ON THE CONSTRUCTION OF GROUP DIVISIBLE INCOMPLETE BLOCK DESIGNS[1]

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1. Summary. It has been shown in [1] that all partially balanced incomplete block (PBIB) designs with two associate classes, can be divided into a small number of types according to the nature of the association relations among the treatments. One simple and important type is the group divisible (GD). The combinatorial properties of GD designs have been studied in [2] and the analysis along with that for other types is given in [1]. Here we give methods of constructing GD designs. These designs are likely to prove useful in agricultural, genetic and industrial experiments.

2. Introduction. An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be group divisible (GD) if the treatments can be divided into m groups, each with n treatments, so that the treatments belonging to the same group occur together in λ_1 blocks and treatments belonging to different groups occur together in λ_2 blocks. If $\lambda_1 = \lambda_2 = \lambda$ (say) then every pair of treatments occurs together in λ blocks and the design reduces to the well known balanced incomplete block (BIB) design.

It has been shown in [2] that the parameters $v, b, r, k, m, n, \lambda_1$ and λ_2 satisfy the following relations and inequalities.

$$v = mn, \qquad bk = vr \tag{2.0}$$

$$\lambda_1(n - 1) + \lambda_2 n(m - 1) = r(k - 1) \tag{2.1}$$

$$Q = r - \lambda_1 \geqq 0, \qquad P = rk - v\lambda_2 \geqq 0. \tag{2.2}$$

The GD designs were divided into three classes: (a) Singular GD designs characterized by $Q = 0$, (b) Semi-regular GD designs characterized by $Q > 0$, $P = 0$, (c) Regular GD designs characterized by $Q > 0$, $P > 0$. The combinatorial properties of each class were separately studied. These will be referred to at appropriate places so far as they are relevant to the problem of construction of GD designs, which is the main concern of this paper. We shall confine ourselves to the practically useful range $v \geqq 10$, $r \leqq 10$, $k \leqq 10$, and choose λ_1 and λ_2 not to exceed 3, except for a few singular and semi-regular designs of special interest.

As noted in [1] GD designs besides being a sub-class of PBIB designs [3], [4] with two associate classes, can also be regarded as a sub-class of inter- and intra-group balanced incomplete block (IIGBI) designs [5].

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3. Some types of balanced incomplete block (BIB) designs.

(a) The construction of GD designs can in many instances be made to depend on known solutions for BIB designs [6], [7], [8], [9]. We shall here bring together certain results with a view to subsequent use. The parameters of a BIB design will be denoted by a starred letter in order to distinguish them from the parameters of GD designs. Thus the number of treatments will be denoted by v^*, the number of blocks by b^*, the number of replications of each treatment by r^*, the number of treatments in each block by k^*, and the number of times any two treatments occur together in a block by λ^*. The design is said to be resolvable [10] if the blocks can be grouped in such a way that each group contains a complete replication.

(b) The simplest type of BIB design is the unreduced type with $k = 2$, the blocks of which are obtained by taking all possible pairs of t treatments. The parameters are

$$v^* = t, \quad b^* = t(t-1)/2, \quad r^* = t-1, \quad k^* = 2, \quad \lambda^* = 1 \tag{3.0}$$

We shall later use the fact that when t is even, the solution can be expressed in a resolvable form. For example, if $t = 6$, then we can write the 15 blocks as

$$\begin{array}{ccc} (1, 4), & (2, 3), & (0, \infty) \\ (2, 0), & (3, 4), & (1, \infty) \\ (3, 1), & (4, 0), & (2, \infty) \\ (4, 2), & (0, 1), & (3, \infty) \\ (0, 3), & (1, 2), & (4, \infty) \end{array} \tag{3.1}$$

where the treatments are 0, 1, 2, 3, 4 and ∞, and the three blocks in any particular row of (3.1) give a complete replication. In the general case when $t = 2u$ the solution can be generated by developing the initial blocks

$$(1, 2u-2), (2, 2u-3), \cdots, (u-1, u), (0, \infty) \quad \text{mod } (2u-1), \tag{3.2}$$

the treatment ∞ remaining unchanged. The designs (3.0) will be referred to as belonging to series (u).

(c) BIB designs with parameters

$$v^* = s^2, \quad b^* = s^2 + s, \quad r^* = s+1, \quad k^* = s, \quad \lambda^* = 1 \tag{3.3}$$

may be said to belong to the orthogonal series 1 ($OS1$). They are also called balanced lattices [11], and can be obtained from a complete set of orthogonal Latin squares [6], [7]. They can, however, be more readily obtained by using certain difference sets [12] due to one of the authors, which have been given in Table I, and whose use is explained below.

For example let $s = 4$. If we develop the difference set for $s = 4$, mod $(s^2 - 1)$, we get fifteen blocks of the BIB design

$$v^* = 16, \quad b^* = 20, \quad r^* = 5, \quad k^* = 4, \quad \lambda^* = 1. \tag{3.35}$$

They are given by the columns of the scheme

(3.4)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
3	4	5	6	7	8	9	10	11	12	13	14	0	1	2
4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
12	13	14	0	1	2	3	4	5	6	7	8	9	10	11.

The remaining blocks are obtained by starting with the block $0, s + 1, 2(s + 1)$, ∞ and deriving other blocks by adding $1, 2, \cdots, s$ to the treatments of this block, remembering that ∞ is invariant under addition. Thus 5 other blocks are given by the columns of the scheme

(3.45)

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
∞	∞	∞	∞	∞.

TABLE I

Difference sets for generating BIB designs belonging to the orthogonal series OS 1

s	Difference set	Modulus
2	1, 2	mod (3)
3	1, 6, 7	mod (8)
4	1, 3, 4, 12	mod (15)
5	1, 3, 16, 17, 20	mod (24)
7	1, 2, 5, 11, 31, 36, 38	mod (48)
8	1, 6, 8, 14, 38, 48, 49, 52	mod (63)
9	1, 13, 35, 48, 49, 66, 72, 74, 77	mod (80)

The design is resolvable, the ith replication being obtained by taking the ith block from (3.45), and the ith and every succeeding $(s + 1)$st block from (3.4). We may thus rearrange the twenty blocks and get the design in the form, where the replications are separated by vertical lines,

(3.5)

1	6	11	0	2	7	12	1	3	8	13	2	4	9	14	3	5	10	0	4
3	8	13	5	4	9	14	6	5	10	0	7	6	11	1	8	7	12	2	9
4	9	14	10	5	10	0	11	6	11	1	12	7	12	2	13	8	13	3	14
12	2	7	∞	13	3	8	∞	14	4	9	∞	0	5	10	∞	1	6	11	∞.

(d) BIB designs with parameters

$$v^* = b^* = s^2 + s + 1, \qquad r^* = k^* = s + 1, \qquad \lambda^* = 1 \tag{3.6}$$

may be said to belong to the orthogonal series 2 (0S 2). The solution for any design of 0S 2 can be obtained from the corresponding design of 0S 1 by taking

$s + 1$ new treatments, and by adding the ith new treatment to each block of the ith replication, and finally adding a new block containing all the new treatments. A solution is, however, more readily obtained by using the following difference sets due to Singer [13], which have been given in Table II.

Thus the blocks for the BIB design

$$v^* = b^* = 13, \quad r^* = k^* = 4, \quad \lambda^* = 1 \tag{3.65}$$

obtained by using the difference set corresponding to $s = 3$, are given by the columns of the scheme

$$\begin{array}{ccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 \\ 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8. \end{array} \tag{3.7}$$

TABLE II

Difference sets for generating BIB designs belonging to the orthogonal series OS 2

s	Difference set	Modulus
2	0, 1, 3	mod (7)
3	0, 1, 3, 9	mod (13)
4	0, 1, 4, 14, 16	mod (21)
5	0, 1, 3, 8, 12, 18	mod (31)
7	0, 1, 3, 13, 32, 36, 43, 52	mod (57)
8	0, 1, 3, 7, 15, 31, 36, 54, 63	mod (73)
9	0, 1, 3, 9, 27, 49, 56, 61, 77, 81	mod (91)
11	0, 1, 3, 12, 20, 34, 38, 81, 88, 94, 104, 109	mod (133)

(e) BIB designs which are resolvable or (and) for which $\lambda = 1$ are especially important for the construction of GD designs. We present in Table III designs of the type for which $r^* \leqq 11$, and which do not belong to the series u, OS 1 or OS 2 already considered. The reference to the series is in the notation used in [8]. In each case the complete solution can be developed from certain initial blocks.

Designs marked + are resolvable. For (2), (5) and (9) the initial blocks provide a complete replication. Hence, in developing, the replications remain separate. For (6) the first seven blocks provide a complete replication and when developed yield replications I–VII. Each of the other three initial blocks when developed yields a complete replication. The solutions given here have been taken or adapted from [8], [14] and [15]. In developing the initial blocks the suffixes and ∞ should be kept invariant. (For the use of binary symbols see Section 6.)

TABLE III

BIB designs which are resolvable or (and) for which $\lambda = 1$.

Serial no.	Series	v^*	b^*	r^*	k^*	λ^*	Initial blocks	Modulus
(1)	T_2	13	26	6	3	1	(1,3,9), (2,6,5)	mod (13)
(2)+	T_1	15	35	7	3	1	$(1_1,2_1,4_1)$, $(3_1,1_2,5_2)$, $(6_1,2_2,3_2)$, $(5_1,4_2,6_2)$, $(0_1,0_2,\infty)$	mod (7)
(3)	F_1	25	50	8	4	1	(00,01,41,13), (00,32,21,02)	mod (5, 5)
(4)	T_2	19	57	9	3	1	(1,7,11), (2,14,3), (4,9,6)	mod (19)
(5)+	F_2	28	63	9	4	1	$(01_1,02_1,10_2,20_2)$, $(21_1,12_1,22_2,11_2)$, $(01_2,02_2,10_3,20_3)$, $(21_2,12_2,22_3,11_3)$, $(01_3,02_3,10_1,20_1)$, $(21_3,12_3,22_1,11_1)$, $(00_1,00_2,00_3,\infty)$	mod (3, 3)
(6)+	(T_1)	21	70	10	3	1	$(0_1,0_2,0_3)$, $(1_1,2_1,4_1)$, $(1_2,2_2,4_2)$, $(1_3,2_3,4_3)$, $(3_1,5_2,6_3)$, $(3_2,5_3,6_1)$, $(3_3,5_1,6_2)$ Reps I–VII; $(1_1,2_3,4_2)$ Rep VIII; $(1_2,2_1,4_3)$ Rep IX; $(1_3,2_2,4_1)$ Rep X	mod (7)
(7)	(G_1)	41	82	10	5	1	(1,37,16,18,10), (8,9,5,21,39)	mod (41)
(8)	(G_2)	45	99	11	5	1	$(01_1,02_1,10_3,20_3,00_2)$, $(21_1,12_1,22_3,11_3,00_2)$, $(01_2,02_2,10_4,20_4,00_3)$, $(21_2,12_2,22_4,11_4,00_3)$, $(01_3,02_3,10_5,20_5,00_4)$, $(21_3,12_3,22_5,11_5,00_4)$, $(01_4,02_4,10_1,20_1,00_5)$, $(21_4,12_4,22_1,11_1,00_5)$, $(01_5,02_5,10_2,20_2,00_1)$, $(21_5,12_5,22_2,11_2,00_1)$, $(00_1,00_2,00_3,00_4,00_5)$	mod (3, 3)
(9)+	(B_1)	8	14	7	4	3	$(0,1,2,4)$, $(3,5,6,\infty)$	mod (7)

4. Construction of singular GD designs. It has been shown in [2] that if in a BIB design with parameters v^*, b^*, r^*, k^*, λ^* we replace each treatment by a group of n treatments, we get a singular GD design with parameters

$$v = nv^*, \quad b = b^*, \quad r = r^*, \quad k = nk^*,$$
$$m = v^*, \quad n = n, \quad \lambda_1 = r^*, \quad \lambda_2 = \lambda^*. \tag{4.0}$$

Conversely, every singular GD design is obtainable in this way from a corresponding BIB design. The problem of constructing singular GD designs, therefore, offers no difficulty. However, if r^* and λ^* differ too much, then in the derived GD design, the accuracy of the within group and between group comparisons will appreciably differ. We give in Table IV some cases of practical interest.

A singular GD design may be considered to belong to the same series as the corresponding BIB design. The series has been shown along with the serial number in Table IV. It is clear that if a BIB design is resolvable the same is true of a GD design derived from it. Resolvability has been denoted by +.

As an example consider design (11) of Table IV. The blocks of the corresponding BIB design are given by (3.7). Replacing each treatment i by two treatments

TABLE IV

Parameters of some singular GD designs, and the corresponding BIB designs from which they are derivable

Serial no. and series	Parameters of BIB design					Parameters of corresponding singular GD design							
	v^*	b^*	r^*	k^*	λ^*	v	b	r	k	m	n	λ_1	λ_2
(1) $u+$	4	6	3	2	1	12	6	3	6	6	3	3	1
(2) $u+$	4	6	3	2	1	16	6	3	8	6	4	3	1
(3) $u+$	4	6	3	2	1	20	6	3	10	6	5	3	1
(4) $OS\ 2$	7	7	3	3	1	14	7	3	6	7	2	3	1
(5) $OS\ 2$	7	7	3	3	1	21	7	3	9	7	3	3	1
(6) u	5	10	4	2	1	10	10	4	4	5	2	4	1
(7) u	5	10	4	2	1	15	10	4	6	5	3	4	1
(8) u	5	10	4	2	1	20	10	4	8	5	4	4	1
(9) $OS\ 1+$	9	12	4	3	1	18	12	4	6	9	2	4	1
(10) $OS\ 1+$	9	12	4	3	1	27	12	4	9	9	3	4	1
(11) $OS\ 2$	13	13	4	4	1	26	13	4	8	13	2	4	1

i_1 and i_2, we see that the blocks of the GD design under consideration, are given by the columns of the scheme

$$
\begin{array}{ccccccccccccc}
0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 7_1 & 8_1 & 9_1 & 10_1 & 11_1 & 12_1 \\
0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 7_2 & 8_2 & 9_2 & 10_2 & 11_2 & 12_2 \\
1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 7_1 & 8_1 & 9_1 & 10_1 & 11_1 & 12_1 & 0_1 \\
1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 7_2 & 8_2 & 9_2 & 10_2 & 11_2 & 12_2 & 0_2 \\
3_1 & 4_1 & 5_1 & 6_1 & 7_1 & 8_1 & 9_1 & 10_1 & 11_1 & 12_1 & 0_1 & 1_1 & 2_1 \\
3_2 & 4_2 & 5_2 & 6_2 & 7_2 & 8_2 & 9_2 & 10_2 & 11_2 & 12_2 & 0_2 & 1_2 & 2_2 \\
9_1 & 10_1 & 11_1 & 12_1 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 7_1 & 8_1 \\
9_2 & 10_2 & 11_2 & 12_2 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 7_2 & 8_2 .
\end{array} \tag{4.1}
$$

The treatments i_1 and i_2 belong to the same group ($i = 0, 1, \cdots, 12$). They occur together in the same block four times. Two treatments not belonging to the same group occur together in a block just once.

5. Method of "omitting varieties" for the generation of GD designs. Consider a BIB design with parameters

$$(5.0) \qquad v^*, \quad b^*, \quad r^*, \quad k^*, \quad \lambda^* = 1.$$

A particular treatment θ occurs in r blocks. The remaining $v^* - 1 = r^*(k^* - 1)$ treatments can be divided into r^* groups, each containing $k^* - 1$ treatments, two treatments belonging to the same group if they occur together in the same block with θ. If we form a new design by omitting the treatment θ, and all the blocks containing it, we evidently get a GD design with parameters

$$(5.1) \qquad \begin{array}{llll} v = v^* - 1, & b = b^* - r^*, & r = r^* - 1, & k = k^*, \\ m = r^*, & n = k^* - 1, & \lambda_1 = 0, & \lambda_2 = 1. \end{array}$$

THEOREM 1. *By omitting a particular treatment θ from a BIB design with parameters* (5.0), *we obtain a GD design with parameters* (5.1). *Two treatments belong to the same group if they occur together in the same block as θ.*

In particular, if we start with a BIB design belonging to the orthogonal series OS 1, with parameters given by (3.3), we get a series of regular GD designs with parameters

$$(5.2) \qquad v = b = s^2 - 1, \quad r = k = s, \quad m = s + 1, \quad n = s - 1, \quad \lambda_1 = 0, \quad \lambda_2 = 1.$$

The method of obtaining the blocks of a design of OS 1 using the difference sets in Table I has already been explained. To get the corresponding design of (5.2), it is convenient to omit the treatment ∞. Thus taking $s = 4$, the blocks of the GD design

$$(5.3) \qquad v = b = 15, \quad r = k = 4, \quad m = 5, \quad n = 3, \quad \lambda_1 = 1, \quad \lambda_2 = 0$$

are given by the columns of the scheme (3.4), and the groups are given by the columns of (3.45), if the last row containing only ∞ is omitted.

The BIB designs (1)–(8) of Table III may also be employed to generate corresponding GD designs. For example the blocks of

$$(5.4) \qquad v^* = 13, \quad b^* = 26, \quad r^* = 6, \quad k^* = 3, \quad \lambda^* = 1$$

obtained by developing the initial blocks given in Table III are

$$(5.5) \qquad \begin{array}{cccccccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 \\ 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4. \end{array}$$

Omitting the treatment 0, the blocks of the GD design

$$(5.6) \qquad \begin{array}{llll} v = 12, & b = 20, & r = 5, & k = 3, \\ m = 6, & n = 2, & \lambda_1 = 0, & \lambda_2 = 1 \end{array}$$

are given by the columns of the scheme

$$(5.7) \qquad \begin{array}{cccccccccccccccccccc} 1 & 2 & 3 & 4 & 6 & 7 & 8 & 9 & 10 & 12 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 11 & 12 & 1 \\ 3 & 4 & 5 & 6 & 8 & 9 & 10 & 11 & 12 & 1 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 2 & 3 & 5 \\ 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 7 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 4, \end{array}$$

and the groups are given by the columns in

$$(5.8) \qquad \begin{matrix} 5 & 11 & 2 & 9 & 10 & 4 \\ 7 & 6 & 8 & 12 & 1 & 3. \end{matrix}$$

We give in Table V parameters of BIB designs with $\lambda = 1$, together with the parameters of GD designs derivable from them by omitting a variety. Designs of the orthogonal series OS 2, and the semi-regular GD designs derivable from them have not been included, as the latter will be obtained in Section 7, as members of a more general class.

TABLE V

Parameters of BIB designs with $\lambda = 1$ not belonging to the series OS 2 and GD designs derivable from them by "omitting varieties"

Serial no. and series		Parameters of BIB design					Parameters of GD design							
		v^*	b^*	r^*	k^*	λ^*	v	b	r	k	m	n	λ_1	λ_2
(1)	OS 1	16	20	5	4	1	15	15	4	4	5	3	0	1
(2)	OS 1	25	30	6	5	1	24	24	5	5	6	4	0	1
(3)	OS 1	49	56	8	7	1	48	48	7	7	8	6	0	1
(4)	OS 1	64	72	9	8	1	63	63	8	8	9	7	0	1
(5)	OS 1	81	90	10	9	1	80	80	9	9	10	8	0	1
(6)	T_2	13	26	6	3	1	12	20	5	3	6	2	0	1
(7)	T_1	15	35	7	3	1	14	28	6	3	7	2	0	1
(8)	F_1	25	50	8	4	1	24	42	7	4	8	3	0	1
(9)	T_2	19	57	9	3	1	18	48	8	3	9	2	0	1
(10)	F_2	28	63	9	4	1	27	54	8	4	9	3	0	1
(11)	T_1	21	70	10	3	1	20	60	9	3	10	2	0	1
(12)	G_1	41	82	10	5	1	40	72	9	5	10	4	0	1
(13)	G_2	45	99	11	5	1	44	88	10	5	11	4	0	1

6. Method of differences for generating GD designs.

(a) The method of differences has been extensively used in [8] and [9] for the construction of BIB designs. We shall here adapt it to the construction of GD designs. Consider a module M with a finite number of elements. To each element let there correspond h treatments, the treatments corresponding to the element x being

$$(6.0) \qquad x_1, x_2, \cdots, x_h .$$

Thus there are $v = gh$ treatments. Treatments denoted by symbols with the same lower suffix i may be said to belong to the ith class.

Let $x_i^{(u)}$ and $x_j^{(v)}$ be two different treatments of the ith and jth classes respectively, where $x^{(u)}$ and $x^{(v)}$ are elements of M. Let

$$x^{(u)} - x^{(v)} = d, \qquad x^{(v)} - x^{(u)} = -d. \tag{6.1}$$

We then say that the pair of treatments $x_i^{(u)}$ and $x_j^{(v)}$ give rise to the difference d of the type $[i, j]$ and difference $-d$ of the type $[j, i]$. When $i = j$ the differences are called "pure" and when $i \neq j$ the differences are called "mixed". The differences d of the type $[i, j]$ and $-d$ of type $[j, i]$ are said to be "complementary" to one another. Thus every pair of treatments gives rise to a pair of complementary differences, one difference corresponding to each order of writing the treatments. Clearly there are h different types of pure differences and $h(h - 1)$ different types of mixed differences. Since every nonzero element of M can appear in a pure difference, and every element (zero or nonzero) in a mixed difference, the total number of different possible differences is

$$h(g - 1) + h(h - 1)g = v(v - 1)/g. \tag{6.15}$$

If θ is an arbitrary element of M and

$$x^{(\alpha)} = x^{(u)} + \theta, \qquad x^{(\beta)} = x^{(v)} + \theta, \tag{6.2}$$

then the pair of treatments $x_i^{(\alpha)}$ and $x_j^{(\beta)}$ give rise to the same pair of complementary differences as $x_i^{(u)}$ and $x_j^{(v)}$. Since θ can take g different values, we get g pairs of treatments giving rise to differences d and $-d$ of types $[i, j]$ and $[j, i]$ respectively, and it is easy to see that there are no other treatment pairs which give rise to the same differences. The $v(v - 1)/2$ treatment pairs thus give rise to just $v(v - 1)/g$ differences, which checks with (6.15).

Given an initial block B containing k treatments we can get g blocks by developing it in the following manner. Let θ be any arbitrary element of M. Then we get a new block B_θ corresponding to θ by replacing each treatment x_i in the initial block by x_i' where $x' = x + \theta$. By varying θ we get all the g required blocks. The initial block B gives rise to $k(k - 1)$ differences namely, the differences which arise from the $k(k - 1)/2$ pairs of treatments which can be formed from the treatments in B. If any pair of treatments occurs in B, then all the g pairs of treatments which give rise to the same differences as the given pair, occur in the corresponding positions in the blocks developed from B.

(b) THEOREM 2. *Let M be a module with m elements and to each element of M let there correspond n treatments. Let it be possible to find t initial blocks*

$$B_1, B_2, \cdots, B_t$$

each containing k treatments, and an initial group G containing n treatments such that

(i) *the $n(n - 1)$ differences arising from G are all different, and*

(ii) *among the $k(k - 1)t$ differences arising from the initial blocks each difference occurs λ_2 times, except those which arise from G, each of which occurs λ_1 times.*

Then by developing the initial blocks $B_1, B_2, \cdots, B_t$ we get the GD design with parameters $v = mn$, $b = mt$, $r = kt/n$, k, m, n, λ_1, λ_2, the group being obtained by developing the initial group G.

PROOF. Two treatments belong to the same group if and only if the differences arising from them occur among those arising from G. By the conditions of the theorem and what has been said before any such pair will occur among the developed blocks λ_1 times, and all other pairs will occur λ_2 times. Also in the developed blocks each treatment must occur in $(n - 1)\lambda_1 + n(m - 1)\lambda_2$ pairs. But if this treatment occurs in r blocks then this number of pairs is also $r(k - 1)$. Hence r must be the same for all treatments and is given by

$$(n - 1)\lambda_1 + n(m - 1)\lambda_2 = r(k - 1). \tag{6.25}$$

Again the total number of pairs in all the developed blocks is $mk(k - 1)t/2$ and this must equal $mnr(k - 1)/2$ since each treatment occurs in $r(k - 1)$ pairs. Hence $r = kt/n$. This completes the proof.

In particular let M be the module of residue classes mod (m), and let the initial group G consist of treatments

$$0_1, 0_2, \cdots, 0_n. \tag{6.3}$$

Then to get a GD design with parameters $v, b, r, k, m, n, \lambda_1, \lambda_2$ we have to find t initial blocks such that among the $k(k - 1)t$ differences arising from these blocks each pure difference and each nonzero mixed difference arises just λ_2 times, and each zero mixed difference arises λ_1 times. The designs (1)–(7) of Table VI have been obtained by using this special case of Theorem 2. For example for design (2) of Table VI, the complete set of blocks obtained by developing the given initial blocks mod (7), are given by the columns of the scheme

$$\begin{matrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 \\ 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 1_1 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_2 \\ 4_1 & 5_1 & 6_1 & 0_1 & 1_1 & 2_1 & 3_1 & 4_2 & 5_2 & 6_2 & 0_2 & 1_2 & 2_2 & 3_2 \\ 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1. \end{matrix} \tag{6.4}$$

The groups obtained by developing the initial group are given by the columns of the scheme

$$\begin{matrix} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 \\ 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2. \end{matrix} \tag{6.5}$$

(c) The scope of the method of differences can be further extended by using the concept of "partial cycle" (P.C.), (cf. [14]). We shall illustrate the use of this concept by considering a specific example.

Let M be the module of residue classes mod (15), and to each element of M let there correspond a unique treatment. Consider the set of treatments (0,3,6,9,12). This set cannot form an initial group for the purposes of Theorem 2, since the differences arising are not all different but are the elements 3,6,9, 12 each repeated five times. We however note that if we develop this set, then the complete cycle of 15 sets consists of the three sets (0,3,6,9,12), (1,4,7,10,13) and (2,5,8,11,14) each repeated five times. We can therefore say that the complete cycle is divisible into 5 equal parts. If we take only a partial cycle, namely 1/5 of

TABLE VI

GD designs which can be generated by the method of differences

Serial no.	Parameters v m	b n	r λ_1	k λ_2	Initial group	Initial blocks	Modulus
(1)	14 7	28 2	6 0	3 1	$(0_1,0_2)$	$(1_1,6_1,0_2)$, $(2_1,5_1,0_2)$, $(3_1,4_1,0_2)$, $(1_2,2_2,4_2)$	mod (7)
(2)	14 7	14 2	4 0	4 1	$(0_1,0_2)$	$(1_1,2_1,4_1,0_2)$, $(1_2,2_2,4_2,0_1)$	mod (7)
(3)	26 13	52 2	8 0	4 1	$(0_1,0_2)$	$(1_1,3_1,9_1,0_2)$, $(2_1,6_1,5_1,0_2)$, $(1_2,3_2,9_2,0_1)$, $(2_2,6_2,5_2,0_1)$	mod (13)
(4)	18 9	54 2	9 2	3 1	$(0_1,0_2)$	$(0_1,3_2,1_2)$, $(0_1,4_2,0_2)$, $(0_1,5_2,8_2)$, $(0_1,6_2,7_2)$, $(0_1,1_1,4_1)$, $(0_1,2_1,2_2)$	mod (9)
(5)	30 15	75 2	10 2	4 1	$(0_1,0_2)$	$(0_1,2_1,14_1,4_2)$, $(0_2,2_2,14_2,4_1)$, $(0_1,4_1,10_1,1_2)$, $(0_2,4_2,10_2,1_1)$, $(0_1,8_1,0_2,8_2)$	mod (15)
(6)	39 13	78 3	10 2	5 1	$(0_1,0_2,0_3)$	$(1_1,3_1,9_1,0_2,0_3)$, $(2_1,6_1,5_1,0_2,0_3)$, $(1_2,3_2,9_2,0_3,0_1)$, $(2_2,6_2,5_2,0_3,0_1)$, $(1_3,3_3,9_3,0_1,0_2)$, $(2_3,6_3,5_3,0_1,0_2)$	mod (13)
(7)	10 5	20 2	8 0	4 3	$(0_1,0_2)$	$(0_1,1_2,2_2,4_2)$, $(0_2,1_1,2_1,4_1)$, $(0_1,2_2,3_2,4_2)$, $(0_2,2_1,3_1,4_1)$	mod (5)
(8)	16 4	32 4	6 0	3 1	(0,4,8,12) $\frac{1}{4}$ P.C.	(0,1,10), (0,2,5)	mod (16)
(9)	24 4	72 6	9 0	3 1	(0,4,8,12,16,20) $\frac{1}{6}$ P.C.	(0,1,11), (0,2,7), (0,3,9)	mod (24)
(10)	15 5	30 3	6 0	3 1	(0,5,10) $\frac{1}{3}$ P.C.	(0,6,8), (0,11,14)	mod (15)
(11)	15 3	45 5	9 2	3 1	(0,3,6,9,12) $\frac{1}{5}$ P.C.	(0,6,12), (0,3,4), (0,2,7)	mod (15)
(12)	12 6	12 2	4 2	4 1	(0,6) $\frac{1}{2}$ P.C.	(0,1,4,6)	mod (12)
(13)	12 4	36 3	9 0	3 2	(0,4,8) $\frac{1}{3}$ P.C.	(0,1,3), (0,1,6), (0,2,5)	mod (12)

TABLE VI—*Cont.*

Serial no.	Parameters v m	 b n	 r λ_1	 k λ_2	Initial group	Initial blocks	Modulus
(14)	26 13	26 2	9 0	9 3	(0,13) ½ P.C.	(0,1,2,8,11,18,20,22,23)	mod (26)
(15)	35 5	70 7	10 2	5 1	(00,10,20,30,40,50,60) 1/7 P.C.	(10,20,40,01,04), (10,20,40,02,03)	mod (7, 5)
(16)	33 3	33 11	7 2	7 1	(00,10,20,30,40,50, 60,70,80,90,t0) 1/11 P.C.	(10,40,50,90,30,01,02)	mod (11, 3)
(17)	15 3	30 5	8 1	4 2	(00,10,20,30,40) 1/5 P.C.	(00,40,21,22), (00,20,11,12)	mod (5, 3)
(18)	15 5	30 3	10 2	5 3	(00,10,20) 1/3 P.C.	(00,10,21,22,24), (00,10,21,22,23)	mod (3, 5)
(19)	24 3	60 8	10 2	4 1	(00,30,60,90, 01,31,61,91) 1/8 P.C.	(00,10,40,91) C.C. (00,20,50,31) C.C. (00,60,01,61) ½ P.C.	mod (12, 2)
(20)	24 6	80 4	10 0	3 1	(00,20,40,60) ¼ P.C.	(00,10,61) C.C. (00,50,71) C.C. (00,11,42) C.C. (00,01,02) 1/3 P.C.	mod (8, 3)
(21)	12 3	30 4	10 2	4 3	(00,01,30,31) ¼ P.C.	(00,20,30,11) C.C. (00,10,50,41) C.C. (00,20,01,21) ½ P.C.	mod (6, 2)

the complete cycle for our groups, we see that any two treatments, the differences arising from which are 3,12 or 6,9, occur together just once in a group.

We now note that among the 18 differences arising from the initial blocks

(6.6) $$(0,6,12),\ (0,3,4),\ (0,2,7)$$

the elements 3,6,9,12 each occur twice, and the other nonzero elements, namely 1,2,4,5,7,8,10,11,13,14 each occur once. If, therefore, we develop these initial blocks mod (15) we get design (11) of Table VI, the groups consisting of 1/5 of the complete cycle obtained by developing the initial group (0,3,6,9,12). This is denoted by writing 1/5 P.C. after (0,3,6,9,12) in the column 3 of Table VI.

We may now state the following obvious generalization of Theorem 2.

THEOREM 3. *Let M be a module with cm elements and to each element of M let there correspond n/c treatments (c is supposed to be a divisor of n). Let it be possible to find t initial blocks each containing k treatments, and an initial group G containing n treatments such that:*

(i) *The differences arising from G consist of $n(n-1)/c$ different differences each repeated c times, the complete cycle of G being divisible into c equal parts.*

(ii) *Among the $k(k-1)t$ differences arising from the initial blocks each difference occurs λ_2 times, except the $n(n-1)/c$ differences arising from G, each of which occurs λ_1 times.*

Then by developing the initial blocks $B_1, B_2, \cdots, B_t$ we get the GD design with parameters $v = mn$, $b = mct$, $r = kct/n$, k, m, n, λ_1, λ_2, the groups being $1/c$th part of the complete cycle obtained by developing G.

In particular let $c = n$, and let M be the module of residue classes mod (mn), one treatment corresponding to each element of M. Let G be

$$(0, m, 2m, \cdots, m(n-1)). \tag{6.7}$$

Then the differences arising from G are the $n - 1$ elements $m, 2m, \cdots, (n-1)m$ each repeated n times. The complete cycle of G is divisible into n equal parts and we can get $1/n$ part of this cycle, by adding $0, 1, \cdots, m-1$ to the elements of G and taking residues mod (mn). This gives us the m groups. If it is possible to find the initial blocks $B_1, B_2, \cdots, B_t$ each with k treatments, such that the differences arising from them consist of the elements $m, 2m, \cdots, (n-1)m$ each repeated λ_1 times, and all other nonzero elements of M each repeated λ_2 times, then by developing $B_1, B_2, \cdots, B_t$ we get the blocks of the GD design with parameters $v = mn$, $b = mnt$, $r = kt$, m, n, λ_1, λ_2. Designs (8)–(14) of Table VI have all been obtained in this manner.

(d) In applying the method of differences, the use of systems of double modulus (u, v) is often advantageous. The elements of such a system are binary symbols xy, where x is a residue class mod (u) and y is a residue class mod (v). In adding two elements, we add the components separately and reduce the first component mod (u) and the second component mod (v).

In applying Theorem 3, using systems of double modulus we shall take $u = n$, $v = m$, so that M is a system of double modulus (n, m). We shall illustrate by considering design (18) of Table VI, where $m = 5$, $n = 3$. The initial group G is (00,10,20), and consists of all elements of M for which the second component is zero. The complete cycle of G consists of 15 groups divisible into 3 equal parts. One of these parts is obtained by adding to G all the element of M for which the first component is zero. The groups of this "partial cycle" are taken as our groups They are given by the columns of

$$\begin{matrix} 00 & 01 & 02 & 03 & 04 \\ 10 & 11 & 12 & 13 & 14 \\ 20 & 21 & 22 & 23 & 24. \end{matrix} \tag{6.8}$$

The fact that the groups are obtained by taking only ⅓ of the complete cycle obtainable from G is denoted by writing ⅓ P.C. after (00,10,20) in column 3 of Table VI. The differences arising from G are all the nonnull elements of M for which the second component is zero, each repeated 3 times. If we now note that among the forty differences arising from the initial blocks (00,10,21,22,24), (00,10,21,22,23) the elements 10,20 of M each occur twice, and the other nonnull elements of M each occur thrice, it follows from Theorem 3 that on developing these initial blocks we shall obtain all the blocks of design (18) of Table VI. Designs (15)–(18) of Table VI have all been obtained in this manner. In design (16), t stands for 10.

(e) Finally instead of considering only complete cycles developed from initial blocks, we may also allow partial cycles. This will be illustrated by considering design (20) of Table VI. M is here a system of double modulus (8,3). The initial group G consists of $n = 4$ elements (00,20,40,60). The differences arising from G are the elements 20,40,60 each occurring four times. For our groups we therefore take $\frac{1}{4}$ part of the complete cycle obtained by developing G. Our blocks should be such that two treatments differing by ± 20 or ± 40 should not be in the same block, but any two treatments the difference of which is anything else should occur in a block just once. Now the differences arising from the initial blocks (00,10,61), (00,50,71), (00,11,42) are all the elements of M (occurring once) except 20,40,60,01,02. Hence by developing these initial blocks we would get all pairs of treatments occurring together except those which differ by ± 20, ± 40, ± 01. We can therefore complete the solution by adding the initial block (00,01,02) and taking $\frac{1}{3}$ of the complete cycle obtainable from it, since the differences arising from it are 01 and 02 each repeated thrice. Designs (19) and (21) of Table V have also been obtained in a similar manner. The letters C.C. after an initial block mean that we have to take the complete cycle developed from it, whereas $1/n$ P.C. after an initial block means that only $1/n$ part of the complete cycle has to be taken. Of course this notation has been used only for those designs in which some of the initial blocks have partial cycles.

It should be noted that Theorem 3 when properly interpreted remains valid even when some of the initial blocks have partial cycles. If $1/s$ part of the cycle arising from a block is taken, then this block counts only as $1/s$ blocks, and the differences arising from it count only as $k(k-1)/s$ differences (i.e., every set of s identical differences counts only as one). Thus in design (20) of Table VI, the number of initial blocks is $t = 10/3$ since only $\frac{1}{3}$ of the cycle of the last initial block is taken. Since to each element there corresponds only one treatment $c = n$, the relation $r = kct/n$ is seen to remain valid. The $k(k-1)t$ differences arising from the initial blocks are the 6×3 differences arising from the first three initial blocks, together with the two differences arising from the last initial block.

7. Construction of semi-regular GD designs with $\lambda_1 = 0$.

(a) For a semi-regular GD design $P = rk - v\lambda_2 = 0$ by definition. Hence from (2.0) and (2.1)

$$r = \lambda_2 n - \lambda_1(n-1). \tag{7.0}$$

In this section we shall consider the case $\lambda_1 = 0$. This leads to $r = \lambda_2 n$, $k = m$. Hence the parameters of the design can be written as

$$v = mn, \qquad b = n^2\lambda_2, \qquad r = n\lambda_2, \qquad k = m, \qquad m, n, \lambda_1 = 0, \lambda_2. \tag{7.1}$$

We shall first establish the equivalence of the design (7.1) with an orthogonal array $A = [\lambda_2 n^2, m, n, 2]$ of strength 2, which may be defined as a matrix $A = (a_{ij})$, with m rows and $\lambda_2 n^2$ columns for which each element a_{ij} is one of the integers $0, 1, 2, \cdots, n-1$, and which has the orthogonality property that for any two rows, say i and u, the pairs $(a_{ij}, a_{uj}), j = 1, 2, \cdots, \lambda_2 n^2$ occurring in the corresponding columns consist of all possible ordered pairs of the integers 0, 1, $2, \cdots, n-1$, each repeated λ_2 times. It follows that each of the integers $0, 1, 2, \cdots, n-1$ appears $n\lambda_2$ times in each row of A. Orthogonal arrays have been studied by Plackett and Burman, Rao, Bush and one of the authors (Bose), [16], [17], [18], [19], [20], [21], [22].

THEOREM 4. *The existence of a semi-regular GD design with parameters* (7.1) *implies the existence of an orthogonal array* $A = [\lambda_2 n^2, m, n, 2]$ *of strength* 2, *and conversely.*

PROOF. Replace any integer x appearing in the ith row of A by the treatment $(i-1)n + x$. The ith row of the derived scheme now contains the treatments

$$(i-1)n, (i-1)n+1, \cdots, (i-1)n+n-1. \tag{7.15}$$

We shall show that the columns of the derived scheme give the blocks of the GD design (7.1), where the ith group of treatments is (7.15) Treatments belonging to the ith group occur only in the ith row of the derived scheme. Hence two treatments belonging to different groups never occur together in the same block (column). Also from the orthogonality property of A it follows that any two treatments belonging to different groups occur together in λ_2 blocks. This proves our statement.

Conversely, suppose there exists a semi-regular GD design with parameters (7.1). Let the ith group of treatments be given by (7.15), $i = 1, 2, \cdots, m$. It has been shown in [2] that each block of a semi-regular GD design contains the same number of treatments from each group. Since $k = m$ in the present case, each block contains just one treatment from each block. We can now exhibit the blocks of (7.1) as the columns of a rectangular scheme in which the treatments of the ith group occupy the ith row. Replacing the treatment

$$(i-1)n + x$$

of the ith group by x, $x = 1, 2, \cdots, n-1$, $i = 1, 2, \cdots, m$. We then get an orthogonal array A of size $\lambda_2 n^2$, m constraints, n levels and strength 2. This proves the equivalence of the orthogonal array A and the GD design (7.1).

COROLLARY. *The existence of GD design* (7.1) *implies the existence of the GD design with parameters*

$$\begin{array}{llll} v = m_1 n, & b = n^2\lambda_2, & r = n\lambda_2, & k = m_1, \\ m_1, & n, & \lambda_1 = 0, & \lambda_2 = 1 \end{array} \tag{7.2}$$

where $m_1 < m$.

If the GD design (7.1) is written in a form in which the columns give the blocks, and the treatments of the ith group appear only in the ith row, then to get the blocks of (7.2), we have simply to discard the last $m - m_1$ rows.

(b) In special cases the blocks of GD designs with parameters (7.1) can be obtained more expeditiously by using affine resolvable BIB designs or finite geometries rather than by directly using orthogonal arrays.

A resolvable BIB design is said to be affine resolvable if any two blocks of different replications have exactly the same number of treatments in common. It has been shown by one of the authors (Bose) [10], that the necessary and sufficient condition for a resolvable BIB design to be affine resolvable is

$$b^* = v^* + r^* - 1. \tag{7.25}$$

In this case the number of treatments common to blocks of two different replications is ${k^*}^2/v^*$, which must therefore be integral. The connection between orthogonal arrays and affine resolvable BIB designs was noticed by Plackett and Burman [19].

It is clear that if we dualize an affine resolvable BIB design with parameters $v^*, b^*, r^*, k^*, \lambda^*$, we get a semi-regular GD design with parameters

$$\text{(7.3)}\qquad \begin{gathered} v = b^*, \quad b = v^*, \quad r = k^*, \quad k = r^*, \\ m = r^*, \quad n = b^*/r^*, \quad \lambda_1 = 0, \quad \lambda_2 = {k^*}^2/v^*. \end{gathered}$$

In particular the BIB designs (3.3) belonging to the series $0S$ 1 are affine resolvable and lead by dualization to the blocks of the GD design

$$\text{(7.35)}\qquad \begin{gathered} v = s^2 + s, \quad b = s^2, \quad r = s, \quad k = s + 1, \\ m = s + 1, \quad n = s, \quad \lambda_1 = 0, \quad \lambda_2 = 1. \end{gathered}$$

From this we can get the blocks for (cf. Theorem 4, Corollary)

$$\text{(7.4)}\qquad \begin{gathered} v = ms, \quad b = s^2, \quad r = s, \quad k = m, \\ m, \quad n = s, \quad \lambda_1 = 0, \quad \lambda_2 = 1, \end{gathered}$$

where $m < s + 1$.

It will appear that we can express the blocks of (7.4) in a resolvable form. This will be illustrated by considering the special case $s = 4$. The columns of scheme (3.5) give the blocks of the BIB design $v^* = 16$, $b^* = 20$, $r^* = 5$, $k^* = 4$, $\lambda^* = 1$ in a resolvable form. Let us write down the dual of this design. The blocks of the dual corresponding to the treatments of the original can now be numbered 0, 1, 2, $\cdots$, 14 and ∞. Also the treatments of the dual corresponding to the blocks of the original can be numbered 1, 2, $\cdots$, 20, and can be divided into five groups corresponding to the replications. If in the original (3.5), the treatment i occurs in the block j in the dual we put the treatment j in the block i. The blocks of the dual are then given by the columns of the following scheme, where the last column corresponds to the block ∞.

(7.45)

4	1	2	1	1	4	2	3	2	2	4	3	1	3	3	4
7	8	5	6	5	5	8	6	7	6	6	8	7	5	7	8
11	11	12	9	10	9	9	12	10	11	10	10	12	11	9	12
13	15	15	16	13	14	13	13	16	14	15	14	14	16	15	16
19	17	19	19	20	17	18	17	17	20	18	19	18	18	20	20.

Finally, we rearrange the blocks so that all blocks containing the same treatment of the last group come together, and arrive at the scheme

(7.5)

1	4	3	2	2	4	1	3	4	2	1	3	1	2	3	4
8	5	6	7	8	6	7	5	7	5	6	8	5	6	7	8
11	9	12	10	9	10	12	11	11	12	9	10	10	11	9	12
15	14	13	16	13	15	14	16	13	15	16	14	13	14	15	16
17	17	17	17	18	18	18	18	19	19	19	19	20	20	20	20.

Taking only the first m rows of the scheme (7.5) the columns give the blocks of the semi-regular GD design

$$v = 4m, \qquad b = 16, \qquad r = 4, \qquad k = m, \tag{7.55}$$
$$m, \qquad n = 4, \qquad \lambda_1 = 0, \qquad \lambda_2 = 1$$

when $m < 5$, the design is in a resolvable form the replications being separated by the vertical lines.

(c) The connection between orthogonal arrays and finite geometries is given in [22]. We shall now illustrate the use of finite geometries in obtaining the blocks of semi-regular GD designs.

Consider the finite projective geometry PG(3, p^n), where p is a prime, and set $s = p^n$. There are exactly $s^2 + s + 1$ lines passing through any point O. Let us choose O = (0,0,0,1). Choose any $m \leqq s^2 + s + 1$ lines through O, and let the points other than O on these lines correspond to the treatments. We then have ms treatments divided into m groups, the s treatments corresponding to points on the same line forming a group. There are s^3 planes not passing through O. Each of these planes intersects a line through O in a unique point. Hence if we take these planes for blocks, then each block would contain exactly one treatment from each group. Also any treatment is contained in s^2 blocks. Two treatments belonging to the same group do not occur together in any block, but the points corresponding to two treatments of different groups are joined by a line through which s of the planes chosen for blocks pass. Hence two treatments not belonging to the same group occur together in s blocks. We thus get a semi-regular GD design with parameters

$$v = ms, \qquad b = s^3, \qquad r = s^2, \qquad k = m, \tag{7.6}$$
$$m, \qquad n = s, \qquad \lambda_1 = 0, \qquad \lambda_2 = s;$$

where $m \leqq s^2 + s + 1$.

We shall now show that if $m \leqq s^2$, then the blocks can be obtained in a resolvable form. Choose any plane through O, say $x_3 = 0$, and call it the fundamental plane. There are s^2 lines on the fundamental plane not passing through O. Through each of these lines there pass s planes chosen as blocks, which obviously give a complete replication provided that none of the m lines, the points of which (other than O) give the treatments, lie on the fundamental plane. Since there are s^2 lines through O not lying on the fundamental plane, we can get the blocks of (7.6) in a resolvable form if $m \leqq s^2$.

Again if $s^2 < m \leqq s^2 + s$, we can divide the blocks into s sets of s^2 each, such that the blocks of any set give s complete replications. This can be done by taking a fundamental line, say $x_2 = 0$, $x_3 = 0$. Let the lines whose points correspond to the treatments be different from the fundamental line. Then the s^2 blocks corresponding to planes passing through the same point of the fundamental line give s complete replications.

The equation of any plane not passing through O may be put in the form $ax_1 + bx_2 + cx_3 + x_4 = 0$ where a, b, c are elements of the Galois field GF(p^n). Varying a, b, c we get all the s^3 planes. The s planes, for which a and b remain fixed but c takes the s different possible values, give a complete replication (when none of the lines, whose points correspond to the treatments, lie in $x_3 = 0$), and the s^2 planes, for which a remains fixed, but b and c take all possible values, give a set of s complete replications (when $x_2 = 0$, $x_3 = 0$ is not one of the lines whose points correspond to the treatments). After the blocks have been calculated the points representing the treatments may be identified with the treatments $1, 2, \cdots, ms$.

Using PG(3, 2) we find that, if we retain only the first m rows of the scheme (7.7), then the columns represent the 8 blocks of the semi-regular GD design

(7.65)
$$v = 2m, \quad b = 8, \quad r = 4, \quad k = m, \qquad m, \quad n = 2, \quad \lambda_1 = 0, \quad \lambda_2 = 2.$$

The vertical lines separate the replications.

(7.7)

1	2	1	2	1	2	1	2
3	4	3	4	4	3	4	3
5	6	6	5	5	6	6	5
7	8	8	7	8	7	7	8
9	9	10	10	9	9	10	10
11	11	12	12	12	12	11	11
13	13	13	13	14	14	14	14

The groups for (7.65) are given by the first m columns of

(7.75)
$$\begin{array}{ccccccc} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14. \end{array}$$

Similarly using PG(3, 3) we find that, if we retain only the first m rows of the scheme (7.83), then the columns represent the 27 blocks of the semi-regular GD design

(7.8)
$$v = 3m, \quad b = 27, \quad r = 9, \quad k = m, \qquad m, \quad n = 3, \quad \lambda_1 = 0, \quad \lambda_2 = 3.$$

As before the vertical lines separate the replications.

	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
	4	5	6	4	5	6	4	5	6	5	6	4	5	6	4	5	6	4	6	4	5	6	4	5	6	4	5
	7	8	9	7	8	9	7	8	9	9	7	8	9	7	8	9	7	8	8	9	7	8	9	7	8	9	7
	10	11	12	11	12	10	12	10	11	10	11	12	11	12	10	12	10	11	10	11	12	11	12	10	12	10	11
	13	14	15	14	15	13	15	13	14	14	15	13	15	13	14	13	14	15	15	13	14	13	14	15	14	15	13
	16	17	18	17	18	16	18	16	17	18	16	17	16	17	18	17	18	16	17	18	16	18	16	17	16	17	18
	19	20	21	21	19	20	20	21	19	19	20	21	21	19	20	20	21	19	19	20	21	21	19	20	20	21	19
(7.83)	22	23	24	24	22	23	23	24	22	23	24	22	22	23	24	24	22	23	24	22	23	23	24	22	22	23	24
	25	26	27	27	25	26	26	27	25	27	25	26	26	27	25	25	26	27	26	27	25	25	26	27	27	25	26
	28	28	28	29	29	29	30	30	30	28	28	28	29	29	29	30	30	30	28	28	28	29	29	29	30	30	30
	31	31	31	32	32	32	33	33	33	32	32	32	33	33	33	31	31	31	33	33	33	31	31	31	32	32	32
	34	34	34	36	36	36	35	35	35	35	35	35	34	34	34	36	36	36	36	36	36	35	35	35	34	34	34
	37	37	37	37	37	37	37	37	37	38	38	38	38	38	38	38	38	38	39	39	39	39	39	39	39	39	39.

The groups for (7.8) are given by the first m columns of

	1	4	7	10	13	16	19	22	25	28	31	34	37
(7.86)	2	5	8	11	14	17	20	23	26	29	32	35	38
	3	6	9	12	15	18	21	24	27	30	33	36	39.

(d) Whenever an orthogonal array $[\lambda_2 n^2, m, n, 2]$ of strength 2 is directly available we can use it for obtaining the blocks of (7.1). The procedure to be followed has already been explained in the proof of Theorem 4.

Using the array [18, 7, 3, 2] given in [22], we get the blocks of the semi-regular GD design

$$v = 3m, \quad b = 18, \quad r = 6, \quad k = m, \tag{7.9}$$
$$m, \quad n = 3, \quad \lambda_1 = 0, \quad \lambda_2 = 2;$$

where $m \leqq 7$, by retaining only the first m rows of the scheme.

	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
	4	5	6	4	5	6	5	6	4	6	4	5	5	6	4	6	4	5
	7	8	9	8	9	7	7	8	9	9	7	8	9	7	8	8	9	7
(7.92)	10	11	12	12	10	11	12	10	11	10	11	12	11	12	10	11	12	10
	13	14	15	14	15	13	15	13	14	14	15	13	13	14	15	15	13	14
	16	17	18	18	16	17	17	18	16	17	18	16	18	16	17	16	17	18
	19	19	19	19	19	19	20	20	20	20	20	20	21	21	21	21	21	21.

As before, the blocks are given by columns, and the vertical lines divide complete replications. Thus the design is resolvable for $m \leqq 6$. The groups are given by the first m columns of

	1	4	7	10	13	16	19
(7.94)	2	5	8	11	14	17	20
	3	6	9	12	15	18	21.

Similarly using the array [32,9,4,2] given in [22] we can get the blocks of the semi-regular GD design

$$v = 4m, \quad b = 32, \quad r = 8, \quad k = m, \tag{7.96}$$
$$m, \quad n = 4, \quad \lambda_1 = 0, \quad \lambda_2 = 2$$

if $m \leqq 9$. The design can be obtained in a resolvable form if $m \leqq 8$.

Plackett and Burman [19] have given orthogonal arrays $[4\lambda, 4\lambda - 1, 2, 2]$ for all integral $\lambda \leqq 25$, except $\lambda = 23$. These may be used to obtain the blocks of the corresponding singular GD designs with parameters

TABLE VII

Parameters of semi-regular GD designs with $\lambda_1 = 0, \lambda_2 \leqq 3, r \leqq 10$

Serial no.	Parameters v	b	r	k	m	n	λ_1	λ_2	Maximum m	Maximum m for resolvability
(1)	$3m$	9	3	m	m	3	0	1	4	3
(2)	$4m$	16	4	m	m	4	0	1	5	4
(3)	$5m$	25	5	m	m	5	0	1	6	5
(4)	$6m$	36	6	m	m	6	0	1	3	2
(5)	$7m$	49	7	m	m	7	0	1	8	7
(6)	$8m$	64	8	m	m	8	0	1	9	8
(7)	$9m$	81	9	m	m	9	0	1	10	9
(8)	$10m$	100	10	m	m	10	0	1	3	2
(9)	$2m$	8	4	m	m	2	0	2	7	4
(10)	$3m$	18	6	m	m	3	0	2	7	6
(11)	$4m$	32	8	m	m	4	0	2	9	8
(12)	$5m$	50	10	m	m	5	0	2	6	5
(13)	$2m$	12	6	m	m	2	0	3	11	2
(14)	$3m$	27	9	m	m	3	0	3	13	9

$$v = 2m, \quad b = 4\lambda, \quad r = 2\lambda, \quad k = m, \tag{7.98}$$
$$m, \quad n = 2, \quad \lambda_1 = 0, \quad \lambda_2 = \lambda;$$

where $m \leqq 4\lambda - 1$. Of course only small values of λ and m yield designs of practical interest.

We present in Table VII the parameters of semi-regular GD designs for which $r \leqq 10, \lambda_1 = 0, \lambda_2 \leqq 3$, and the blocks for which can be obtained by the methods discussed in this section. The parameter m has been kept arbitrary, but the maximum value of m for which the design exists and also the maximum value of m for which the design can be obtained in a resolvable form has been given.

Number (12) is the duplicate of number (3), that is, is obtained by repeating each block of (3) twice. Numbers (4) and (8) can be obtained by first writing down the orthogonal array $[n^2, 3, n, 2]$ corresponding to an $n \times m$ Latin square

$n = 6, 10$, as it is well known that a set of $m - 2$ mutually orthogonal $n \times n$ Latin squares is equivalent to an orthogonal array $[n^2, m, n, 2]$, (cf. [18], [21]).

8. Construction of semi-regular GD designs for which $\lambda_1 \neq 0$, $\lambda_2 \neq 0$. Now $P = rk - v\lambda_2 = 0$ by definition, and $k = cm$ since each block contains the same number of treatments from each group [2]. Using (2.0) and (2.1), the eight parameters of the design can be expressed in terms of m, n, λ_2 and c only. Thus the parameters are

$$v = mn, \qquad b = n^2\lambda_2/c^2, \qquad r = n\lambda_2/c, \qquad k = cm, \tag{8.0}$$

$$m,\ n,\ \lambda_1 = n(c-1)\lambda_2/(n-1)c,\ \lambda_2. \tag{8.1}$$

Also as proved in [2] for a semi-regular GD design,

$$b \geqq v - m + 1. \tag{8.2}$$

$$m \leqq \frac{b-1}{n-1} = \frac{n^2\lambda_2 - c^2}{c^2(n-1)}. \tag{8.3}$$

TABLE VIIIA

Parameters of semi-regular GD designs with $\lambda_1 \neq 0$, $r \leqq 10$

Serial no.	Parameters								Maximum m
	v	b	r	k	m	n	λ_1	λ_2	
(1)	$4m$	12	6	$2m$	m	4	2	3	3
(2)	$3m$	9	6	$2m$	m	3	3	4	4
(3)	$6m$	20	10	$3m$	m	6	4	5	3

The values of n, c and λ_2 must be such as to make b, r and λ_1 integral, but m may be any integer subject to (8.3). It follows that, if $\lambda_1 \neq 0$, the only semi-regular GD designs in the range $r \leqq 10$ are those listed in Table VIIIA.

It is clear that, if the blocks and groups for the above designs can be obtained for the maximum value of m, then for any smaller value of m we have only to discard some of the groups and the treatments belonging to them. The groups and blocks for the designs in Table VIIIA are given in Table VIIIB (for the maximum value of m).

Here the groups have been given in full, and only the blocks have to be developed. The validity of the solution follows from the notion of differences developed in [8] and explained in section 6(a) of the present paper. For illustration we shall consider design (3) of Tables VIIIA and B, when m has the maximum value 3, and prove that the initial blocks shown give rise to it when developed.

The 18 treatments form three groups shown in the 2nd column of Table VIIIB. The 15 treatments other than ∞_1, ∞_2, ∞_3 fall into three classes according to the suffix carried (cf. section 6(a)). We shall distinguish three different types of pairs.

(i) Pairs of the type (∞_i, ∞_j); $i \neq j$; $i, j = 1, 2, 3$. Each of the three pairs (∞_1, ∞_2), (∞_2, ∞_3), (∞_3, ∞_1) occurs in just one initial block shown in the 3rd column of Table VIIIB. Since ∞ and the suffixes remain invariant when the blocks are developed, each of these pairs occurs five times in the completed design, as it should since $\lambda_2 = 5$ and ∞_i and ∞_j $(i \neq j)$ belong to different groups.

(ii) Pairs of the type (∞_i, u_j); $i, j = 1, 2, 3$; where u is an element of the field of residue classes, mod (5). When developed, the pair (∞_i, u_j) gives rise to five pairs, of which one component is ∞_i and the second component varies over all the five treatments of the jth class. In the initial blocks, ∞_i occurs with just 4 treatments of the jth class, if $i \neq j$, and 5 elements of the jth class, if $i \neq j$. It follows that any pair (∞_i, u_j) occurs 4 times in the completed design if $i = j$

TABLE VIIIB

Blocks and groups for semi-regular GD designs with $\lambda_1 \neq 0$, $r \leqq 10$

Serial no.	Groups	Initial blocks	Modulus
(1)	$(00_1,01_1,10_1,11_1)$ $(00_2,01_2,10_2,11_2)$ $(00_3,01_3,10_3,11_3)$	$(00_1,01_1; 00_2,10_2; 00_3,11_3)$ $(00_1,11_1; 00_2,01_2; 00_2,10_3)$ $(00_1,10_1; 00_2,11_2; 00_3,01_3)$	mod (2, 2)
(2)	$(0_1,1_1,2_1)$ $(0_2,1_2,2_2)$ $(0_3,1_3,2_3)$ $(\infty_1,\infty_2,\infty_3)$	$(0_1,1_1; 0_2,2_2; 0_3,2_3; \infty_2,\infty_3)$ $(0_1,2_1; 0_2,1_2; 0_3,2_3; \infty_3,\infty_1)$ $(0_1,2_1; 0_2,2_2; 0_3,1_3; \infty_1,\infty_2)$	mod (3)
(3)	$(0_1,1_1,2_1,3_1,4_1,\infty_1)$ $(0_2,1_2,2_2,3_2,4_2,\infty_2)$ $(0_3,1_3,2_3,3_3,4_3,\infty_3)$	$(0_1,1_1,2_1; 1_2,3_2,4_2; 0_3,1_3,2_3)$ $(\infty_1,3_1,4_1; \infty_2,0_2,2_2; 0_3,1_3,2_3)$ $(\infty_1,0_1,2_1; 0_2,1_2,2_2; \infty_3,0_3,2_3)$ $(1_1,3_1,4_1; \infty_2,3_2,4_2; \infty_3,0_3,2_3)$	mod (5)

and 5 times if $i \neq j$. This is as it should be, since ∞_i and u_j do or do not belong to the same group according as $i = j$ or $i \neq j$ and $\lambda_1 = 4$, $\lambda_2 = 5$.

(iii) Pairs of the type of (u_i, u_j); $i, j = 1, 2, 3$, where u is an element of the field of residue classes mod (5). It can be verified that leaving out ∞_1, ∞_2, ∞_3 the initial blocks give rise to each pure difference 4 times and each mixed difference 5 times. Hence in the completed design any pair (u_i, u_j) occurs 4 times if $i = j$ and 5 times if $i \neq j$, as it should, since u_i and u_j do or do not belong to the same group according as $i = j$ or $i \neq j$.

Again it is easy to see that each of the treatments ∞_1, ∞_2, ∞_3 occurs 10 times in the completed design, since each of these occurs twice in the initial blocks. The other treatments also occur 10 times in the completed design, since each class is represented 10 times in the initial blocks. This completes the proof.

If, in design (3) of Table VIIIA, $m = 2$, then the corresponding blocks can be obtained by developing the initial blocks shown in Table VIIIB, after dropping the treatments with suffix 3. It should be noted that the first two initial blocks now give a complete replication, and the same is true of the last two initial blocks. Hence the blocks are obtained in a resolvable form.

9. GD designs derivable by replication addition and subtraction. Consider a BIB design with parameters v^*, b^*, r^*, k^*, λ^* in which v^* is divisible by k^*, and suppose that either a resolvable solution is known, or at least a solution is known in a form where there are v^*/k^* blocks which give a complete replication. Then we can get a GD design with parameters

$$(9.0) \quad v = v^*, \qquad b = tb^* + a(v^*/k^*), \qquad r = tr^* + a, \qquad k = k^*$$

$$(9.1) \quad m = v^*/k^*, \qquad n = k^*, \qquad \lambda_1 = t\lambda^* + a, \qquad \lambda_2 = t\lambda^*$$

in the following manner. Choose a set of v^*/k^* blocks giving a complete replication. Repeat the BIB design t times, and then add the chosen set of blocks a times. Then we get a GD design with parameters given by (9.0) and (9.1), for which the groups are given by the chosen set of blocks.

When the BIB design is repeated t times, the chosen set of blocks is also repeated t times. Hence instead of adding the chosen set of blocks a times, we could delete the chosen set of blocks a_1 times ($a_1 \leqq t$). This would give a GD design with parameters (9.0) and (9.1) with $a = -a_1$. If the original BIB design is resolvable, then the derived GD design is also resolvable.

For example, if we start with the BIB designs of the series OS 1 whose parameeters are given by (3.3), we get resolvable GD designs with parameters

$$(9.2) \quad v = s^2, \qquad b = t(s^2 + s) + as, \qquad r = t(s + 1) + a, \qquad k = s$$

$$(9.3) \quad m = s, \qquad n = s \qquad \lambda_1 = t + a, \qquad \lambda_2 = t$$

where $a \geqq -t$, and s is a prime or a prime power. As an illustration let $s = 4$, $t = 1, a = -1$. The blocks of the BIB design $v^* = 16$, $b^* = 20$, $r^* = 5$, $k^* = 4$, $\lambda^* = 1$ are given in a resolvable form by (3.5). Hence the blocks of the GD design with parameters

$$v = 16, \qquad b = 16, \qquad r = 4, \qquad k = 4,$$

$$m = 4, \qquad n = 4, \qquad \lambda_1 = 0, \qquad \lambda_2 = 1$$

are obtained by taking any four replications from (3.5); the remaining replication then gives the groups.

The blocks of BIB designs belonging to the series OS 1 can be obtained in a resolvable form as explained in Section 3, by using the difference sets in Table I. The blocks for all other BIB designs occurring in Table IX can be found in Table III, being in a resolvable form in every case except $v^* = 45$, $b^* = 99$, $r^* = 11$, $k^* = 5$, $\lambda^* = 1$. In this case the block $(00_1, 00_2, 00_3, 00_4, 00_5)$, when developed mod (3, 3), provides a complete replication.

10. Extension of GD designs. Suppose that there exists a resolvable group divisible design with parameters

$$(10.0) \qquad v = k\alpha, \quad b = r\alpha, \quad r, \ k, \ m, \ n, \ \lambda_1, \ \lambda_2 = 1$$

TABLE IX

Parameters of GD designs derivable from BIB designs by replication addition or subtraction

Serial no. and series		Parameters of BIB design					Auxiliary parameters		Parameters of GD design							
		v^*	b^*	r^*	k^*	λ^*	t	a	v	b	r	k	m	n	λ_1	λ_2
(1)	$OS\ 1+$	16	20	5	4	1	1	−1	16	16	4	4	4	4	0	1
(2)	$OS\ 1+$	16	20	5	4	1	1	1	16	24	6	4	4	4	2	1
(3)	$OS\ 1+$	16	20	5	4	1	1	2	16	28	7	4	4	4	3	1
(4)	$OS\ 1+$	16	20	5	4	1	2	−2	16	32	8	4	4	4	0	2
(5)	$OS\ 1+$	16	20	5	4	1	2	−1	16	36	9	4	4	4	1	2
(6)	$OS\ 1+$	25	30	6	5	1	1	−1	25	25	5	5	5	5	0	1
(7)	$OS\ 1+$	25	30	6	5	1	1	1	25	35	7	5	5	5	2	1
(8)	$OS\ 1+$	25	30	6	5	1	1	2	25	40	8	5	5	5	3	1
(9)	$OS\ 1+$	25	30	6	5	1	2	−2	25	50	10	5	5	5	0	2
(10)	$OS\ 1+$	49	56	8	7	1	1	−1	49	49	7	7	7	7	0	1
(11)	$OS\ 1+$	49	56	8	7	1	1	1	49	63	9	7	7	7	2	1
(12)	$OS\ 1+$	49	56	8	7	1	1	2	49	70	10	7	7	7	3	1
(13)	$OS\ 1+$	64	72	9	8	1	1	−1	64	64	8	8	8	8	0	1
(14)	$OS\ 1+$	64	72	9	8	1	1	1	64	72	10	8	8	8	2	1
(15)	$OS\ 1+$	81	90	10	9	1	1	−1	81	81	9	9	9	9	0	1
(16)	T_1+	15	35	7	3	1	1	−1	15	30	6	3	5	3	0	1
(17)	T_1+	15	35	7	3	1	1	1	15	40	8	3	5	3	2	1
(18)	T_1+	15	35	7	3	1	1	2	15	45	9	3	5	3	3	1
(19)	F_2+	28	63	9	4	1	1	−1	28	54	8	4	7	4	0	1
(20)	F_2+	28	63	9	4	1	1	1	28	69	10	4	7	4	2	1
(21)	T_1+	21	70	10	3	1	1	−1	21	63	9	3	7	3	0	1
(22)	G_2	45	99	11	5	1	1	−1	45	90	10	5	9	5	0	1

so that the b blocks are divisible into r sets of α blocks, each set giving a complete replication. Let

$$(10.1) \qquad \begin{aligned} v' &= r, \quad b' = \frac{r(r-\alpha)}{k+1}, \quad r' = r - \alpha, \quad k' = k + 1, \\ m' &= \frac{r}{n}, \quad n' = n, \quad \lambda_1' = \lambda_1, \quad \lambda_2' = 1. \end{aligned}$$

Then clearly

$$v' = m'n', \qquad b'k' = v'r', \tag{10.2}$$

and it follows from (2.0) and (2.1) that

$$\lambda_1'(n' - 1) + \lambda_2'n'(m' - 1) = r'(k' - 1). \tag{10.25}$$

Hence if b' and m' are integers, the parameters v', b', r', k', m', n', λ_1', λ_2' given by (10.1) can be the parameters of a GD design. Suppose a combinatorial solution of this design is available. We shall show that in this case we can build up a solution of the GD design with parameters

$$\begin{aligned} v'' &= v + v', & b'' &= b + b', & r'' &= r, & k'' &= k + 1, \\ m'' &= m + m', & n'' &= n, & \lambda_1'' &= \lambda_1, & \lambda_2'' &= 1. \end{aligned} \tag{10.3}$$

Let the treatments in (10.0) and (10.1) be different so that there are altogether $v + v'$ treatments. To each block in the ith replication of (10.0) adjoin the ith treatment of (10.1), $(i = 1, 2, \cdots, r)$. To the design (10.0) so extended, add all the blocks of (10.1). This gives us a combinatorial solution of (10.3) where the groups are the groups of (10.0) and (10.1) taken together. It is easy to see that the necessary conditions are satisfied. This method may be called the method of extension.

As an illustration we shall build up the solution of the GD design

$$\begin{aligned} v &= 12, & b &= 24, & r &= 6, & k &= 3, \\ m &= 6, & n &= 2, & \lambda_1 &= 2, & \lambda_2 &= 1, \end{aligned} \tag{10.4}$$

starting from a solution of

$$\begin{aligned} v &= 6, & b &= 18, & r &= 6, & k &= 2, \\ m &= 3, & n &= 2, & \lambda_1 &= 2, & \lambda_2 &= 1 \end{aligned} \tag{10.45}$$

which can be obtained by adding one complete replication, say the last, to the solution (3.1) of the BIB design (3.0). Here $\alpha = 3$, and we see from (10.1) that for extension we require a solution of

$$\begin{aligned} v' &= 6, & b' &= 6, & r' &= 3, & k' &= 3, \\ m' &= 3, & n' &= 2, & \lambda_1' &= 2, & \lambda_2' &= 1. \end{aligned} \tag{10.5}$$

It is seen from Theorem 2 that a solution of this is obtainable by developing mod (6) the initial block (0,1,3). However to keep the treatments of (10.5) distinct from those of (10.45) we may replace the ith treatment of (10.5) by a_i. Proceeding as explained, the blocks of (10.4) are given by the columns of the scheme

$$\begin{array}{cccccccccccccccccccccccc} 1 & 2 & 0 & 2 & 3 & 1 & 3 & 4 & 2 & 4 & 0 & 3 & 0 & 1 & 4 & 0 & 1 & 4 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 4 & 3 & \infty & 0 & 4 & \infty & 1 & 0 & \infty & 2 & 1 & \infty & 3 & 2 & \infty & 3 & 2 & \infty & a_1 & a_2 & a_3 & a_4 & a_5 & a_0 \\ a_0 & a_0 & a_0 & a_1 & a_1 & a_1 & a_2 & a_2 & a_2 & a_3 & a_3 & a_3 & a_4 & a_4 & a_4 & a_5 & a_5 & a_5 & a_3 & a_4 & a_5 & a_0 & a_1 & a_2, \end{array} \tag{10.6}$$

and the groups are given by the columns of

$$(10.65)\qquad \begin{matrix} 0 & 1 & 4 & a_0 & a_1 & a_2 \\ 3 & 2 & \infty & a_3 & a_4 & a_5 . \end{matrix}$$

Again we can build up the solution of the GD design

$$(10.7)\qquad \begin{matrix} v = 24, & b = 54, & r = 9, & k = 4, \\ m = 8, & n = 3, & \lambda_1 = 3, & \lambda_2 = 1 \end{matrix}$$

by starting with the design

$$(10.8)\qquad \begin{matrix} v = 15, & b = 45, & r = 9, & k = 3, \\ m = 5, & n = 3, & \lambda_1 = 3, & \lambda_2 = 1 \end{matrix}$$

which is design (18) of Table IX, and use for extension the solution of

$$(10.9)\qquad \begin{matrix} v' = 9, & b' = 9, & r' = 4, & k' = 4, \\ m' = 3, & n' = 3, & \lambda_1' = 3, & \lambda_2' = 1 \end{matrix}$$

which can be obtained by developing mod (9) the initial block (0,1,3,6).

11. Addition of GD designs. The method of addition consists of getting a new GD design by taking together the blocks of two suitable GD designs with the same v and k. It may be regarded as a slight generalization of the method of replication addition discussed in Section 9. This will be explained by two examples.

(a) If in (7.4) we put $m = s - 1$, we get the GD design

$$(11.0)\qquad \begin{matrix} v = s^2 - s, & b = s^2, & r = s, & k = s - 1, \\ m = s - 1, & n = s, & \lambda_1 = 0, & \lambda_2 = 1, \end{matrix}$$

a solution of which is available if s is a prime or a prime power.

If we take the s blocks formed by taking all possible combinations of $s - 1$ treatments from the ith group, we get an unreduced BIB design with parameters

$$(11.15)\qquad v^* = b^* = s, \quad r^* = k^* = s - 1, \quad \lambda^* = s - 2.$$

Repeating this for each group and taking together all the BIB designs so formed we get the GD design

$$(11.2)\qquad \begin{matrix} v = s^2 - s, & b = s^2 - s, & r = s - 1, & k = s - 1, \\ m = s - 1, & n = s, & \lambda_1 = s - 2, & \lambda_2 = 0. \end{matrix}$$

Taken by itself this is a disconnected design in the sense explained in [23] and [24], and any contrast between treatments of different groups is nonestimable. But if we take together the blocks of (11.0) and (11.2) we get the GD design

$$(11.35)\quad \begin{array}{llll} v = s^2 - s, & b = 2s^2 - s, & r = 2s - 1, & k = s - 1, \\ m = s - 1, & n = s, & \lambda_1 = s - 2, & \lambda_2 = 1. \end{array}$$

As an illustration we give below the blocks for the case $s = 4$ (Design (3) of Table X).

$$(11.4)\quad \begin{array}{cccccccccccccccccccccccccccc} 1 & 4 & 3 & 2 & 2 & 4 & 1 & 3 & 4 & 2 & 1 & 3 & 1 & 2 & 3 & 4 & 2 & 1 & 1 & 1 & 6 & 5 & 5 & 5 & 10 & 9 & 9 & 9 \\ 8 & 5 & 6 & 7 & 8 & 6 & 7 & 5 & 7 & 5 & 6 & 8 & 5 & 6 & 7 & 8 & 3 & 3 & 2 & 2 & 7 & 7 & 6 & 6 & 11 & 11 & 10 & 10 \\ 11 & 9 & 12 & 10 & 9 & 10 & 12 & 11 & 11 & 12 & 9 & 10 & 10 & 11 & 9 & 12 & 4 & 4 & 4 & 3 & 8 & 8 & 8 & 7 & 12 & 12 & 12 & 11. \end{array}$$

The corresponding groups are given by the columns of the scheme

$$(11.45)\quad \begin{array}{ccc} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12. \end{array}$$

TABLE X

Parameters of GD designs obtainable by extension and addition

Serial no.	Parameters							
	v	b	r	k	m	n	λ_1	λ_2
(1)	12	24	6	3	6	2	2	1
(2)	24	54	9	4	8	3	3	1
(3)	12	28	7	3	3	4	2	1
(4)	20	45	9	4	4	5	3	1
(5)	12	32	8	3	2	6	2	1

The first 16 blocks of (11.4) are obtained by taking the first three rows of (7.5), whereas the remaining 12 blocks are obtained by taking all combinations of three treatments from each group.

By taking $s = 5$ in (11.35) we get design (4) of Table X.

(b) Suppose we have solutions available for GD designs with parameters

$$(11.5)\quad v = mn, \; b, \; r, \; k, \; m, \; n, \; \lambda_1, \; \lambda_2$$

$$(11.6)\quad v' = mn/\alpha, \; b', \; r', \; k' = k, \; m' = m/\alpha, \; n' = n, \; \lambda_1', \; \lambda_2'$$

where m' and α are integers, and

$$(11.65)\quad \lambda_1 + \lambda_1' = \lambda_2 + \lambda_2' = \lambda_1'' \text{ (say).}$$

The m groups of (11.5) can be divided into α sets each of m' groups. With the v' treatments occurring in any such set we can write down a solution for (11.6).

If we do this for each set and add the $\alpha b'$ blocks so obtained to the blocks of (11.5) we get the solution of a GD design with parameters

$$(11.7)\quad \begin{array}{llll} v'' = mn, & b'' = b + b'\alpha, & r'' = r + r', & k'' = k, \\ m'' = \alpha, & n'' = m'n, & \lambda_1'', & \lambda_2'' = \lambda_2 \end{array}$$

where the treatments occurring in a set now belong to the same group. Obviously every treatment occurs $r + r'$ times in the final design, but we have to show that any two treatments belonging to the same set occur together λ_1'' times, and any two treatments belonging to different sets occur together λ_2 times.

If two treatments belong to the same set, they either occur together in the same group or in different groups. In the first case they occur together in λ_1 blocks obtained from (11.5) and in λ_1' blocks obtained from (11.6). In the second case they occur together in λ_2 blocks obtained from (11.5) and λ_2' blocks obtained from (11.6). It follows from (11.65) that in either case they occur together λ_1'' times.

Again if two treatments belong to different sets they will occur together in λ_2 blocks obtained from (11.5) and in no blocks obtained from (11.6). This completes the proof.

As an illustration let us start with the GD design with parameters

$$(11.75)\quad \begin{array}{llll} v = 12, & b = 20, & r = 5, & k = 3, \\ m = 6, & n = 2, & \lambda_1 = 0, & \lambda_2 = 1, \end{array}$$

the blocks of which are given by the columns of (5.7) and the groups by (5.8).

Let us take $\alpha = 2$, and let the first three groups belong to the first set and the last three groups to the second set. Also as noted in Section 10 a solution of the GD design with parameters

$$(11.8)\quad \begin{array}{lll} v' = b' = 6, & r' = k' = 3, & m' = 3, \\ n' = 2, & \lambda_1' = 2, & \lambda_2' = 1 \end{array}$$

is given by the last six columns of (10.6), and the groups by the last three columns of (10.65). We note that $\lambda_1 + \lambda_1' = \lambda_2 + \lambda_2' = 2$. Hence we can build up a solution of

$$(11.85)\quad \begin{array}{llll} v = 12, & b = 32, & r = 8, & k = 3, \\ m = 2, & n = 6, & \lambda_1 = 2, & \lambda_2 = 1 \end{array}$$

by adding to the solution of (11.75) a solution of (11.8) twice over identifying a_0, a_1, a_2, a_3, a_4, a_5, once with 5, 11, 2, 7, 6, 8 and next with 9, 10, 4, 12, 1, 3 respectively. Thus the 32 blocks of (11.85) are given by the 20 columns of the scheme (5.7) together with the twelve columns of the following scheme

(11.9)

5	11	2	7	6	8	9	10	4	12	1	3
11	2	7	6	8	5	10	4	12	1	3	9
7	6	8	5	11	2	12	1	3	9	10	4.

The first group consists of the treatments 5, 11, 2, 7, 6, 8 and the second group consists of 9, 10, 4, 12, 1, 3.

The parameters of GD designs obtainable by extension and addition are shown in Table X.

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