEMMA. *Let n and ν be arbitrary integers and I an arbitrary time interval of length t_I. The probability of the occurrence of exactly ν events of the sequence $\{f^{(n)}\}$ in the interval I is*

$$P(\nu) = \exp(-m\theta t_I) \cdot (m\theta t_I)^\nu / \nu!. \tag{13}$$

The proof is well known since the distribution of the number of events of $\{f^{(n)}\}$ in I is the compound distribution of the Poisson probability $\{\exp(-mt_I)\}(mt_I)^\mu/\mu!$ for the number μ of events of the sequence (1) in I and the binomial probability $\binom{\mu}{\nu}\theta^\nu(1 - \theta)^{\mu-\nu}$ for the number ν of those μ events selected for $\{f^{(n)}\}$; (see Feller [3]).

Let Δ denote the random interval length between successive events of the sequence $\{f^{(n)}\}$. By (13) the probability that at least one event of $\{f^{(n)}\}$ occurs in the interval $(0, t)$ is $1 - \exp(-mt\theta)$. Thus the conditional distribution of Δ subject to the condition $\Delta < u$ is

$$(14) \qquad P(\Delta \leqq t \mid \Delta < u; m, \theta) = \begin{cases} \dfrac{1 - \exp(-m\theta t)}{1 - \exp(-m\theta u)}, & 0 < t < u. \\ 1, \ t \geqq u. \end{cases}$$

The transform of (14) is

$$(15) \qquad m\theta\{1 - \exp[-(m\theta + s)u]\}/[1 - \exp(-m\theta u)]\cdot(m\theta + s).$$

In the sequence $\{f^{(n)}\}$ suppose that exactly k successive events $f_1^{(n)}, f_2^{(n)}, \cdots, f_k^{(n)}$ occur separated successively by times less than u and that $f_{k+1}^{(n)}$ succeeds $f_k^{(n)}$ by time u or more. The probability of such an occurrence is

$$(16) \qquad [1 - \exp(-m\theta u)]^k \exp(-m\theta u)$$

and by the general rule of selection, the conditional distribution of $\tau_n - u$ subject to the above occurrence is the k-fold convolution of (14). It follows that the transform $\psi(s)$ is the sum over $k = 0, 1, 2, \cdots$ of the product of (16) by the kth power of (15). This summation readily gives

$$(17) \qquad \psi(s) = (m\theta + s)\cdot\exp(-m\theta u)/\{m\theta\cdot\exp[-(m\theta + s)u] + s\}.$$

For more detail in the above argument, see Feller [2] where the case $\theta = 1$ is treated. In deriving (17), it was assumed without mentioning that $\theta > 0$. The result persists for $\theta = 0$ since in that case it reduces to $\psi(s) \equiv 1$ which means that $P(\tau_n = u) = 1$ in agreement with the Type I rule of selection.

Inversion of the transform $[\psi(s)]^{N-1}/s$ gives the cumulative distribution $G_{N-1}(t; m, \theta)$ of η_{N-1} defined in (3). The result is

$$(18) \qquad \begin{aligned} G_{N-1}(t; m, \theta) = {} & \exp[-(N - 1)m\theta u] \sum_{n=0}^{M} (-1)^n \binom{N - 2 + n}{n} \\ & \exp(-nm\theta u) \sum_{r=0}^{N-1} \binom{N - 1}{r} \frac{[m\theta(t - nu)]^{n+r}}{(n + r)!}, \end{aligned}$$

where M is the largest integer such that $Mu < t$.

It is interesting to note that Raff [8] has used a special case of (18) as a waiting time distribution in a certain traffic problem.

By (10), the convolution

$$(19) \qquad H_N(t; m, \theta) = \int_0^t F_M(t - x; m)\, dG_{N-1}(x; m, \theta)$$

using (12) and (18) gives the distribution function (4). The results of the convolution (19) are so complicated as to be almost useless. In performing the integration, it must be noted that G_{N-1} is discontinuous at $t = 0$ and that the integral is in the Stieltjes sense. Taking these facts into account, it will be convenient for later purposes to write (19) in the form

$$H_N(t; m, \theta) = \exp[-(N - 1)m\theta u]F_N(t; m) + [1 - \exp\{-(N - 1)m\theta u\}]Q_N(t; m, \theta) \tag{20}$$

where $Q_N(t; m, \theta)$ is the conditional distribution of ζ_N subject to the condition that $\eta_{N-1} > 0$. The acutal form of Q_N will not be used in the sequel. The interested reader can easily calculate it.

In the case of a Type I counter, $\theta = 0$ and the right member of (20) reduces to (12). If $\theta > 0$, but $(N - 1)mu$ is small, (20) is approximated by (12) so that the counter behaves nearly like a Type I. Whether or not the accuracy of the approximation is sufficient is a matter of judgement for the individual reader. The remarks following Theorem 3 in Section 5 below give further aid in such judgement. The asymptotic results derived in the next section apply to cases in which N/mu is large.

3. Asymptotic percentage points of $\mathbf{H_N(t; m, \theta)}$. Let p be an assigned number in the range $0 < p < 1$ and define the percentage point t_p by means of

$$p = P(\zeta_N \leqq t_p; m, \theta) = H_N(t_p; m, \theta). \tag{21}$$

Exact calculation of t_p appears feasible only in cases where H_N reduces to (12). Define the parameters

$$M = mu, \qquad \lambda = N/M. \tag{22}$$

In a well designed counting experiment λ will usually be quite large. Under this condition approximate normalization of H_N is permissible for the calculation of t_p.

The distribution function of the random variable $m\zeta_N/\lambda$ will be denoted by $H_N^*(t: m, \theta)$. Its transform is given by

$$\chi_N^*(s) = \chi_N(ms/\lambda) \tag{23}$$

where (10) is to be used on the right. The cumulants of the distribution H_N^* are readily found from (23) on inserting (11) and (17) in (10). They are

$$\begin{aligned} \kappa_1 &= M + (M\lambda - 1)(\lambda\theta)^{-1}(e^{M\theta} - 1 - M\theta), \\ \kappa_r &= (r - 1)!\Big[M\lambda^{1-r} - (M\lambda - 1)(\lambda\theta)^{-r} \\ &\quad \cdot \Big\{1 - \sum_{j=1}^{r} \frac{r}{j} \cdot \frac{1}{(r - j)!}(-jM\theta)^{r-j}e^{jm\theta}\Big\}\Big], \qquad r > 1. \end{aligned} \tag{24}$$

Let x_p and t^* be defined by

$$p = (2\pi)^{-1/2}\int_{-\infty}^{x_p} \exp(-\tfrac{1}{2}x^2)\, dx = H_N^*(\kappa_1 + \kappa_2^{1/2}t_p^*; m, \theta).$$

Also, define the constants l_r, $r = 1, 2, 3, \cdots$, as

$$l_1 = l_2 = 0, \qquad l_r = \kappa_r/\kappa_2^{\frac{1}{2}r}, \qquad r \geqq 3. \tag{25}$$

An application of the formula (6.75) in Kendall [5] gives

$$(26)\quad t_p^* = x_p + \frac{l_3}{6}(x_p^2 - 1) + \frac{l_4}{64}(x_p^3 - 3x_p) - \frac{l_3^2}{36}(2x_p^3 - 5x_p) + O(\lambda^{-3/2}).$$

Only a part of Kendall's formula has been retained. The interested reader may obtain the terms of orders $O(\lambda^{-3/2})$ and $O(\lambda^{-2})$ on recognizing that Kendall's symbols x and ξ are respectively the t_p^* and x_p of (26). (The reader should be warned that the reference formula contains several misprints in the third edition of Kendall's book.) The percentage point t_p defined in (21) is given by (26) and

$$(27)\qquad mt_p/\lambda = \kappa_1 + \kappa_2^{1/2} t_p^* .$$

4. Confidence intervals for m; Type I counter. The general methods to be employed in this section are described in Cramér [1] and Kendall [6].

The object of most counting experiments is the estimation of the mean rate m of occurrence per unit time of the events of the sequence $\{f\}$. The fundamental resolving time u of the counter is usually regarded as known; it will be small. For radioactivity counters u is of the order of magnitude of 10^{-4} to 10^{-8} second depending upon the refinement of the counter. It will be convenient to establish confidence intervals for the parameter M defined in (22).

A precise argument can be given establishing confidence intervals for M for a I counter. The argument is difficult to make precise for a general type of counter.

In a great many counting experiments the product $(N - 1)mu$ will be small enough that (20) may be regarded as essentially equal to (12); that is, that the counter is of Type I. In this case the percentage point t_p defined in (21) is given by the exact equation

$$(28)\qquad p = \frac{1}{2\Gamma(N)} \int_0^{2m t_p} [\exp(-\tfrac{1}{2}x)] \cdot (\tfrac{1}{2}x)^{N-1}\, dx.$$

It will be recognized that $2mt_p$ is the percentage point of the chi square distribution with $2N$ degrees of freedom. In this case (26) and (27) give

$$(29)\quad Mt_p = Nu\{1 + x_p/N^{\frac{1}{2}} + (x_p^2 - 1)/3N + (x_p^3 - 7x_p)/36N^{3/2}\} + O(\lambda^{-1}).$$

Clearly, t_p is a monotone decreasing function of M; let this dependence be denoted by $t_p(M)$.

Suppose now that the counting experiment is of the fixed count type. Then N is assigned and ζ_N is the observed random variable. The equation

$$(30)\qquad t_p(L_p) = \zeta_N$$

defines a new random variable L_p such that, no matter what the true value of M is,

$$(31)\qquad P(L_p < M; M, \theta) = 1 - p.$$

Let p_1 and p_2, $p_1 < p_2$, be two values of p. By (28) $t_{p_1} < t_{p_2}$ and by (29) and (30) $L_{p_1} < L_{p_2}$. It follows by well known rules of probability theory that

$$P(L_{p_1} < M < L_{p_2}; M, \theta) = p_2 - p_1 .$$

This proves

THEOREM 1. (*Type I counter; fixed count experiment.*) *Let N be an assigned integer and p_1, p_2 assigned probabilities with $0 \leqq p_1 < p_2 \leqq 1$. An interval of 100 $(p_2 - p_1)$ percent confidence for $M = mu$ is given by*

$$L_{p_1} < M < L_{p_2} \tag{32}$$

in which the L_p are defined by

$$L_p\zeta_N = Nu\{1 + x_p/N^{\frac{1}{2}} + (x_p^2 - 1)/3N + (x_p^3 - 7x_p)/36N^{3/2}\} + O(\lambda^{-1}), \quad p = p_1, p_2, \tag{33}$$

and where ζ_N is defined by the last equation in (3).

Continuing with the Type I rule of selection ($\theta = 0$), suppose that a fixed time experiment is to be used in counting. Since the random variable $n(T)$ of this case is discrete valued, exact percentage point functions analogous to the $t_p(M)$ defined in (28) are not obtainable. The following procedure is based upon percentage points for $n(T)$ in terms of M so chosen that the confidence intervals derived from them have confidence levels greater than or equal to $p_2 - p_1$, the inequality on the probability being as close to an equality as is possible.

Assign p_1 and p_2, $0 \leqq p_1 < p_2 \leqq 1$, and let two functions $N_{p_i}(M)$, $i = 1, 2$, be defined as follows. Let $N_{p_1}(M)$ be the smallest integer such that

$$P\{n(T) \geqq N_{p_1}(M); T, M\} \leqq p_1 \tag{34}$$

and let $N_{p_2}(M)$ be the largest integer such that

$$P\{n(T) \geqq N_{p_2}(M); T, M\} \geqq p_2 . \tag{35}$$

These functions are the analogues of the percentage points $t_{p_i}(M)$ used in the previous case. It will be shown presently that (29) may again be used. By (6) and the Type I assumption, for any integer $N \geqq 1$,

$$\begin{aligned} P\{n(T) \geqq N; T, M\} &= F_N\{T - (N-1)u; m\} \\ &= \frac{1}{2\Gamma(N)} \int_0^{2m[T-(N-1)u]} \exp\left(-\tfrac{1}{2}x\right)\left(\tfrac{1}{2}x\right)^{N-1} dx, \end{aligned} \tag{36}$$

and $P\{n(T) \geqq 0; T, M\} = 1$. Using the integral (36), define two sequences $M_k^{(i)}$, $k = 0, 1, 2, \cdots, i = 1, 2$, by the equations

$$\begin{aligned} &M_0^{(i)} = 0, i = 1, 2, \\ &p_i = P\{n(T) \geqq k; T, M_k^{(i)}\}, \qquad i = 1, 2, \quad k \geqq 1. \end{aligned} \tag{37}$$

It is clear from the integrals involved that for each pair (i, k), $M_k^{(i)} < M_{k+1}^{(i)}$ and for any $k \geqq 1$, $M_k^{(1)} < M_k^{(2)}$. It follows at once from the definition (34) that

$$N_{p_1}(M) = k \text{ if } M_{k-1}^{(1)} < M \leqq M_k^{(1)}, \qquad k = 1, 2, 3, \cdots \tag{38}$$

and similarly from the definition (35) that

$$N_{p_2}(M) = k \text{ if } M_k^{(2)} \leqq M < M_{k+1}^{(2)}, \qquad k = 0, 1, 2, \cdots. \tag{39}$$

These two functions are monotone nondecreasing in M; the first is continuous on the left and the second continuous on the right. They are the stairstep percentage point functions familiar in the theory of confidence intervals for the parameter of a discontinuous distribution.

Define two new random variables as functions of $n(T)$ as follows: if $n(T) = k$, $k = 0, 1, 2, \cdots$,

$$\begin{aligned} L_{p_1}^*[n(T)] &= M_k^{(1)} \\ L_{p_2}^*[n(T)] &= M_{k+1}^{(2)}. \end{aligned} \tag{40}$$

Let x be an arbitrary number in the range $M_{k-1}^{(1)} < x \leqq M_k^{(1)}$. The inequality $L_{p_1}^* < x$ is equivalent to $n(T) \leqq k - 1$. Thus, by (38), $P\{L_{p_1}^* < x; T, M\} = P\{n(T) \leqq k - 1; T, M\} = 1 - P\{n(T) \geqq N_{p_1}(x); T, M\}$ for x in the above range. For an appropriate integer k, one may choose $x = M$ whatever M may be. Then by (34)

$$P\{L_{p_1}^* < M; T, M\} \geqq 1 - p_1$$

whatever M may be. Now let x be arbitrarily chosen in the range $M_k^{(2)} \leqq x < M_{k+1}^{(2)}$. By a similar argument $P\{L_{p_2}^* \leqq x; T, M\} = 1 - P\{n(T) \geqq N_{p_2}(x); T, M\}$. Again, using $x = M$ with an appropriate choice of k, (35) shows that

$$P\left\{L_{p_2}^* \leqq M; T, M\right\} \leqq 1 - p_2$$

regardless of what M may be. It follows that

$$P\{L_{p_1}^* < M < L_{p_2}^*; T, M\} \geqq p_2 - p_1.$$

This proves

THEOREM 2. (*Type I counter; fixed time experiment.*) *Let an interval of observation* $0 < t \leqq T$ *be assigned and let* $0 \leqq p_1 < p_2 \leqq 1$. *A confidence interval for the parameter* $M = mu$ *for confidence level at least* 100 $(p_2 - p_1)$ *percent is given by*

$$L_{p_1}^* < M < L_{p_2}^* \tag{41}$$

where the limits are the random variables defined by (40) *and* (37). (Results similar to this are given for the Poisson distribution by Garwood [4] and for the binomial distribution by Kendall [6]. It seems to have gone unnoticed that strict inequalities are obtainable.)

Practical calculations of the confidence limits (41) are effected as follows. Comparison of the integrals (28) and (36) indicates that an asymptotic formula for

$M_N^{(i)}$ is obtained by equating the quantity $M_N^{(i)} \cdot [T - (N - 1)u]$ to the right member of (29) with p replaced by p_i. Thus the limits (41) may be obtained from (33) with the following changes. First replace ζ_N in (33) by $T - (N - 1)\, u$ and then replace N throughout the resulting equation by $n(T)$ to obtain $L_{p_1}^*$ and by $n(T) + 1$ to obtain $L_{p_2}^*$. Example 2 of Section 6 illustrates this computation.

Central intervals ($p_1 = 1 - p_2$) obtained from Theorem 1 are optimal in the sense described by Kendall, [6] sections 19.10 through 19.12. Whether or not this is true for Theorem 2 is not clear to the authors. In Theorem 1 the random variable ζ_N defined in (8) reduces to the sum of N independent quantities T_j, $j = 0, 1, \cdots, N$-1, whose common probability density is the derivative $m \cdot \exp(-mt)$ of (7) on $t > 0$. The likelihood $L = m^N \exp(-\sum_{j=0}^{N-1} T_j)$ of the T_j satisfies the conditions of the above reference. It follows from Kendall's discussion that the central confidence intervals $(N/\zeta_N)(1 \pm x_{p_1}/N^{\frac{1}{2}})$ for m are asymptotically (relative to N) shortest on the average in a general class of intervals obtained by use of the central limit theorem. These optimal intervals agree asymptotically with the results of Theorem 1.

5. Confidence intervals for m; general counter type. Precise proofs of theorems analogous to Theorems 1 and 2 for the case of the general counter model would be very difficult. The distribution (19) being regarded as unusable, the entire argument must be based upon the asymptotic formulas (26), (27) for the percentage points t_p of (19). The vague nature of the error estimate in those formulas prohibits precise arguments and results. Further formal manipulations for which no general error estimate seems possible will be introduced presently.

Consider the general counter type. The formula (29) used in connection with the Type I counter will be replaced by

$$Mt_p = \lambda u[\kappa_1 + \kappa_2^{1/2}\, t_p^*] \tag{42}$$

in which t_p^* is given by (26) with the full forms of the cumulants (24) used. In this case, since $\theta > 0$, the ratios (25) depend upon M in a complicated manner. Indeed, the dependence of the right member of (42) upon M is so formidable as to make the formula almost useless for any general considerations. A simplification of (42) will be given below for small values of M. In most well designed counting experiments M will be quite small; the experimenter has some latitude in the choice of values of u to effect this. The restriction $M \leqq 0.1$ will not exclude many experiments; for example, in radioactivity counting with the best available counter, $u = 10^{-8}$ second and the restriction amounts to the requirement that the source emit no more than 10^7 particles per second in the direction of the counter. Fairly extensive numerical calculations of $t_p(M)$ performed by the authors indicate that an expansion of the right member of (42) as a series of powers of M retaining powers through M^4 gives a simple formula for $t_p(M)$ that should be accurate to within one tenth of one percent in most cases provided that $M \leqq 0.1$; the error decreases rapidly as M tends to zero. The case $M > 0.1$ will be mentioned briefly later.

The cumulants (24) are readily expressed as power series in M:

$$\kappa_r = M(r-1)!\,\lambda^{1-r} + (M\lambda - 1)(\lambda\theta)^{-r} \sum_{n=r+1}^{\infty} A_n^{(r)}(M\theta)^n,$$

(43)

$$A_n^{(r)} = (r!/n!) \sum_{k=0}^{r-1} (-1)^k \binom{n}{k} (r-k)^{n-1}, \qquad r = 1, 2, 3, \cdots.$$

By formal manipulations of these series one reduces (42) to

$$Mt_p = \{a_0 + a_1M^2 + a_2M^3 + a_3M^4 + \cdots\} + 0(\lambda^{-1}) \tag{44}$$

where the coefficients a_i are given by

(45)

$$a_0 = Nu\{1 + x_p/N^{1/2} + (x_p^2 - 1)/3N + (x_p^3 - 7x_p)/36N^{3/2}\},$$

$$a_1 = (N-1)u\theta/2,$$

$$a_2 = \frac{(N-1)u}{6}\{\theta^2 + x_p\theta/N^{1/2} - 2(x_p^2 - 1)\theta/3N + (13x_p^3 - 19x_p)\theta/36N^{3/2}\},$$

$$a_3 = \frac{(N-1)u}{6}\{\theta^3/4 + x_p\theta^2/N^{1/2} + (3 - 8\theta)\theta(x_p^2 - 1)/12N + [(13\theta - 12)x_p^3 - (19\theta - 30)x_p]\theta/36N^{3/2}\}.$$

To within the accuracy of the terms retained in (44) and (45), the derivative of $t_p(M)$ will be negative over the entire range $0 < M \leqq 0.1$ provided that a_2 and a_3 are positive and

$$a_1 + 0.2a_2 + 0.03a_3 < 100a_0. \tag{46}$$

If this is satisfied, $t_p(M)$ will be a monotone decreasing function of M in the indicated range. For instance, it is easy to show that (46) is satisfied for all values of θ in $0 \leqq \theta \leqq 1$ if x_p and N satisfy $1 \leqq x_p^2 \leqq N/4$. The investigation required by a given case is easily made.

Writing (44) in the form

$$-a_0 = t_pM + a_1M^2 + a_2M^3 + a_3M^4 + \cdots,$$

standard inversion formulas for power series may be applied to calculate M in terms of t_p and the a_i. The result of inverting the equation $t_p(L_p) = \zeta_N$ appears in (47) below.

An extension of Theorem 1 to the general counter is now immediate. Extension of Theorem 2 is then accomplished by the substitutions described immediately below the statement of Theorem 2. Thus

THEOREM 3. *(General counter.) Let* p_1 *and* p_2 *be assigned,* $0 \leqq p_1 < p_2 \leqq 1$. *If* $0 < M \leqq 0.1$ *and* (46) *is satisfied, an approximate confidence interval for the parameter* $M = mu$ *for confidence level* $100(p_2 - p_1)$ *percent is given by*:

(i) *for a fixed count experiment with count* N, *the interval* (32) *with the* L_{p_i} *obtained from*

$$L_p = \frac{a_0}{\zeta_N}\left\{1 + \frac{a_0 a_1}{\zeta_N^2} + \frac{a_0^2 a_2}{\zeta_N^3} + \frac{2a_0^2 a_1^2 + a_0^3 a_3}{\zeta_N^4} + \frac{5a_0^3 a_1 a_2}{\zeta_N^5} + \frac{5a_0^3 a_1^3}{\zeta_N^6} + \cdots\right\} + O(\lambda^{-1}), \qquad p = p_1, p_2; \tag{47}$$

(ii) *for a fixed time experiment in the interval* $0 < t \leqq T$, *the interval* (41) *with the* $L_{p_i}^*$ *obtained from* (47) *as follows. First replace* ζ_N *by* $T - (N - 1)u$ *and* L_p *by* L_p^*. *Then, for* $L_{p_1}^*$ *replace* N *by the count* $n(T)$ *and for* $L_{p_2}^*$ *replace* N *by* $n(T) + 1$.

If M is small and N is large, one expects each of the quantities a_r/ζ_N in (47) to be near the value M. In such cases the approximation

$$L_p \cong \frac{a_0}{\zeta_N}\left\{1 + \frac{a_0 a_1}{\zeta_N^2}\right\} \cong \frac{a_0}{\zeta_N}\left\{1 + \frac{N(N-1)u\theta}{2\zeta_N^2}\right\} \tag{48}$$

for (47) will suffice for practical purposes. This will likely be the situation in the great majority of counting experiments where confidence intervals are desired for the parameter m. Indeed, the term $a_0 a_1/\zeta_N^2$ in the bracket in (48) will usually be quite small compared with unity. These remarks indicate the extent to which the Type I counter assumption is justified; the examples in the next section should clarify these remarks.

The power series manipulations used in obtaining (44) and its inverse are valid for values of M greater than 0.1 and the monotoneity condition is easily extended. The authors cannot recommend the results for accuracy. The reader who may be interested in counting experiments for which $M > 0.1$ should graph $t_p(M)$ using the closed forms (24) of the cumulants κ_r in (26), (27) and (42) for values of the various parameters that are of interest. The range of monotoneity of $t_p(M)$ will then be evident and inversion of $t_p(L_p) = \zeta_N$ is easily performed from such graphs. It does not seem feasible to provide a sufficient number of such graphs in this paper to cover the multiplicity of conditions that might arise in counting experiments.

The complexity of the distribution (19) bars any discussion of minimum average length confidence intervals.

6. Examples. The three examples given below illustrate the use of Theorems 1, 2 and 3. In the first example the rate of $m = 2$ particles per second is typical of radio-isotope tracer experiments. The extremely low rate of four particles in eight hours of Example 2 might be found in a cosmic ray count while the very high rate of $m = 2500$ particles per second in Example 3 might be found in a nuclear physics laboratory.

In designing a counting experiment it is important to make use of Theorem 1 whenever possible in order to utilize its statistical and practical efficiency. This is brought out in the examples. In both of Examples 1 and 2 it turns out that confidence intervals for m are essentially independent of the counter type. Example 2 illustrates the construction used in the proof of Theorem 2 and compares precise results with asymptotic results. Example 3 shows the effects of variations in the resolving time u and the sample size N.

EXAMPLE 1. (Fixed count.) Suppose that the counting rate is expected to be about $m = 2$ particles per second. It is desired to obtain a 95.46 per cent ($x_{p_2} = -x_{p_1} = 2$) central confidence interval for m using a counter for which $u = 10^{-4}$ second and a five minute observation interval. Fix $N = 600$ and suppose that $\zeta_N = 290$ seconds is observed. Since $(N - 1)M$ is approximately 0.12, it appears from (20) that Theorem 3 should be used. By (45), keeping five decimal places of accuracy in the computation of the $a_i/u\zeta_N$, one finds $a_0/\zeta_N = 2.06897 (1.00167 \pm 0.08163)u$, $a_1/\zeta_N = 1.03276u \cdot \theta$, and $a_2/\zeta_N = 0.34425(\theta^2 - 0.00334 \cdot \theta \pm 0.08176 \cdot \theta)$, the upper signs being used for L_{p_2} and the lower for L_{p_1}. Since $u = 10^{-4}$, it is clear at once that the bracket in (47) is essentially unity. Thus the confidence interval is $1.90354 < m < 2.24132$. Note that the result is the same as would have been obtained from Theorem 1; this is true only to within the accuracy of the compuatation.

EXAMPLE 2. (Fixed time.) Suppose that m is expected to be about 1/120 particle per minute and that $u = 10^{-6}$ minute. Fix $T = 500$ minutes and suppose $n(T) = 4$ observed. Here, (20) indicates that the counter may be assumed to be of type I so Theorem 2 may be used. By (41), the limits are $L^*_{p_1} = M_4^{(1)}$ and $L^*_{p_2} = M_5^{(2)}$ on $M = mu$ and by (36) and (37), $2M_k^{(i)} \cdot [(T/u) - (k - 1)]$ is that value below which $100p_i$ per cent of the chi square distribution of $2k$ degrees of freedom lies. For a central 90 per cent interval $2M_4^{(1)} \cdot (5 \cdot 10^8 - 3) = 2.733$ and $2M_5^{(2)} \cdot (5 \cdot 10^8 - 4) = 18.307$. From these values, $0.00273 < m < 0.01831$. The asymptotic result obtained from (29) using $x_{p_2} = -x_{p_1} = 1.645$ is $0.00275 < m < 0.01832$ which agrees very well with the precise result.

EXAMPLE 3. (Fixed count, high rate.) Suppose that m is expected to be about 2500 particles per second. Three cases are considered:

(i) $u = 4 \cdot 10^{-5}$ second, $N = 75{,}000$ and $\zeta_N = 30.1$ seconds,

(ii) $u = 4 \cdot 10^{-5}$ second, $N = 300{,}000$ and $\zeta_N = 120.4$ seconds, and

(iii) $u = 4 \cdot 10^{-8}$, $N = 75{,}000$ and $\zeta_N = 30.1$ seconds. Examination of (20) indicates that Theorem 3 must be used in all cases. As in Example 1, 95.46 per cent central confidence intervals are given. Keeping the same order of accuracy as in Example 1, one finds for cases (i) and (iii) that $a_0/\zeta_N = 2491.7\ (1.00001 \pm 0.00730)u$, $a_1/\zeta_N = 1245.8u \cdot \theta$, and $a_2/\zeta_N = 415.3(\theta^2 \pm 0.01460 \cdot \theta)u$. For case (ii), replace the quantities enclosed in parentheses in a_0/ζ_N and a_2/ζ_N above by (1 ± 0.00365) and $(\theta^2 \pm 0.0073 \cdot \theta)$ respectively. Applying these in (47) and dropping terms in the bracket there that vanish to five decimal places, the results are:

(i) $2473.5 + 12.2\theta + 0.5\theta^2 < m < 2509.9 + 12.6\theta + 0.5\theta^2$

(ii) $2482.6 + 12.4\theta + 0.5\theta^2 < m < 2500.8 + 12.5\theta + 0.5\theta^2$

(iii) $2473.5 < m < 2509.9.$

The value of θ will certainly be unknown in most counting experiments. Indeed θ may be a purely fictitious parameter that should only be used to tie together the extreme cases of the Type I and Type II selection rules. Cases (i) and (ii) show a dependence of the limits on θ that is substantial relative to the length of the confidence intervals. Increasing N shortens the interval but has only a very slight effect upon the θ dependence.

REFERENCES

[1] H. CRAMÉR, *Mathematical Methods in Statistics*, Princeton University Press, 1946.

[2] W. FELLER, "On probability problems in the theory of counters," *Courant Anniversary Volume*, 1948.

[3] W. FELLER, *Probability Theory and Its Applications*, John Wiley and Sons, 1950.

[4] F. GARWOOD, "Fiducial limits for the Poisson distribution," *Biometrika*, Vol. 28 (1936), pp. 437–442.

[5] M. G. KENDALL, *The Advanced Theory of Statistics*, Vol. 1, Charles Griffen and Co., 4th edition, London, 1948.

[6] M. G. KENDALL, *The Advanced Theory of Statistics*, Vol. 2, Charles Griffen and Co., London, 1948.

[7] S. MALMQUIST, "A statistical problem connected with the counting of radioactive particles, *Ann. Math. Stat.*, Vol. 18 (1947), pp. 255–265.

[8] M. S. RAFF, "The distribution of blocks in an uncongested stream of automobile traffic," *J. Amer. Stat. Assoc.*, Vol. 46 (1951), pp. 114–123.